Feng and Pelicer *Boundary Value Problems* (2015) 2015:206
DOI 10.1186/s13661-015-0468-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global existence and exponential stability for a nonlinear Timoshenko system with delay

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Abstract

This paper is concerned with a nonlinear Timoshenko system modeling clamped thin elastic beams with time delay. The delay is defined on a feedback term associated to the equation for rotation angle. Under suitable assumptions on the data, we establish the well-posedness of the problem with respect to weak solutions. We also establish the exponential stability of the system under the usual equal wave speeds assumption.

MSC: 35B40

Keywords: Timoshenko system; time delay; global existence; exponential stability

1 Introduction

In this paper, we are concerned with a Timoshenko system with time delay,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(x,t-\tau) + f(\psi) = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,1) \times \mathbb{R}^+$. When $\mu_1 = \mu_2 = f = 0$, this system was proposed by Timoshenko [1] as a model for vibrations of a thin elastic beam of length 1. Here, $\varphi = \varphi(x,t)$ denotes the transverse displacement of the beam, $\psi = \psi(x,t)$ denotes the rotation angle of the beam's filament and ρ_1, ρ_2, k, b are positive constants related to physical properties of the beam. In the system, $\mu_1\psi_t$ represents a frictional damping and $f(\psi)$ is a forcing term. The time delay is given by $\mu_2\psi_t(x,t-\tau)$, where μ_1, μ_2, τ are positive constants.

To the system we add the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0, \qquad \varphi_t(x,0) = \varphi_1, \qquad \psi(x,0) = \psi_0, \qquad \psi_t(x,0) = \psi_1, \\ \psi_t(x,t-\tau) = f_0(x,t-\tau), \quad t \in (0,\tau), \end{cases} \tag{1.2}$$

where f_0 is prescribed, and the Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad \forall t \geq 0. \tag{1.3}$$

We observe that our problem is set in a context where: (a) the damping is defined only on the equation for rotation angle; (b) the presence of a time delay; (c) exponential stability

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under a nonlinear forcing. Under this scenario we briefly comment some of related early works.

For partially damped Timoshenko systems, an important result was presented by Soufyane [2, 3]. He showed that the linear system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0,$$
$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \psi_t = 0$$

is exponentially stable if and only if

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}. \tag{1.4}$$

This assumption, which means that both waves on the system have equal propagation speed, was later extended to several other problems based on Timoshenko systems. We refer the reader to the references [4–13] among others.

On the other hand, dynamics of delay systems have been a major research subject in differential equations (see, *e.g.*, [14, 15]). It is known that a time delay on the feedback term (internal or at the boundary) in a wave equation can destabilize the system, depending on the weight of each term, as discussed in Datko *et al.* [16] and Nicaise and Pignotti [17, 18]. Following that context, Said-Houari and Laskri [13] studied the stability of system (1.1) with $f(\psi) = 0$. They proved that, under condition (1.4) and $\mu_2 < \mu_1$, the system is exponentially stable.

In the present paper our objective is to extend the result of Said-Houari and Laskri [13] to a nonlinear framework by adding a forcing term $f(\psi)$. The rest of the paper is organized as follows. In Section 2, we present some preliminary remarks and the main results. In Section 3, we prove the well-posedness of system (1.1)-(1.3) by using semigroup theory. In Section 4, we prove the exponential stability of system (1.1)-(1.3) by using energy methods.

2 Preliminaries and main results

In this paper we use standard Lebesgue and Sobolev spaces

$$L^q(0,1), \quad 1 \le q \le \infty, \text{and } H_0^1(0,1).$$

In the case $q = 2$ we write $\|u\|$ instead of $\|u\|_2$.

Now we give some hypotheses on the forcing term $f(\psi(x,t))$. We assume $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\left|f(\psi^2) - f(\psi^1)\right| \le k_0\left(\left|\psi^1\right|^\theta + \left|\psi^2\right|^\theta\right)\left|\psi^1 - \psi^2\right| \quad \text{for all } \psi^1, \psi^2 \in \mathbb{R}, \tag{2.1}$$

where $k_0 > 0$, $\theta > 0$. In addition we assume that

$$0 \le \hat{f}(\psi) \le f(\psi)\psi \quad \text{for all } \psi \in \mathbb{R}, \tag{2.2}$$

with $\hat{f}(z) = \int_0^z f(s)\,ds$.

