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New existence theory of positive solutions to fourth order p-Laplacian elasticity problems

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Abstract

Using a new technique for dealing with the bending term of the fourth order p-Laplacian elasticity problems, several new and more general results are obtained for the existence of at least single, twin or triple positive solutions. It is interesting to point out that estimates on the norms of these solutions will also be provided.

Keywords: fourth order p-Laplacian elasticity problems; transformation technique; positive solutions; estimate on the norm

1 Introduction

It is well known that many nonlinear problems can be reduced to the search for solutions of various differential equations. Fixed point methods are often used for proving the existence of such solutions. An overview of such results can be found in Guo *et al.* [1], in Guo and Lakshmikantham [2], and in Demling [3].

At the same time, we notice that some new techniques via appropriate transformation are proved to be very effective in studying the solvability of differential equations. Such techniques have attracted the attention of Zhang *et al.* [4] and Wong [5], *etc.* In [4], Zhang *et al.* considered the existence of positive solutions of the following problems:

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \ldots, x^{(n-2)}(t)) = \theta, & t \in J, t \neq t_k, k = 1, 2, \ldots, m, \\ \Delta x^{(n-1)}|_{t=t_k} = -I_k(x^{(n-2)}(t_k)), & k = 1, 2, \ldots, m, \\ x^{(i)}(0) = \theta, & i = 0, 1, \ldots, n-3, \\ x^{(n-2)}(0) = x^{(n-2)}(1) = \int_0^1 g(t) x^{(n-2)}(t) \, dt, \end{cases} \quad (1.1)$$

where $J = [0, 1]$.

First, by means of the linear transformation

$$x^{(n-2)}(t) = y(t),$$

the authors converted (1.1) into

$$\begin{cases} x^{(n-2)}(t) = y(t), & t \in J, \\ x^{(i)}(0) = \theta, & i = 1, 2, \ldots, n-3, \end{cases}$$

and

$$\begin{cases} -y''(t) = f(t, x(t), x'(t), \ldots, x^{(n-2)}(t)), & t \in J, t \neq t_k, \\ \Delta y'|_{t=t_k} = -I_k(y(t_k)), & k = 1, 2, \ldots, m, \\ y(0) = y(1) = \int_0^1 g(t)y(t)\, dt. \end{cases} \tag{1.2}$$

Then it follows from Lemma 2.3 in [4] that they converted the results obtained for problem (1.2) to the counterpart for problem (1.2).

In [5], Wong transformed the following problems:

$$\begin{cases} (-1)^m y^{(2m+1)}(t) = F(t, y(t), y'(t)), & 0 < t < 1, \\ y(0) = 0, \quad y^{(2k-1)}(0) = y^{(2k-1)}(1) = 0, & 0 \le k \le m \end{cases} \tag{1.3}$$

into

$$\begin{cases} y'(t) = x(t), & t \in J, \\ y(0) = 0, \end{cases}$$

and

$$\begin{cases} (-1)^m x^{(2m)}(t) = F(t, y(t), y'(t)), & 0 < t < 1, \\ x^{(2k-2)}(0) = x^{(2k-2)}(1) = 0, & 1 \le k \le m \end{cases} \tag{1.4}$$

by using

$$y'(t) = x(t).$$

So the existence of a solution of the complementary Lidstone boundary value problem (1.3) follows from the existence of a solution of the Lidstone boundary value problem (1.4). From the proof of the main results of [5], we notice that the above paper requires F to satisfy some assumptions of monotonicity which are essential for the technique used.

Being directly inspired by [4, 5], in the present paper, by using a transformation technique and a fixed point method, we shall investigate the existence and multiplicity of positive solutions for the fourth order p-Laplacian elasticity problems

$$\begin{cases} (\phi_p(y''(t)))'' = F(t, y(t), y''(t)), & 0 < t < 1, \\ ay(0) - by'(0) = \int_0^1 g(s)y(s)\, ds, \\ ay(1) + by'(1) = \int_0^1 g(s)y(s)\, ds, \\ \phi_p(y''(0)) = \phi_p(y''(1)) = \int_0^1 h(s)\phi_p(y''(s))\, ds, \end{cases} \tag{1.5}$$

where $a, b > 0$, $\phi_p(t) = |t|^{p-2}t$, $p > 1$, $\phi_q = \phi_p^{-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, $F : [0, 1] \times R \times R \to R$ is continuous.

Problem (1.5) occurs in beam theory; see [6, 7], for example for a beam with small deformation; a beam of a material which satisfies a nonlinear power-like stress and strain law; a beam with two-sided links which satisfies a nonlinear power-like elasticity law. If $p = 2$, $g = 0$ and $h = 0$, in mechanics, problem (1.5) is called the cantilever beam equation [8, 9]. The equation describes the deflection of an elastic beam fixed at the right and freed at the left.

Most research papers on fourth order elasticity problems consider nonlinear terms that as regards F in (1.5) involve y only, and derivative-dependent nonlinearities are seldom tackled; see [10–18] to name a few. Besides achieving new results, the presence of the derivative y'' in the nonlinear term F requires a special technique to tackle the problem.

The main features of this paper are as follows. First, compared with [6–18], besides achieving new results, estimates on the norms of these solutions will also be provided. Second, we transform problem (1.5) into a differential systems without bending term, *i.e.,* the technique to deal with bending term is completely different from that of [7, 19–21]. Finally, it is pointed out that we do not need any monotone assumption on F, which is weaker than the corresponding assumptions on F in [5].

The rest of the paper is organized as follows: In Section 2, we first present some definitions and lemmas which are needed throughout this paper and then state several fixed point results: Krasnosel'skii's fixed point theorem in a cone, a new fixed point theorem due to Avery and Henderson and Leggett-Williams' fixed point theorem. In Section 3, we use Krasnosel'skii's fixed point theorem to obtain the existence of at least one or two positive solutions of problem (1.5). Section 4 will further discuss the existence of twin positive solutions of problem (1.5). Two new results will be presented by new fixed point theorem due to Avery and Henderson. Section 5 is due to develop existence criteria for (at least) three positive solutions of problem (1.5). In particular, our results in these sections are new when $p = 2$ (the semilinear case). Finally, in Section 6, an example is included to verify the theoretical results. To the best of the authors' knowledge, it is the first paper where the transformation technique and fixed point method are applied to the fourth order p-Laplacian elasticity problems.