In order to deal with the delay feedback term, motivated by [13, 17, 18], we define the following new dependent variable:

$$z(x,\rho,t)=\psi_t(x,t-\tau\rho),\quad x\in(0,1),\rho\in(0,1),t>0. \tag{2.3}$$

Then it is easy to verify

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\quad \text{in } (0,1)\times(0,1)\times(0,\infty). \tag{2.4}$$

Thus, equations (1.1) are transformed to

$$\begin{cases} \rho_1\varphi_{tt}(x,t)-k(\varphi_x+\psi)_x(x,t)=0, \\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+k(\varphi_x+\psi)(x,t) \\ \qquad +\mu_1\psi_t(x,t)+\mu_2 z(x,1,t)+f(\psi(x,t))=0, \\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0, \end{cases} \tag{2.5}$$

with $x\in(0,1)$, $\rho\in(0,1)$ and $t>0$, and the initial and boundary conditions are

$$\begin{cases} \varphi(x,0)=\varphi_0,\quad \varphi_t(x,0)=\varphi_1,\quad \psi(x,0)=\psi_0,\quad \psi_t(x,0)=\psi_1,\quad x\in(0,1) \\ z(x,\rho,0)=f_0(x,-\rho\tau),\quad (x,t)\in(0,1)\times(0,\tau), \\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0,\quad t>0, \\ z(x,0,t)=\psi_t(x,t),\quad x\in(0,1),t>0. \end{cases} \tag{2.6}$$

First of all, we shall show the well-posedness of problem (2.5)-(2.6).

Before using the semigroup theory, we introduce two new dependent variables $u=\varphi_t$ and $v=\psi_t$, then problem (2.5)-(2.6) is reduced to the following problem for an abstract first-order evolutionary equation:

$$\begin{cases} \frac{dU}{dt}(t)=\mathcal{A}U+F,\quad t>0, \\ U(0)=U_0=(\varphi_0,\varphi_1,\psi_0,\psi_1,f_0(\cdot,-\tau))^T, \end{cases} \tag{2.7}$$

where $U=(\varphi,u,\psi,v,z)^T$, and

$$\mathcal{A}U=\begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_{xx}+\psi_x) \\ v \\ \frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+\psi)-\frac{\mu_1}{\rho_2}v-\frac{\mu_2}{\rho_2}z(\cdot,1) \\ -\frac{1}{\tau}z_\rho \end{pmatrix},\qquad F=\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\rho_2}f(\psi) \\ 0 \end{pmatrix},$$

with the domain

$$D(\mathcal{A})=\bigl\{(\varphi,u,\psi,v,z)^T\in H: v=z(\cdot,0),\ \text{in } (0,1)\bigr\}, \tag{2.8}$$

where

$$\begin{aligned} H={}&\bigl(H^2(0,1)\cap H_0^1(0,1)\bigr)\times H_0^1(0,1)\times\bigl(H^2(0,1)\cap H_0^1(0,1)\bigr) \\ &\times H_0^1(0,1)\times L^2\bigl(0,1;H_0^1(0,1)\bigr). \end{aligned}$$

We define the energy space $\mathscr{H}$ by

$$\mathscr{H} := H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2\big((0,1) \times (0,1)\big). \tag{2.9}$$

For $U = (\varphi, u, \psi, v, z)^T$, $\overline{U} = (\overline{\varphi}, \overline{u}, \overline{\psi}, \overline{v}, \overline{z})^T$ and for ξ a positive constant satisfying

$$\tau\mu_2 \le \xi \le \tau(2\mu_1 - \mu_2), \tag{2.10}$$

we equip $\mathscr{H}$ with the inner product

$$\begin{aligned} \langle U, \overline{U} \rangle_{\mathscr{H}} = {} & \int_0^1 \big[\rho_1 u\overline{u} + \rho_2 v\overline{v} + k(\varphi_x + \psi)(\overline{\varphi}_x + \overline{\psi}) + b\psi_x\overline{\psi}_x\big]\,dx \\ & + \xi \int_0^1 \int_0^1 z(x,\rho)\overline{z}(x,\rho)\,d\rho\,dx. \end{aligned} \tag{2.11}$$

Now we give the result of the well-posedness of solutions to problem (2.7).

Theorem 2.1 *Assume that* (2.1)-(2.2) *and* $\mu_2 \le \mu_1$ *hold, then we have the following results.*

(i) *If* $U_0 \in \mathscr{H}$, *then problem* (2.7) *has a unique mild solution* $U \in C([0,\infty), \mathscr{H})$ *with* $U(0) = U_0$.

(ii) *If* U_1 *and* U_2 *are two mild solutions of problem* (2.7), *then there exists a positive constant* $C_0 = C(U_1(0), U_2(0))$ *such that*

$$\big\| U_1(t) - U_2(t) \big\|_{\mathscr{H}} \le e^{C_0 T} \big\| U_1(0) - U_2(0) \big\|_{\mathscr{H}} \quad \text{for any } 0 \le t \le T. \tag{2.12}$$

(iii) *If* $U_0 \in D(\mathcal{A})$, *then the above mild solution is a strong solution.*

The functional energy of solutions of problem (2.5)-(2.6) is defined by

$$\begin{aligned} E(t) = {} & \frac{1}{2} \int_0^1 \big[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k|\varphi_x + \psi|^2 + b\psi_x^2\big]\,dx \\ & + \frac{\xi}{2} \int_0^1 \int_0^1 z^2(x,\rho,t)\,d\rho\,dx + \int_0^1 \hat{f}\big(\psi(t)\big)\,dx. \end{aligned} \tag{2.13}$$

Below we shall give the stability result.