2 Preliminaries

To establish the existence of positive solutions of problem (1.5), let us list the following assumptions, which will hold throughout this paper:

(H$_1$) $F \in C([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty))$;
(H$_2$) $g, h \in L^1[0,1]$ are nonnegative, $\mu \in [0,a)$, $\nu \in [0,1)$, where

$$\mu = \int_0^1 g(s)\,ds, \qquad \nu = \int_0^1 h(s)\,ds.$$

We shall reduce elasticity problem (1.5) to a system without the bending term. To this goal, first, by means of the transformation

$$\phi_p\big(y''(t)\big) = -x(t).$$

It follows from (1.5) that Lemma 2.1 holds.

Lemma 2.1 (See [22]) *Assume that* (H$_2$) *holds. Then for any $x \in C[0,1]$, the boundary value problem*

$$\begin{cases} -y''(t) = \phi_q(x(t)), & 0 < t < 1, \\ ay(0) - by'(0) = \int_0^1 g(s)y(s)\,ds, \\ ay(1) + by'(1) = \int_0^1 g(s)y(s)\,ds, \end{cases} \qquad (2.1)$$

has a unique solution y given by

$$y(t) = \int_0^1 H_1(t,s)\phi_q\big(x(s)\big)\,ds, \tag{2.2}$$

where

$$H_1(t,s) = G_1(t,s) + \frac{1}{a-\mu}\int_0^1 G_1(s,v)g(v)\,dv, \tag{2.3}$$

$$G_1(t,s) = \frac{1}{d}\begin{cases}(b+as)(b+a(1-t)), & 0 \le s \le t \le 1,\\ (b+at)(b+a(1-s)), & 0 \le t \le s \le 1,\end{cases} \tag{2.4}$$

$$d = a(a+2b).$$

Taking into account (2.1) *and* (2.2), *problem* (1.5) *reduces to the following problem*:

$$\begin{cases} x''(t) = -F(t, \int_0^1 H_1(t,s)\phi_q(x(s))\,ds, -\phi_q(x(t))), & 0 < t < 1,\\ x(0) = x(1) = \int_0^1 h(s)x(s)\,ds.\end{cases} \tag{2.5}$$

Lemma 2.2 (See [22]) *Assume that* (H$_1$)-(H$_2$) *hold. Then the boundary value problem* (2.5) *has a unique solution x given by*

$$x(t) = \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right)ds, \tag{2.6}$$

where

$$H_2(t,s) = G_2(t,s) + \frac{1}{1-v}\int_0^1 G_2(s,v)h(v)\,dv, \tag{2.7}$$

$$G_2(t,s) = \begin{cases}s(1-t), & 0 \le s \le t \le 1,\\ t(1-s), & 0 \le t \le s \le 1.\end{cases} \tag{2.8}$$

If problem (2.5) *has a solution* x^*, *then by* (2.2), *problem* (1.5) *has a solution given by*

$$y^*(t) = \int_0^1 H_1(t,s)\phi_q\big(x^*(s)\big)\,ds.$$

So the existence of a solution of problem (1.5) *follows from the existence of a solution of problem* (2.5).

Lemma 2.3 (See [22]) *Let* (H$_2$) *hold. Then we have the following results*:

$$G_1(t,s) > 0, \qquad H_1(t,s) > 0, \quad \forall t,s \in [0,1], \tag{2.9}$$

$$\frac{b^2}{d} \le G_1(t,s) \le G_1(s,s) \le \frac{(a+b)^2}{d}, \quad \forall t,s \in [0,1], \tag{2.10}$$

$$\rho_1 \le H_1(t,s) \le H_1(s,s) \le \rho_2, \quad \forall t,s \in [0,1], \tag{2.11}$$

where

$$\rho_1 = \frac{b^2 \gamma_1}{a + 2b}, \qquad \rho_2 = \frac{\gamma_1(a + b)^2}{a + 2b}, \qquad \gamma_1 = \frac{1}{a - \mu}.$$

Lemma 2.4 (See [22]) *Let* (H_2) *hold. Then we obtain the following results*:

$$G_2(t,s) \geq 0, \qquad H_2(t,s) \geq 0, \quad \forall t,s \in [0,1], \tag{2.12}$$

$$G_2(t,s) > 0, \qquad H_2(t,s) > 0, \quad \forall t,s \in (0,1), \tag{2.13}$$

$$G_2(t,t)G_2(s,s) \leq G_2(t,s) \leq G_2(s,s) \leq \frac{1}{4}, \quad \forall t,s \in [0,1], \tag{2.14}$$

$$\rho G_2(s,s) \leq H_2(t,s) \leq \gamma_2 G_2(s,s) \leq \frac{1}{4}\gamma_2, \quad \forall t,s \in [0,1], \tag{2.15}$$

where

$$\rho = \frac{\int_0^1 G_2(\tau,\tau)h(\tau)\,d\tau}{1 - \nu}, \qquad \gamma_2 = \frac{1}{1 - \nu}.$$

It is clear from (2.2) that $\|y^*\| \leq \rho_2 \phi_q(\|x^*\|)$, moreover, if x^* is positive, so is y^*.

Let $J = [0,1]$, $E = C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{t\in J} |x(t)|$.

Define an operator $T: E \rightarrow E$ as follows:

$$(Tx)(t) = \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q(x(\tau))\,d\tau, -\phi_q(x(s))\right)ds. \tag{2.16}$$

Let K be a cone in E which is defined as

$$K = \{x \in E : x(t) \geq 0, \text{and } x(t) \geq \delta\|x\| \text{ for } t \in J\}, \tag{2.17}$$

where

$$\delta = \frac{\rho}{\gamma_2}.$$

Lemma 2.5 *Let* (H_1)-(H_2) *hold. Then we have* $T(K) \subset K$, *and* $T: K \rightarrow K$ *is completely continuous*.

Proof In view of condition (2.15), we see that

$$\|Tx\| = \max_{t\in J} \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q(x(\tau))\,d\tau, -\phi_q(x(s))\right)ds$$

$$\leq \int_0^1 \gamma_2 G_2(s,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q(x(\tau))\,d\tau, -\phi_q(x(s))\right)ds.$$

Moreover, it follows from (2.15) that

$$(Tx)(t) = \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right)ds$$

$$\geq \int_0^1 \rho G_2(s,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right)ds$$

$$\geq \delta\|Tx\|.$$

This proves that $T(K) \subset K$.

Next by standard methods and the Ascoli-Arzela theorem one can prove that $T: K \to K$ is completely continuous. So this is omitted. □

In the rest of this section, we state several fixed point theorems which we needed later.

Lemma 2.6 (See [1–3]) *Let P be a cone of real Banach space E, and Ω_1 and Ω_2 be two bounded open sets in E such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let operator $A: P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the two conditions*

   (i) $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$,
   (ii) $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$

is satisfied. Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Remark 2.1 To make clear what $\bar{\Omega}_2, \partial\Omega_2, \partial\Omega_1$, and $\Omega_2 \setminus \bar{\Omega}_1$ mean, we give typical examples of Ω_1 and Ω_2, *e.g.*,

$$\Omega_1 = \big\{x \in C[a,b] : \|x\|_\infty < r\big\}, \qquad \Omega_2 = \big\{x \in C[a,b] : \|x\|_\infty < R\big\}$$

with $0 < r < R$, where $\|x\|_\infty = \sup_{t \in J} |x(t)|$.

Given a nonnegative continuous functional ψ on a cone P of a real Banach space, we define, for each $r > 0$, the set

$$P(\psi, r) = \big\{u \in P : \psi(u) < r\big\}.$$

Lemma 2.7 (See [23]) *Let P be a cone in a real Banach space. Let α and γ be increasing nonnegative continuous functional on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq H\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exists a completely continuous operator $A: \overline{P(\gamma, c)} \to P$ and $0 < r_1 < r_2 < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \quad \text{for } 0 \leq \lambda \leq 1 \text{ and } x \in \partial P(\theta, r_2),$$

and

   (i) $\gamma(Ax) > c$ *for all $x \in \partial P(\gamma, c)$;*
   (ii) $\theta(Ax) < r_2$ *for all $x \in \partial P(\theta, r_2)$;*
   (iii) $P(\alpha, r_1) \neq \emptyset$ *and $\alpha(Ax) > r_1$ for $x \in \partial P(\alpha, r_1)$.*

Then *A has at least two positive solutions x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ satisfying*

$$r_1 < \alpha(x_1), \quad with \ \theta(x_1) < r_2, \quad and \quad r_2 < \theta(x_2) \quad with \ \gamma(x_2) < c.$$

Let $0 < r_1 < r_2$ be given and let β be a nonnegative continuous concave functional on the cone K. Define the convex sets K_{r_1}, $K(\beta, r_1, r_2)$ by

$$K_{r_1} = \left\{ x \in K : \|x\| < r_1 \right\},$$

$$K(\beta, r_1, r_2) = \left\{ x \in K : r_1 \leq \beta(x), \|x\| \leq r_2 \right\}.$$

Finally we state the Leggett-Williams fixed point theorem [24].

Lemma 2.8 (See [24]) *Let K be a cone in a real Banach space E, $A : \bar{K}_c \to \bar{K}_c$ be completely continuous and β be a nonnegative continuous concave functional on K with $\beta(x) \leq \|x\|$ for all $x \in K_{r_1}$. Suppose there exist $0 < d < r_1 < r_2 \leq c$ such that*
  (i) *$\{ x \in K(\beta, r_1, r_2) : \beta(x) > r_1 \} \neq \emptyset$ and $\beta(Ax) > r_1$ for $x \in K(\beta, r_1, r_2)$;*
  (ii) *$\|Ax\| < d$ for $\|x\| \leq d$;*
  (iii) *$\beta(Ax) > r_1$ for $x \in K(\beta, r_1, c)$ with $\|Ax\| > r_2$.*
Then A has at least three positive solutions x_1, x_2, x_3 satisfying

$$\|x_1\| < d, \qquad r_1 < \beta(x_2), \qquad \|x_3\| > d \quad and \quad \beta(x_3) < r_1.$$

3 Single or twin solutions of problem (1.5)

In this section, we apply Lemma 2.6 to establish the existence of positive solutions for problem (1.5). We begin by introducing the following conditions on $F(t, u, v)$:

(H_3) There exist two positive constants r, R with $(\rho_2 + 1)\phi_q(r) < (\rho_1 + 1)\phi_q(\delta R)$ such that

$$F(t, u, v) \leq \frac{1}{\eta} r, \quad \forall t \in J, |u| + |v| \leq (\rho_2 + 1)\phi_q(r), \tag{3.1}$$

$$F(t, u, v) \geq \frac{1}{\rho \beta \delta} R, \quad \forall t \in J, |u| + |v| \geq (\rho_1 + 1)\phi_q(\delta R), \tag{3.2}$$

where

$$\eta = \frac{1}{4}\gamma_2, \qquad \sigma = \rho_1 \delta^{q-1}, \qquad \beta = \int_0^1 s(1-s) \, ds = \frac{1}{6}.$$

Theorem 3.1 *Assume that (H_1)-(H_3) hold. Then we have the following conclusions:*
  (i) *Problem (2.5) has (at least) one positive solution $x \in K$ such that*

$$\delta r \leq x(t) \leq \frac{1}{\delta} R, \quad t \in J. \tag{3.3}$$

  (ii) *Problem (1.5) has (at least) one positive solution y such that*

$$\begin{cases} y(t) = \int_0^1 H_1(t, s)\phi_q(x(s)) \, ds, \quad t \in J; \\ \|y\| \leq \rho_2 \phi_q(\|x\|); \\ y(t) \geq \sigma \phi_q(\|x\|), \quad t \in J. \end{cases} \tag{3.4}$$

We further have

$$\sigma \phi_q(r) \le y(t) \le \rho_2 \phi_q\left(\frac{1}{\delta}R\right), \quad t \in J. \tag{3.5}$$

Proof Let T be the cone preserving, completely continuous operator that was defined by (2.16).