Theorem 2.2 *Assume that* (2.1)-(2.2) *and* $\mu_2 < \mu_1$ *hold. Assume that* (1.4) *also holds. Then, with respect to mild solutions, there exist* $C > 0$ *and* $\eta > 0$ *such that*

$$E(t) \le Ce^{-\eta t}, \quad t \ge 0. \tag{2.14}$$

3 The well-posedness

In this section, we shall study the well-posedness of solutions to problem (2.5)-(2.6) to complete the proof of Theorem 2.1.

Lemma 3.1 *The energy $E(t)$ defined by* (2.13) *is a nonincreasing function along the solution trajectories, i.e., there exists a positive constant C such that for any $t \geq 0$,*

$$E'(t) \leq -C\int_0^1 \psi_t^2(x,t)\,dx - C\int_0^1 z^2(x,1,t)\,dx \leq 0, \tag{3.1}$$

and there exist two positive constants δ_0 and C_1, independent of initial data in $\mathscr{H}$, such that for any $t \geq 0$,

$$\begin{aligned} E(t) \geq \delta_0\biggl(&\int_0^1 \varphi_t^2\,dx + \int_0^1 \psi_t^2\,dx + \int_0^1 |\varphi_x + \psi|^2\,dx + \int_0^1 \psi_x^2\,dx \\ &+ \int_0^1\int_0^1 z^2(x,\rho)\,d\rho\,dx\biggr) - C_1. \end{aligned} \tag{3.2}$$

Proof Multiplying the first equation in (2.5) by φ_t, the second equation by ϕ_t, integrating the result over $(0,1)$ with respect to x and using Young's inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \bigl[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k|\varphi_x + \psi|^2 + b\psi_x^2\bigr]\,dx\right) \\ &\quad = -\mu_1\int_0^1 \psi_t\,dx - \mu_2\int_0^1 \psi_t z(x,1,t)\,dx \\ &\quad \leq \left(-\mu_1 + \frac{\mu_2}{2}\right)\int_0^1 \psi_t\,dx + \frac{\mu_2}{2}\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{3.3}$$

We multiply the third equation in (2.5) by $\frac{\xi}{\tau}z$ and integrate the result over $(0,1)\times(0,1)$ with respect to ρ and x, respectively, to get

$$\begin{aligned} \frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 z(x,\rho,t)\,d\rho\,dx &= -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx \\ &= \frac{\xi}{2\tau}\int_0^1 \bigl(z^2(x,0,t) - z^2(x,1,t)\bigr)\,dx, \end{aligned}$$

which, together with (3.3), (2.10) and the fact $\frac{d}{dt}\hat{f}(\psi) = f(\psi)\psi_t$, gives us (3.1).

It is easy to get (3.2) by using (2.2) with $\delta_0 = \min\{\frac{1}{2}, \frac{\xi}{2}\}$. The proof is therefore complete. □

Lemma 3.2 *The operator $\mathcal{A}$ defined in* (2.7) *is the infinitesimal generator of a C^0-semigroup in $\mathscr{H}$.*

Proof It follows from (3.1) that for all $U(t) \in D(\mathcal{A})$,

$$\langle \mathcal{A}U, U\rangle_{\mathscr{H}} \leq -C\int_0^1 \psi_t^2(x,t)\,dx - C\int_0^1 z^2(x,1,t)\,dx \leq 0,$$

which implies that the operator $\mathcal{A}$ is a dissipative operator.

Next we will prove that the operator $I - \mathcal{A}$: $D(\mathcal{A}) \to \mathscr{H}$ is onto, that is, given $U^* = (f_1,f_2,f_3,f_4,f_5)^T \in \mathscr{H}$, we seek $U = (\varphi,u,\psi,v,z)^T \in D(\mathcal{A})$ is a solution of $(I - \mathcal{A})U = U^*$. We have

$$\begin{cases} \varphi - u = f_1, \\ u - \frac{k}{\rho_1}(\varphi_x + \psi)_x = f_2, \\ \psi - v = f_3, \\ v - \frac{b}{\rho_2}\psi_{xx} + \frac{b}{\rho_2}(\varphi_x + \psi) + \frac{\mu_1}{\rho_2}v + \frac{\mu_2}{\rho_2}z(\cdot, 1) = f_4, \\ z + \frac{1}{\tau}z_\rho = f_5. \end{cases}$$

These equations can be solved following Said-Houari and Laskri [13] or Nicaise and Pignotti [17].

Then we can infer that the operator $\mathcal{A}$ is m-dissipative in $\mathscr{H}$. Since $D(\mathcal{A})$ is dense in $\mathscr{H}$, thus we can conclude that the operator $\mathcal{A}$ is the infinitesimal generator of a C^0-semigroup in $\mathscr{H}$ by the Lumer-Phillips theorem (see, for example, Pazy [19]). The proof is now complete. □

Lemma 3.3 *The operator F defined in* (2.7) *is locally Lipschitz in $\mathscr{H}$.*

Proof Let $U_1 = (\varphi^1, u^1, \psi^1, v^1, z^1)$ and $U_1 = (\varphi^2, u^2, \psi^2, v^2, z^2)$, then we have

$$\|F(U_1) - F(U_2)\|_{\mathscr{H}} \le \|f(\psi^1) - f(\psi^2)\|_{L^2}.$$

By using (2.1), Hölder's and Poincaré's inequalities, we can obtain

$$\begin{aligned} \|f(\psi^1) - f(\psi^2)\|_{L^2} &\le \left(\|\psi^1\|_{2\theta}^{\theta} + \|\psi^2\|_{2\theta}^{\theta}\right)\|\psi^1 - \psi^2\| \\ &\le C_1\|\psi_x^1 - \psi_x^2\|, \end{aligned}$$

which gives us

$$\|F(U_1) - F(U_2)\|_{\mathscr{H}} \le C_1\|U_1 - U_2\|_{\mathscr{H}}.$$

Then the operator F is locally Lipschitz in $\mathscr{H}$. The proof is hence complete. □

Proof of Theorem 2.1 It follows from Lemmas 3.2-3.3 that the Cauchy problem has a unique local mild solution

$$U(t) = e^{\mathcal{A}t}U_0 + \int_0^t e^{\mathcal{A}(t-s)}F(U(s))\,ds \tag{3.4}$$

defined in a maximal interval $(0, t_{\max})$.

If $t_{\max} < \infty$, then

$$\lim_{t\to\infty}\|U(t)\|_{\mathscr{H}} = +\infty. \tag{3.5}$$

Let $U(t)$ be a mild solution with $U_0 \in D(\mathcal{A})$. By using Theorem 6.1.5 in Pazy [19], we conclude that it is a strong solution. It follows from (3.2) that for all $t \ge 0$,

$$\|U(t)\|_{\mathscr{H}}^2 \le \frac{1}{\delta_0}\left(E(0) + C_1\right),$$

which, by density, holds for mild solutions. Then it is a contradiction with (3.5) and therefore $t_{\max} = \infty$, that is, the solution is global. The proof of (i) of Theorem 2.1 is complete.

It is easy to get inequality (2.12) by using (3.4), the local Lipschitz behavior of F and Gronwall's inequality. Then we can obtain the continuous dependence on the initial data for mild solutions. This proves the item (ii) of Theorem 2.1.

By using Theorem 6.1.5 in Pazy [19] (see also [20]), we know that any mild solutions with initial data in $D(\mathcal{A})$ are strong. Then the proof of Theorem 2.1 is therefore complete. □

4 Exponential stability

In this section, we shall prove Theorem 2.2, which will be divided into the following lemmas.

Lemma 4.1 *Let* $(\varphi, \varphi_t, \psi, \psi_t, z)$ *be the solution of problem* (2.5)-(2.6). *The functional* I_1 *defined by*

$$I_1 = -\int_0^1 (\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t)\,dx - \frac{\mu_1}{2}\int_0^1 \psi^2\,dx \tag{4.1}$$

satisfies that for any $\varepsilon > 0$,

$$\begin{aligned} \frac{d}{dt}I_1(t) \le{}& -\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right)dx + k\int_0^1 |\varphi_x + \psi|^2\,dx + (b + C_1 + \varepsilon)\int_0^1 \psi_x^2\,dx \\ &+ \frac{\mu_2^2}{4\lambda_1\varepsilon}\int_0^1 z^2(x,1,t)\,dx, \end{aligned} \tag{4.2}$$

hereafter $\lambda_1 > 0$ *is the first eigenvalue of* $-\Delta$ *in* $H_0^1(0,1)$.

Proof A straightforward calculation gives

$$\frac{dI_1}{dt} = -\int_0^1 (\rho_1\varphi_{tt}\varphi + \rho_2\psi_{tt}\psi)\,dx - \int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right)dx - \mu_1\int_0^1 \psi\psi_t\,dx.$$

Using (2.5) and integrating by parts, we see that

$$\begin{aligned} \frac{dI_1}{dt} ={}& -\rho_1\int_0^1 \varphi_t^2\,dx - \rho_2\int_0^1 \psi_t^2\,dx + k\int_0^1 |\varphi_x + \psi|^2\,dx + b\int_0^1 \psi_x^2\,dx \\ &+ \mu_2\int_0^1 z(x,1,t)\psi\,dx + \int_0^1 f(\psi)\psi\,dx. \end{aligned} \tag{4.3}$$