Let $x \in K$ with $\|x\| = r$. Then $0 \le \phi_q(x(t)) \le \phi_q(r)$, $t \in J$, and

$$0 \le \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau \le \rho_2\phi_q(r).$$

And hence for $x \in K$ with $\|x\| = r$, we have

$$\left|\int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau\right| + \big|-\phi_q\big(x(t)\big)\big| \le (\rho_2 + 1)\phi_q(r).$$

Then it follows from (3.1) that

$$\|Tx\| = \max_{t \in J} \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right)ds$$

$$\le \eta\frac{1}{\eta}r = r. \tag{3.6}$$

Now if we let $\Omega_1 = \{x \in K : \|x\| < r\}$, then (3.6) shows that

$$\|Tx\| \le \|x\|, \quad x \in \partial\Omega_1. \tag{3.7}$$

Further, let

$$R_1 = \frac{1}{\delta}R, \tag{3.8}$$

and

$$\Omega_2 = \big\{x \in K : \|x\| < R_1\big\}.$$

Then $x \in K$ and $\|x\| = R_1$ implies

$$x(t) \ge \delta x(s), \quad t,s \in J,$$

that is,

$$x(t) \ge \delta R_1 = R, \quad t \in J.$$

Hence, $\phi_q(x(t)) \ge \phi_q(R)$ for all $t \in J$, and

$$\int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau \ge \rho_1 \int_0^1 \phi_q\big(x(\tau)\big)\,d\tau \ge \rho_1\phi_q\big(\delta\|x\|\big).$$

So

$$\left| \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau \right| + \left| -\phi_q\big(x(t)\big) \right| \geq (\rho_1 + 1)\phi_q(\delta R).$$

Using condition (3.2), it follows from $x \in K$ and $\|x\| = R_1$ that

$$\|Tx\| = \max_{t\in J} \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right)ds$$

$$\geq \rho\beta \frac{1}{\rho\beta\delta}R = \frac{R}{\delta} = R_1,$$

that is, $x \in \partial\Omega_2$ implies

$$\|Tx\| \geq \|x\|. \tag{3.9}$$

It follows from Lemma 2.6 that problem (2.5) has (at least) one positive solution $x \in \bar{\Omega}_2 \setminus \Omega_1$ satisfying (3.3).

It is observed from (2.2) that problem (1.5) has (at least) one positive solution y such that

$$y = \int_0^1 H_1(t,s)\phi_q\big(x(s)\big)\,ds, \quad t \in J \text{ and } \|y\| \leq \rho_2\phi_q\big(\|x\|\big).$$

Moreover, since $x \in K$, we get for $t \in J$,

$$y(t) = \int_0^1 H_1(t,s)\phi_q\big(x(s)\big)\,ds$$

$$\geq \rho_1 \int_0^1 \phi_q\big(x(s)\big)\,ds$$

$$\geq \rho_1\phi_q\big(\delta\|x\|\big)$$

$$= \sigma\phi_q\big(\|x\|\big).$$

Then we get (3.4).

Further, it follows from (3.3) and (3.4) that (3.5) holds. □

In Theorem 3.2 we assume the following condition on $F(t,u,v)$:

(H$_4$) There exist two positive constants r, R with $(\rho_2 + 1)\phi_q(r) < R$ such that:

$$F(t,u,v) \geq \frac{1}{\rho_1\phi_p(\rho_1+1)\beta\delta}\phi_p\big(|u|+|v|\big), \quad \forall t \in J, |u|+|v| \leq (\rho_2+1)\phi_q(r), \tag{3.10}$$

$$F(t,u,v) \leq \frac{1}{2\eta\phi_p(\rho_2+1)}\phi_p\big(|u|+|v|\big), \quad \forall t \in J, |u|+|v| \geq R, \tag{3.11}$$

and we write

$$M = \max_{\forall t\in J, |u|+|v|\leq R} F(t,u,v). \tag{3.12}$$

Theorem 3.2 *Assume that* (H_1)-(H_2) *and* (H_4) *hold. Then we have the following conclusions*:

  (i) *Problem (2.5) has (at least) one positive solution* $x \in K$ *such that*

$$\delta r \le x(t) \le \max\left\{2\eta M, \phi_p\left(\frac{R}{\phi_q(\delta)(\rho_1 + 1)}\right)\right\}, \quad t \in J. \tag{3.13}$$

  (ii) *Problem (1.5) has (at least) one positive solution* y *such that (3.4) holds. We further have*

$$\sigma\delta\phi_q(r) \le y(t) \le \max\left\{\rho_2\phi_q(2\eta M), \frac{\rho_2 R}{\phi_q(\delta)(\rho_1 + 1)}\right\}, \quad t \in J. \tag{3.14}$$

Proof Let $x \in K$ with $\|x\| = r$. Then $0 \le \phi_q(x(t)) \le \phi_q(r)$, $t \in J$, and

$$0 \le \int_0^1 H_1(s, \tau)\phi_q\big(x(\tau)\big)\, d\tau \le \rho_2\phi_q(r).$$

And hence for $x \in K$ with $\|x\| = r$, we have

$$0 \le \left|\int_0^1 H_1(s, \tau)\phi_q\big(x(\tau)\big)\, d\tau\right| + \left|-\phi_q\big(x(t)\big)\right| \le (\rho_2 + 1)\phi_q(\delta r),$$

and it follows from condition (3.10) that

$$\|Tx\| = \max_{t \in J} \int_0^1 H_2(t, s)F\left(s, \int_0^1 H_1(s, \tau)\phi_q\big(x(\tau)\big)\, d\tau, -\phi_q\big(x(s)\big)\right) ds$$

$$\ge \rho_1\beta \frac{1}{\rho_1\beta\delta\phi_p(\rho_1 + 1)}\phi_p\big[(\rho_1 + 1)\phi_q(\delta r)\big] = r, \tag{3.15}$$

that is, $x \in \partial\Omega_1$ implies that

$$\|Tx\| \ge \|x\|. \tag{3.16}$$

From (3.11) and (3.12), we have

$$F(t, u, v) \le M + \frac{1}{2\eta\phi_p(\rho_2 + 1)}\phi_p\big(|u| + |v|\big), \quad (t, u, v) \in J \times [0, \infty) \times (-\infty, 0]. \tag{3.17}$$