It follows from Young's inequality and Poincaré's inequality that for any $\varepsilon > 0$,

$$\begin{aligned} \int_0^1 \left|z(x,1,t)\psi\right|dx &\le \varepsilon\lambda_1\int_0^1 \psi^2\,dx + \frac{1}{4\varepsilon\lambda_1}\int_0^1 z^2(x,1,t)\,dx \\ &\le \varepsilon\int_0^1 \psi_x^2\,dx + \frac{1}{4\varepsilon\lambda_1}\int_0^1 z^2(x,1,t)\,dx, \end{aligned} \tag{4.4}$$

$$\int_0^1 \left|f(\psi)\psi\right|dx \le \int_0^1 |\psi|^\theta|\psi||\psi|\,dx \le \|\psi\|_{2(\theta+1)}^\theta\|\psi\|_{2(\theta+1)}\|\psi\| \le C_1\int_0^1 \psi_x^2\,dx, \tag{4.5}$$

which, together with (4.3)-(4.4), gives us (4.2). The proof is now complete. □

Lemma 4.2 *Let $(\varphi,\varphi_t,\psi,\psi_t,z)$ be the solution of problem* (2.5)-(2.6). *We define the functional I_2 by*

$$I_2(t)=\int_0^1(\rho_2\psi_t\psi+\rho_1\varphi_t g)\,dx+\frac{\mu_1}{2}\int_0^1\psi^2\,dx,\tag{4.6}$$

where g is the solution of

$$-g_{xx}=\psi_x,\qquad g|_{x=0,1}=0.\tag{4.7}$$

Then the functional I_2 satisfies, for any $\eta,\tilde{\eta}>0$,

$$\begin{aligned}\frac{d}{dt}I_2(t)\le{}&(\mu_2\eta-b)\int_0^1\psi_x^2\,dx+\left(\rho_2+\frac{\rho_2}{4\tilde{\eta}}\right)\int_0^1\psi_t^2\,dx+\frac{\rho_1}{\lambda_1}\tilde{\eta}\int_0^1\varphi_t^2\,dx\\&+\frac{\mu_2}{4\eta\lambda_1}\int_0^1 z^2(x,1,t)\,dx-\int_0^1\hat{f}(\psi)\,dx.\end{aligned}\tag{4.8}$$

Proof We know from (2.5) that

$$\begin{aligned}\frac{d}{dt}I_2(t)={}&-b\int_0^1\psi_x^2\,dx+\rho_2\int_0^1\psi_t^2\,dx-k\int_0^1\psi^2\,dx+k\int_0^1 g_x^2\,dx\\&+\rho_1\int_0^1\varphi_t g_t\,dx-\mu_2\int_0^1\psi z(x,1,t)\,dx-\int_0^1 f(\psi)\psi\,dx.\end{aligned}\tag{4.9}$$

By (4.7), we can get

$$\begin{cases}\int_0^1 g_x^2\,dx\le\int_0^1\psi^2\,dx\le\int_0^1\psi_x^2\,dx,\\ \int_0^1 g_t^2\,dx\le\int_0^1 g_{xt}^2\,dx\le\int_0^1\psi_t^2\,dx.\end{cases}\tag{4.10}$$

Using Young's inequality and Poincaré's inequality, we have

$$\begin{aligned}\mu_2\int_0^1\left|\psi z(x,1,t)\right|dx&\le\mu_2\eta\lambda_1\int_0^1\psi^2\,dx+\frac{\mu_2}{4\eta\lambda_1}\int_0^1 z^2(x,1,t)\,dx\\&\le\mu_2\eta\int_0^1\psi_x^2\,dx+\frac{\mu_2}{4\eta\lambda_1}\int_0^1 z^2(x,1,t)\,dx,\end{aligned}\tag{4.11}$$

$$\begin{aligned}\rho_1\int_0^1|\varphi_t g_t|\,dx&\le\frac{\rho_1}{\lambda_1}\tilde{\eta}\int_0^1\varphi_t^2\,dx+\frac{\rho_1\lambda_1}{4\tilde{\eta}}\int_0^1 g_t^2\,dx\\&\le\frac{\rho_1}{\lambda_1}\tilde{\eta}\int_0^1\varphi_t^2\,dx+\frac{\rho_1}{4\tilde{\eta}}\int_0^1\psi_t^2\,dx.\end{aligned}\tag{4.12}$$

Combining (2.2) and (4.11)-(4.12) with (4.9) and (2.2), we can complete the proof. □

Now we define the following functional:

$$J(t):=\rho_2\int_0^1\psi_t(\varphi_x+\psi)\,dx+\rho_2\int_0^1\psi_x\varphi_t\,dx.\tag{4.13}$$

Then we may get the following lemma.

Lemma 4.3 *Let* $(\varphi, \varphi_t, \psi, \psi_t, z)$ *be the solution of problem* (2.5)-(2.6), *and assume that* (1.4) *holds. Then the functional* $J(t)$ *satisfies, for any* $\varepsilon > 0$,

$$\begin{aligned}\frac{d}{dt}J(t) \le{}& b[\psi_x\varphi_x]_{x=0}^{x=1} - \frac{k}{2}\int_0^1 (\varphi_x + \psi)^2\,dx + \left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right)\int_0^1 \psi_x^2\,dx \\ &+ \left(\rho_2 + \frac{\mu_1^2}{k}\right)\int_0^1 \psi_t^2\,dx + \frac{\mu_2^2}{k}\int_0^1 z^2(x,1,t)\,dx \\ &- \int_0^1 \hat{f}(\psi)\,dx. \end{aligned} \tag{4.14}$$

Proof By taking a derivative of (4.13), we arrive at

$$\begin{aligned}\frac{d}{dt}J(t) ={}& \rho_2\int_0^1 \psi_{tt}(\varphi_x + \phi)\,dx + \rho_2\int_0^1 \psi_t(\varphi_x + \psi)_t\,dx \\ &+ \rho_2\int_0^1 \psi_{xt}\varphi_t\,dx + \rho_2\int_0^1 \psi_x\varphi_{tt}\,dx.\end{aligned}$$

Using (2.5), (1.4) and integration by parts, we get

$$\begin{aligned}\frac{d}{dt}J(t) ={}& b[\psi_x\varphi_x]_{x=0}^{x=1} + \rho_2\int_0^1 \psi_t^2\,dx - k\int_0^1 (\varphi_x + \psi)^2\,dx \\ &- \mu_1\int_0^1 \psi_t(\varphi_x + \psi)\,dx - \mu_2\int_0^1 (\varphi_x + \psi)z(x,1,t)\,dx \\ &- \int_0^1 \varphi_x f(\psi)\,dx - \int_0^1 f(\psi)\psi\,dx.\end{aligned}$$

By using Young's inequality and Poincaré's inequality, we know that for any $\varepsilon > 0$,

$$-\mu_1\int_0^1 \psi_t(\varphi_x + \psi)\,dx \le \frac{k}{4}\int_0^1 (\varphi_x + \psi)^2\,dx + \frac{\mu_1^2}{k}\int_0^1 \psi_t^2\,dx, \tag{4.15}$$

$$-\mu_2\int_0^1 (\varphi_x + \psi)z(x,1,t)\,dx \le \frac{k}{4}\int_0^1 (\varphi_x + \psi)^2\,dx + \frac{\mu_2^2}{k}\int_0^1 z^2(x,1,t)\,dx, \tag{4.16}$$

and

$$\begin{aligned}\int_0^1 \left|\varphi_x f(\psi)\right|dx &\le \|\varphi_x\|\|\psi\|_{2(\theta+1)}^{\theta}\|\psi\|_{2(\theta+1)} \\ &\le \frac{\varepsilon}{2b^2}\int_0^1 \varphi_x^2\,dx + \frac{b^2}{2\varepsilon\lambda_1}\int_0^1 \psi_x^2\,dx \\ &\le \frac{\varepsilon}{b^2}\int_0^1 (\varphi_x + \psi)^2\,dx + \frac{\varepsilon}{b^2}\int_0^1 \psi^2\,dx + \frac{b^2}{2\varepsilon\lambda_1}\int_0^1 \psi_x^2\,dx \\ &\le \frac{\varepsilon}{b^2}\int_0^1 (\varphi_x + \psi)^2\,dx + \left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right)\int_0^1 \psi_x^2\,dx, \end{aligned} \tag{4.17}$$

which, together with (4.15)-(4.16), gives us (4.14). The proof is now complete. □

Next we deal with the boundary term in (4.14). As in [13], we define the function

$$q(x) = -4x + 2, \quad x \in (0,1). \tag{4.18}$$

Lemma 4.4 *Let* $(\varphi, \varphi_t, \psi, \psi_t, z)$ *be the solution of problem* (2.5)-(2.6), *then the following estimate holds for any* $\varepsilon > 0$:

$$\begin{aligned} b[\psi_x \varphi_x]_{x=0}^{x=1} \le & -\frac{\varepsilon\rho_1}{k}\frac{d}{dt}\int_0^1 q\varphi_t\varphi_x\,dx - \frac{\rho_2 b}{4\varepsilon}\frac{d}{dt}\int_0^1 q\psi_t\psi_x\,dx + \frac{2\rho_1\varepsilon}{k}\int_0^1 \varphi_t^2\,dx \\ & + \left(\varepsilon + \frac{b^2}{2} + \frac{b^2}{2\varepsilon} + \frac{b^2}{4\varepsilon^3} + \frac{3b^2}{4} + \frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right)\int_0^1 \psi_x^2\,dx \\ & + \left(\frac{\rho_2 b}{2\varepsilon} + \frac{\mu_1^2}{4\varepsilon^2}\right)\int_0^1 \psi_t^2\,dx + \frac{k^2\varepsilon}{4}\int_0^1 (\varphi_x + \psi)^2\,dx \\ & + \frac{\mu_2^2}{4\varepsilon^2}\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{4.19}$$