Further, let

$$R_2 > \max\left\{2\eta M, \phi_p\left(\frac{R}{\phi_q(\delta)(\rho_1 + 1)}\right)\right\} \tag{3.18}$$

and

$$\Omega_3 = \big\{x \in K : \|x\| < R_2\big\}.$$

Noticing that for $x \in \partial\Omega_3$ we have

$$R < (\rho_1 + 1)\phi_q(\delta R_2) \le \left|\int_0^1 H_1(s, \tau)\phi_q\big(x(\tau)\big)\, d\tau\right| + \left|-\phi_q\big(x(t)\big)\right|.$$

Thus, for $x \in \partial\Omega_3$, it follows from (3.17) that

$$
\begin{aligned}
\|Tx\| &= \max_{t \in J} \int_0^1 H_2(t,s) F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right) ds \\
&\leq \eta \int_0^1 \left(M + \frac{1}{2\eta\phi_p(\rho_2 + 1)}\phi_p\big((\rho_2 + 1)\phi_q(R_2)\big)\right) ds \\
&= \eta M + \frac{1}{2}R_2 \\
&< R_2,
\end{aligned}
$$

that is, $x \in \partial\Omega_3$ implies

$$
\|Tx\| < \|x\|. \tag{3.19}
$$

It now follows from Lemma 2.6 that problem (2.5) has (at least) one positive solution $x \in \bar{\Omega}_3 \setminus \Omega_1$ satisfying (3.13).

It follows from (2.2) that problem (1.5) has (at least) one positive solution y. Similar to the proof of (3.5), one can show that y satisfies (3.14). □

Theorem 3.3 *Assume that* (H_1)-(H_2), *(3.2) of* (H_3) *and (3.10) of* (H_4) *hold. In addition, let* f *satisfies the following condition*:

(H_5) *Let* l, ζ_i $(i = 1, 2)$ *and* L *satisfy*

$$
0 < l < \delta\zeta_1 < \zeta_1 < \delta\zeta_2 < \zeta_2 < L < \infty.
$$

If

$$
\max_{\forall t \in J, \phi_q(\delta\zeta_i) \leq |u| + |v| \leq (\rho_2 + 1)\phi_q(\zeta_i)} F(t, u, v) < \frac{1}{\eta}\zeta_i, \quad i = 1, 2,
$$

then we have the following conclusions:

(i) *Problem (2.5) has (at least) two positive solutions* $x_1, x_2 \in K$ *such that*

$$
\delta l \leq x_1(t) \leq \zeta_1 < \delta\zeta_2 \leq x_2(t) \leq L, \quad t \in J. \tag{3.20}
$$

(ii) *Problem (1.5) has (at least) two positive solutions* y_1, y_2 *such that, for* $i = 1, 2$,

$$
\begin{cases}
y_i(t) = \int_0^1 H_1(t,s)\phi_q(x_i(s))\,ds, \quad t \in J; \\
\|y_i\| \leq \rho_2\phi_q(\|x_i\|); \\
y_i(t) \geq \sigma\phi_q(\|x_i\|), \quad t \in J.
\end{cases} \tag{3.21}
$$

We further have

$$
\begin{cases}
y_1(t) > \sigma\phi_q(l), \quad t \in J; \\
\|y_2\| \leq \rho_2\phi_q(L).
\end{cases} \tag{3.22}
$$

Proof If (3.10) of (H$_4$) holds, similar to the proof of (3.16), we can prove that

$$\|Tx\| \geq \|x\|, \quad x \in K, \|x\| = l. \tag{3.23}$$

If (3.2) of (H$_3$) holds, similar to the proof of (3.9), we have

$$\|Tx\| \geq \|x\|, \quad x \in K, \|x\| = L. \tag{3.24}$$

Finally, we show that

$$\|Tx\| < \|x\|, \quad x \in K, \|x\| = \zeta_i, i = 1, 2. \tag{3.25}$$

In fact, for $x \in K$ with $\|x\| = \zeta_i$ ($i = 1, 2$) then we have

$$x(t) \leq \|x\| = \zeta_i, \quad i = 1, 2,$$

and

$$\rho_1 \phi_q(\delta \zeta_i) \leq \int_0^1 H_1(s, \tau) \phi_q\big(x(\tau)\big) \, d\tau \leq \rho_2 \phi_q(\zeta_i), \quad i = 1, 2.$$

Therefore,

$$\phi_q(\delta \zeta_i) < (\rho_1 + 1)\phi_q(\delta \zeta_i) \leq \left| \int_0^1 H_1(s, \tau) \phi_q\big(x(\tau)\big) \, d\tau \right| + \left| -\phi_q\big(x(s)\big) \right| \leq (\rho_2 + 1)\phi_q(\zeta_i),$$

$$i = 1, 2,$$

and hence it follows from (H$_5$) that

$$\|Tx\| = \max_{t \in J} \int_0^1 H_2(t, s) F\left(s, \int_0^1 H_1(s, \tau) \phi_q\big(x(\tau)\big) \, d\tau, -\phi_q\big(x(s)\big) \right) ds$$

$$< \eta \frac{1}{\eta} \zeta_i = \zeta_i, \quad i = 1, 2,$$

which shows that (3.25) holds.

Applying Lemma 2.6 to (3.23), (3.24) and (3.25) shows that problem (2.5) has (at least) two positive solutions x_1, x_2 with $x_1 \in K_{l, \zeta_1} = \{x \in K, l \leq \|x\| \leq \zeta_1\}$, $x_2 \in K_{\zeta_2, L} = \{x \in K, \zeta_2 \leq \|x\| \leq L\}$. Hence, since for $x_i \in K$ we have $x_i(t) \geq \delta \|x_i\|$, $t \in J$, $i = 1, 2$, it follows that (3.20) holds.

Similar to the proof of (3.4) and (3.5), one can show that (3.21) and (3.22) hold. □

In Theorem 3.4 we assume the following condition on $F(t, u, v)$:

(H$_6$) There exist two constants r, R with $0 < r < \delta R$ such that

$$F(t, u, v) \leq \frac{1}{\eta} r, \quad \forall (t, u, v) \in J \times \big[\rho_1 \phi_q(\delta r), \rho_2 \phi_q(r)\big] \times \big[-\phi_q(r), -\phi_q(\delta r)\big];$$

$$F(t, u, v) \geq \frac{1}{\rho \beta} R, \quad \forall (t, u, v) \in J \times \big[\rho_1 \phi_q(\delta R), \rho_2 \phi_q(R)\big] \times \big[-\phi_q(R), -\phi_q(\delta R)\big].$$

Theorem 3.4 *Assume that* (H_1)-(H_2) *and* (H_6) *hold. Then we have the following conclusions*:

(i) *Problem (2.5) has (at least) one positive solution* $x \in K$ *such that*

$$\delta r \leq x(t) \leq R, \quad t \in J. \tag{3.26}$$

(ii) *Problem (1.5) has (at least) one positive solution* y *such that (3.4) holds. We further have*

$$\sigma \phi_q(r) \leq y(t) \leq \rho_2 \phi_q(R), \quad t \in J. \tag{3.27}$$

Proof For $x \in K$ with $\|x\| = r$, we have $x(s) \in [\delta r, r]$, and

$$\int_0^1 H_1(s, \tau) \phi_q\big(x(\tau)\big) \, d\tau \in \big[\rho_1 \phi_q(\delta r), \rho_2 \phi_q(r)\big],$$

$$-\phi_q\big(x(s)\big) \in \big[-\phi_q(r), -\phi_q(\delta r)\big], \quad \forall s \in J.$$

Then, for $t \in J$, we have

$$(Tx)(t) = \int_0^1 H_2(t, s) F\bigg(s, \int_0^1 H_1(s, \tau) \phi_q\big(x(\tau)\big) \, d\tau, -\phi_q\big(x(s)\big) \bigg) \, ds$$

$$\leq \int_0^1 \gamma_2 G_2(s, s) \, ds \cdot \frac{1}{\eta} r$$

$$= r,$$

i.e. $x \in \partial \Omega_1$ implies

$$\|Tx\| \leq \|x\|. \tag{3.28}$$

On the other hand, for $x \in K$ with $\|x\| = R$, we have $x \in [\delta R, R]$, and $\phi_q(x) \in [-\phi_q(R), -\phi_q(\delta R)]$, $\int_0^1 H_1(s, \tau) \phi_q(x(\tau)) \, d\tau \in [\rho_1 \phi_q(\delta R), \rho_2 \phi_q(R)]$, $\forall s \in J$. Then, for $t \in J$, we have

$$(Tx)(t) = \int_0^1 H_2(t, s) F\bigg(s, \int_0^1 H_1(s, \tau) \phi_q\big(x(\tau)\big) \, d\tau, -\phi_q\big(x(s)\big) \bigg) \, ds$$

$$\geq \int_0^1 \rho G_2(s, s) \, ds \cdot \frac{6}{\rho} R$$

$$= R,$$

i.e. $x \in \partial \Omega_2$ implies

$$\|Tx\| \geq \|x\|. \tag{3.29}$$

Applying Lemma 2.6 to (3.28) and (3.29) shows that T has a fixed point $x \in K \cap (\bar{\Omega}_2 \backslash \Omega_1)$ with (3.26). It is observed from (2.2) that problem (1.5) has one solution y such that

$$y(t) = \int_0^1 H_1(t, s) \phi_q\big(x(s)\big) \, ds, \quad \text{and} \quad y(t) \in \big[\sigma \phi_q(r), \rho_2 \phi_q(R)\big], \quad t \in J. \qquad \square$$

We remark that the condition (H$_6$) in Theorem 3.4 can be replaced by the following condition:

(H$_7$) There exist two constants r, R with $0 < r < \delta R$ such that

$$F(t,u,v) \geq \frac{1}{\rho\beta} r, \quad \forall (t,u,v) \in J \times \left[\rho_1\phi_q(\delta r), \rho_2\phi_q(r)\right] \times \left[-\phi_q(r), \phi_q(\delta r)\right];$$

$$F(t,u,v) \leq \frac{1}{\eta} R, \quad \forall (t,u,v) \in J \times \left[\rho_1\phi_q(\delta R), \rho_2\phi_q(R)\right] \times \left[-\phi_q(R), \phi_q(\delta R)\right].$$

Corollary 3.1 *If the condition* (H$_6$) *in Theorem* 3.4 *is replaced by* (H$_7$), *respectively, then the conclusions of Theorem* 3.4 *also hold.*

4 Further results on twin solutions

In the previous section, we have obtained some results on the existence of at least two positive solutions for problem (1.5). In this section, we will further discuss the existence of two positive solutions for problem (1.5) in the more general case.

For $x \in K$, we define the nonnegative increasing continuous functionals θ, θ_1 and θ_2 by

$$\theta(x) = \min_{t \in J} x(t),$$

$$\theta_1(x) = \max_{t \in [0,\frac{1}{2}]} x(t),$$

and

$$\theta_2(x) = \max_{t \in J} x(t).$$

It is easy to see that, for each $x \in K$,

$$\theta(x) \leq \theta_1(x) \leq \theta_2(x). \tag{4.1}$$

In addition, for each $x \in K$, $\theta(x) = x(1) \geq \delta \|x\|$, which implies that

$$\|x\| \leq \frac{1}{\delta}\theta(x) \quad \text{for all } x \in K. \tag{4.2}$$

Finally, we also note that

$$\theta_1(\lambda x) = \lambda\theta_1(x) \quad \text{for } 0 \leq \lambda \leq 1 \text{ and } x \in \partial K(\theta_1, r),$$

where $r > 0$.

For notational convenience, we denote

$$N = \rho\beta, \qquad N_1 = \eta.$$

We now present the results in this section.

Theorem 4.1 *Suppose that there exist constants $0 < a < b < c$ such that*

$$0 < a < \frac{N}{N_1}b < \frac{\delta N}{N_1}c.$$

Assume F satisfies the following conditions:

(C_1) $F(t, u, v) > \frac{c}{N}$ *for* $(t, u, v) \in J \times [\rho_1\phi_q(c), \rho_2\phi_q(\frac{c}{\delta})] \times [-\phi_q(\frac{c}{\delta}), -\phi_q(c)]$;
(C_2) $F(t, u, v) < \frac{b}{N_1}$ *for* $(t, u, v) \in J \times [0, \rho_2\phi_q(\frac{b}{\delta})] \times [-\phi_q(\frac{b}{\delta}), 0]$;
(C_3) $F(t, u, v) > \frac{a}{N}$ *for* $(t, u, v) \in J \times [\rho_1\phi_q(\delta a), \rho_2\phi_q(a)] \times [-\phi_q(a), -\phi_q(\delta a)]$.