Proof The same argument as in [13], we know that for any $\varepsilon > 0$,

$$b[\psi_x\varphi_x]_{x=0}^{x=1} \le \varepsilon\big[\varphi_x^2(1) + \varphi_x^2(0)\big] + \frac{b^2}{4\varepsilon}\big[\psi_x^2(1) + \psi_x^2(0)\big]. \tag{4.20}$$

By using (2.5), Young's inequality, integration by parts and the following fact

$$\frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x\,dx = \int_0^1 b\rho_2\psi_{tt}\psi_x\,dx + \int_0^1 b\rho_2 q\psi_t\psi_{xt}\,dx,$$

we see that

$$\begin{aligned} \frac{d}{dt}\int_0^1 b\rho_2 q\psi_t\psi_x\,dx \le & -b^2\big[\psi_x^2(1) + \psi_x^2(0)\big] + 2b^2\int_0^1 \psi_x^2\,dx + 2\rho_2 b\int_0^1 \psi_t^2\,dx \\ & + \left(\frac{\varepsilon}{b^2} + \varepsilon^2 k^2\right)\int_0^1 (\varphi_x + \psi)^2\,dx + 2\varepsilon b^2\int_0^1 \psi_x^2\,dx \\ & + \left(b + \frac{b^2}{\varepsilon^2} + \frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right)\int_0^1 \psi_x^2\,dx \\ & + \frac{\mu_1^2}{\varepsilon}\int_0^1 \psi_t^2\,dx + \frac{\mu_2^2}{\varepsilon}\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{4.21}$$

Similarly,

$$\frac{d}{dt}\int_0^1 \rho_1 q\varphi_t\varphi_x\,dx \le -k\big[\varphi_x^2(1) + \varphi_x^2(0)\big] + 3k\int_0^1 \varphi_x^2\,dx + k\int_0^1 \psi_x^2\,dx + 2\rho_1\int_0^1 \varphi_t^2\,dx,$$

which, along with (4.20)-(4.21), gives us (4.19). The proof is now complete. □

In order to handle the term $z(x,\rho,t)$, we introduce the functional

$$I_3(t) := \int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,d\rho\,dx. \tag{4.22}$$

Then we can find the following result in [13].

Lemma 4.5 *Let* $(\varphi,\varphi_t,\psi,\psi_t,z)$ *be the solution of problem* (2.5)-(2.6), *then the following estimate holds*:

$$\frac{d}{dt}I_3(t)\le -I_3(t)-\frac{c}{2\tau}\int_0^1 z^2(x,1,t)\,dx+\frac{1}{2\tau}\int_0^1 \psi_t^2\,dx, \tag{4.23}$$

where c is a positive constant.

Now we define the following Lyapunov functional $\mathscr{L}(t)$ by

$$\begin{aligned}\mathscr{L}(t):={}&ME(t)+\frac{1}{8}I_1(t)+NI_2(t)+J(t)+\frac{\varepsilon}{k}\int_0^1 \rho_1 q\varphi_t\varphi_x\,dx\\&+\frac{\rho_2 b}{4\varepsilon}\int_0^1 q\psi_t\psi_x\,dx+I_3(t).\end{aligned} \tag{4.24}$$

Then we may obtain the following lemma.

Lemma 4.6 *Let* $(\varphi,\varphi_t,\psi,\psi_t,z)$ *be the solution of problem* (2.5)-(2.6). *For M large enough, there exist two positives* γ_1 *and* γ_2 *depending on* M, N *and* ε *such that for any* $t\ge 0$,

$$\gamma_1 E(t)\le \mathscr{L}(t)\le \gamma_2 E(t). \tag{4.25}$$

Proof The same argument as in [13], we can deduce

$$\begin{aligned}\bigl|\mathscr{L}(t)-ME(t)\bigr|\le{}&\alpha_1\int_0^1\varphi_t^2\,dx+\alpha_2\int_0^1\psi_t^2\,dx+\alpha_3\int_0^1(\varphi_x+\psi)^2\,dx\\&+\alpha_4\int_0^1\psi_x^2\,dx+\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dt\\&+\int_0^1\hat{f}(\psi)\,dx,\end{aligned} \tag{4.26}$$

where the positive constants α_i $(i=1,2,3,4)$ are determined as in [13].