Then we have the following conclusions:

(i) *Problem (2.5) has at least two positive solutions x_1 and x_2 such that*

$$a < \theta_2(x_1), \quad \text{with } \theta_1(x_1) < b, \quad \text{and} \quad b < \theta_1(x_2) \quad \text{with } \theta(x_2) < c. \tag{4.3}$$

(ii) *Problem (1.5) has (at least) two positive solutions y_1 and y_2 such that for $i = 1, 2$*

$$\begin{cases} y_i(t) = \int_0^1 H_1(t, s)\phi_q(x_i(s))\, ds, & t \in J; \\ \|y_i\| \le \rho_2\phi_q(\|x_i\|); \\ y_i(t) \ge \sigma\phi_q(\|x_i\|), & t \in J. \end{cases} \tag{4.4}$$

We further have

$$\begin{cases} \|y_1\| > \sigma\phi_q(a); \\ y_2(t) < \rho_2\phi_q(\|x\|_2), & t \in J. \end{cases} \tag{4.5}$$

Proof By the definition of operator T and its properties, it suffices to show that the conditions of Lemma 2.7 hold with respect to T.

We first show that if $x \in \partial K(\theta, c)$, then $\theta(Tx) > c$.

In fact, if $x \in \partial K(\theta, c)$, then $\theta(x) = \min_{t \in J} x(t) = x(1) = c$. Since $x \in K$, one can get $x(t) \ge c$ for $t \in J$. Noticing (4.2), we have

$$c \le x(t) \le \frac{1}{\delta}c, \quad t \in J,$$

which shows that

$$\rho_1\phi_q(c) \le \int_0^1 H_1(t, s)\phi_q\big(x(s)\big)\, ds \le \rho_2\phi_q\left(\frac{1}{\delta}c\right), \qquad -\phi_q\left(\frac{1}{\delta}c\right) \le -\phi_q\big(x(s)\big) \le -\phi_q(c).$$

As a consequence of (C_1)

$$F(t, u, v) > \frac{c}{N} \quad \text{for } t \in J.$$

Also, $Tx \in K$, and hence we get

$$\theta(Tx) = \min_{t \in J}(Tx)(t) = (Tx)(0)$$

$$= \min_{t \in J} \int_0^1 H_2(t, s)F\left(s, \int_0^1 H_1(s, \tau)\phi_q\big(x(\tau)\big), -\phi_q\big(x(s)\big)\right) ds$$

$$> \rho\beta\frac{c}{N}$$

$$= c.$$

Next, we verify that $\theta_1(Tx) < b$ for $x \in \partial K(\theta_1, b)$.

So, let us choose $x \in \partial K(\theta_1, b)$, then

$$\theta_1(x) = \max_{t \in [0,\frac{1}{2}]} x(t) = b.$$

This implies that $0 \leq x(t) \leq b$ for $t \in [0, \frac{1}{2}]$, and, since $x \in K$, we also have

$$x(t) \leq \|x\| \leq \frac{1}{\delta}\theta(x) \leq \frac{1}{\delta}\theta_1(x) = \frac{1}{\delta}b,$$

which implies that

$$0 \leq x(t) \leq \frac{1}{\delta}b, \quad t \in J,$$

and then

$$0 \leq \int_0^1 H_1(t,s)\phi_q(s)\,ds \leq \rho_2\phi_q\left(\frac{1}{\delta}b\right), \qquad -\phi_q\left(\frac{1}{\delta}b\right) \leq -\phi_q(x(s)) \leq 0.$$

It follows from (C$_2$) that

$$F(t,u,v) < \frac{b}{N_1} \quad \text{for } (t,u,v) \in J \times \left[0, \rho_2\phi_q\left(\frac{b}{\delta}\right)\right] \times \left[-\phi_q\left(\frac{b}{\delta}\right), 0\right].$$

Noticing $Tx \in K$, we obtain

$$(Tx)(t) = \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q(x(\tau)), -\phi_q(x(s))\right)ds$$

$$< \eta\frac{b}{N_1}$$

$$= b.$$

Finally, we prove that $K(\theta_2, a) \neq \emptyset$ and $\theta_2(Tx) > a$ for all $x \in \partial K(\theta_2, a)$.

In fact, the constant $\frac{a}{3} \in K(\theta_2, a)$. Moreover, for $x \in \partial K(\theta_2, a)$, we have

$$\theta_2(x) = \max_{t \in J} x(t) = a,$$

which implies that

$$\delta a \leq x(t) \leq a, \quad t \in J,$$

and then

$$\rho_1\phi_q\left(\frac{1}{\delta}a\right) \leq \int_0^1 H_1(t,s)\phi_q(s)\,ds \leq \rho_2\phi_q(a), \qquad -\phi_q(a) \leq -\phi_q(x(s)) \leq -\phi_q\left(\frac{1}{\delta}a\right).$$

Using assumption (C$_3$), we have

$$F(t, u, v) > \frac{a}{N} \quad \text{for } (t, u, v) \in J \times \left[\rho_1 \phi_q(\delta a), \rho_2 \phi_q(a) \right] \times \left[-\phi_q(a), -\phi_q(\delta a) \right].$$

As before $Tx \in K$, and so

$$
\begin{aligned}
\theta_2(Tx) &= \max_{t \in J} (Tx)(t) \\
&= \max_{t \in J} \int_0^1 H_2(t, s) F \left(s, \int_0^1 H_1(s, \tau) \phi_q(x(\tau)), -\phi_q(x(s)) \right) ds \\
&> \rho \eta \frac{a}{N} \\
&= a.
\end{aligned}
$$

Thus, by Lemma 2.7, there exist at least two fixed points of T which are positive solutions x_1 and x_2, belonging to $\overline{K(\theta, c)}$, of problem (2.5) such that

$$a < \theta_2(x_1), \quad \text{with } \theta_1(x_1) < b, \quad \text{and} \quad b < \theta_1(x_2) \quad \text{with } \theta(x_2) < c.$$

Noticing the definition of θ and θ_2, one can show that (4.4) and (4.5) hold. □

5 Triple positive solutions

In this section, we shall study the existence of three positive solutions of problem (1.5). Let σ, N and N_1 be defined in Theorem 3.1 and Theorem 4.1, respectively.