Performing Young's inequality and using the fact

$$\int_0^1\varphi_x^2\,dx\le 2\int_0^1(\varphi_x+\psi)^2\,dx+2\int_0^1\psi^2\,dx,$$

we easily get

$$\begin{aligned}E(t)\ge{}&\frac{1}{4}\min\{1,\xi\}\biggl(\int_0^1\varphi_t^2\,dx+\int_0^1\psi_t^2\,dx+\int_0^1(\varphi_x+\psi)^2\,dx\\&+\int_0^1\psi_x^2\,dx+\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dt+\int_0^1\hat{f}(\psi)\,dx\biggr),\end{aligned} \tag{4.27}$$

It follows from (4.26)-(4.27) that there exists a positive constant $\tilde{C}$ such that

$$\bigl|\mathscr{L}(t)-ME(t)\bigr|\le \tilde{C}E(t). \tag{4.28}$$

Then choosing M so large that $\gamma_1 := M - \tilde{C} > 0$ and $\gamma_2 = M + \tilde{C} > 0$, we complete the proof. □

Proof of Theorem 2.2 It follows from (3.1), (4.2), (4.8), (4.14), (4.19), (4.23) and (4.24) that

$$\begin{aligned}
\frac{d}{dt}\mathscr{L}(t) \le{}& \left[-MC - \frac{\rho_2}{8} + N\left(\rho_2 + \frac{\rho_1}{4\tilde{\eta}}\right) + \frac{\rho_2 b}{2\varepsilon} + \frac{\mu_1^2}{4\varepsilon^2} + \rho_2 + \frac{\mu_1^2}{k} + \frac{1}{2\tau}\right]\int_0^1 \psi_t^2\,dx \\
&+ \left[-MC + \frac{\mu_2^2}{32\lambda_1\varepsilon} + \frac{N\mu_2}{4\eta\lambda_1} + \frac{\mu_2^2}{k} + \frac{\mu_2^2}{4\varepsilon^2} - \frac{c}{2\tau}\right]\int_0^1 z^2(x,1,t)\,dx \\
&+ \left[-\frac{\rho_1}{8} + N\tilde{\eta}\frac{\rho_1}{\lambda_1} + \frac{2\rho_1\varepsilon}{k}\right]\int_0^1 \varphi_t^2\,dx + \left[-\frac{3k}{8} + \frac{k^2\varepsilon}{4}\right]\int_0^1 (\varphi_x + \psi)^2\,dx \\
&+ \left[\frac{b + C_1}{8} + \varepsilon - N(b - \mu_2\eta) + \frac{b^2}{2\varepsilon} + \frac{b^2}{4\varepsilon^3} + \frac{b^2}{2} + \frac{2\varepsilon}{b^2\lambda_1} + \frac{b^2}{\varepsilon\lambda_1}\right]\int_0^1 \psi_x^2\,dx \\
&+ (-N-1)\int_0^1 \hat{f}(\psi)\,dx.
\end{aligned} \tag{4.29}$$

First we choose η so small that

$$\eta \le \frac{b}{2C\mu_2},$$

and then we choose ε small enough so that

$$\varepsilon \le \frac{5k}{128 + 4k^2}.$$

Then we take N so large that

$$\frac{Nb}{4} \ge \frac{b + C_1}{8} + \varepsilon + \frac{b^2}{2\varepsilon} + \frac{b^2}{4\varepsilon^3} + \frac{b^2}{2} + \frac{2\varepsilon}{b^2\lambda_1} + \frac{b^2}{\varepsilon\lambda_1}.$$

After that, we select $\tilde{\eta}$ small enough so that

$$\tilde{\eta} \le \frac{1}{32N}.$$

Then we take M so large that there exists a positive constant δ such that

$$\begin{aligned}
\frac{d}{dt}\mathscr{L}(t) \le{}& -\delta\int_0^1 \left(\varphi_t^2 + \psi_t^2 + (\varphi_x + \psi)^2 + \psi_x^2 + z^2(x,1,t)\right)dx \\
&- \delta\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx - \delta\int_0^1 \hat{f}(\psi)\,dx.
\end{aligned} \tag{4.30}$$

Noting that (2.13), we know that there exists a positive constant β such that

$$\frac{d}{dt}\mathscr{L}(t) \le -\beta E(t), \tag{4.31}$$

which, together with (4.25), yields

$$\frac{d}{dt}\mathscr{L}(t) \le -\frac{\beta}{\gamma_2}\mathscr{L}(t).$$

Then we can get

$$\mathscr{L}(t) \le \mathscr{L}(0)e^{-\frac{\beta}{\gamma_2}t}. \tag{4.32}$$

Using again (4.25), we find that

$$E(t) \le \frac{\gamma_2}{\gamma_1}E(0)e^{-\frac{\beta}{\gamma_2}t},$$

which gives us that the exponential stability holds for any $U_0 \in D(\mathcal{A})$. Noting that $D(\mathcal{A})$ is dense in $\mathscr{H}$, we can extend the energy inequalities to phase space $\mathscr{H}$. Thus we complete the proof of Theorem 2.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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Acknowledgements
This work was supported by the Fundamental Research Funds for the Central Universities with contract No. JBK150128.

Received: 18 June 2015 Accepted: 29 October 2015 Published online: 10 November 2015

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