Theorem 5.1 *Assume that* (H$_1$)-(H$_2$) *hold. Suppose that there exist positive constants* ω_1, ω_2, ω_3 *with*

$$0 < \omega_1 < \phi_p \left(\frac{\rho_1}{\rho_2} \right) \delta \omega_2 < \omega_2 < \frac{\omega_2}{\delta} \leq \omega_3$$

such that the following hold:

(A$_1$) $F(t, u, v) \leq \frac{1}{N_1} \omega_3$ *for* $(t, u, v) \in J \times [0, \rho_2 \phi_q(\omega_3)] \times [-\phi_q(\omega_3), 0]$;

(A$_2$) $F(t, u, v) > \frac{1}{N} \omega_2$ *for* $(t, u, v) \in J \times [\rho_1 \phi_q(\omega_2), \rho_2 \phi_q(\frac{1}{\delta} \omega_2)] \times [-\phi_q(\frac{1}{\delta} \omega_2), -\phi_q(\omega_2)]$;

(A$_3$) $F(t, u, v) < \frac{1}{N_1} \omega_1$ *for* $(t, u, v) \in J \times [0, \rho_2 \phi_q(\omega_1)] \times [-\phi_q(\omega_1), 0]$.

Then we have the following conclusions:

(i) *Problem* (2.5) *has* (*at least*) *three positive solutions* $x_1, x_2, x_3 \in K$ *such that*

$$
\begin{cases}
\|x_1\| < \omega_1; \\
x_2(t) > \omega_2, \quad t \in J; \\
\|x_3\| > \omega_1 \text{ and } \min_{t \in J} x_3(t) < \omega_2.
\end{cases}
\tag{5.1}
$$

(ii) *Problem* (1.5) *has* (*at least*) *three positive solutions* y_1, y_2, y_3 *such that, for* $i = 1, 2, 3$,

$$
\begin{cases}
y_i(t) = \int_0^1 H_1(t, s) \phi_q(x_i(s)) \, ds, \quad t \in J; \\
\|y_i\| \leq \rho_2 \phi_q(\|x_i\|); \\
y_i(t) \geq \rho_1 \phi_q(\delta \|x_i\|), \quad t \in J.
\end{cases}
\tag{5.2}
$$

We further have

$$
\begin{cases}
\|y_1\| < \rho_2 \phi_q(\omega_1); \\
y_2(t) > \rho_1 \phi_q(\omega_2), \quad t \in J; \\
\|y_3\| > \sigma \phi_q(\omega_1).
\end{cases}
\tag{5.3}
$$

Proof Let $x \in \bar{K}_{\omega_3}$, so $0 \leq \phi_q(x(s)) \leq \phi_q(\omega_3)$, $s \in J$, and $0 \leq \int_0^1 H_1(s,\tau)\phi_q(x(\tau))\,d\tau \leq \rho_2 \phi_q(\omega_3)$. By (A$_1$), we see that

$$
\begin{aligned}
\|Tx\| &= \max_{t \in J} \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right)ds \\
&\leq \eta \cdot \frac{1}{N_1}\omega_3 = \omega_3.
\end{aligned}
$$

This proves that $T : \bar{K}(\omega_3) \to \bar{K}(\omega_3)$.

For convenience, we denote $\omega_4 = \frac{\omega_2}{\delta}$. Let $\psi : K \to [0,+\infty)$ be defined by

$$
\psi(x) = \min_{t \in J} x(t).
$$

Clearly, ψ is a nonnegative continuous concave functional on K and $\psi(x) \leq \|x\|$ for all $x \in K$.

Now we need to show that condition (i) of Lemma 2.8 is satisfied. It is obvious that

$$
x(t) = \frac{1}{2}(\omega_2 + \omega_4) \in \left\{ x \in K(\psi, \omega_2, \omega_4) : \psi(x) > \omega_2 \right\}.
$$

Then

$$
\left\{ x \in K(\psi, \omega_2, \omega_4) : \psi(x) > \omega_2 \right\} \neq \emptyset.
$$

Next, let $x \in K(\psi, \omega_2, \omega_4)$. Then for $s \in J$, we have $\omega_2 \leq x(s) \leq \omega_4$, and

$$
\int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau \in \left[\rho_1 \phi_q(\omega_2), \rho_2 \phi_q(\omega_4)\right].
$$

Using condition (A$_2$), it follows that

$$
\begin{aligned}
\psi(Tx) &= \min_{t \in J} \int_0^1 H_2(t,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right)ds \\
&\geq \rho \int_0^1 G_2(s,s)F\left(s, \int_0^1 H_1(s,\tau)\phi_q\big(x(\tau)\big)\,d\tau, -\phi_q\big(x(s)\big)\right)ds \\
&> \rho\beta \cdot \frac{1}{N}\omega_2 \\
&= \omega_2.
\end{aligned}
$$

Therefore, we have shown that $\psi(Tx) > \omega_2$ for all $x \in K(\psi, \omega_2, \omega_4)$.

Next, we shall show that condition (ii) of Lemma 2.8 holds. Take $x \in \bar{K}(\omega_1)$ with $\|x\| < \omega_1$. Then for $s \in J$, we have $x(s) \in [0, \omega_1]$ and

$$\int_0^1 H_1(s, \tau) \phi_q(x(\tau)) \, d\tau \in [0, \rho_2 \phi_q(\omega_1)].$$

By condition (A_3), we have

$$\|Tx\| = \max_{t \in J} \int_0^1 H_2(t, s) F\left(s, \int_0^1 H_1(s, \tau) \phi_q(x(\tau)) \, d\tau, -\phi_q(x(s))\right) ds < \eta \cdot \frac{1}{N_1} \omega_1 = \omega_1.$$

Hence, we have proved that $\|Tx\| < \omega_1$ for all $x \in \bar{K}(\omega_1)$.

Finally, we need to prove that condition (iii) of Lemma 2.8 is satisfied. Let $x \in K(\psi, \omega_2, \omega_3)$ with $\|Tx\| > \omega_4$. Then it follows from (2.16) that

$$\psi(Tx) = \min_{t \in J} \int_0^1 H_2(t, s) F\left(s, \int_0^1 H_1(s, \tau) \phi_q(x(\tau)) \, d\tau, -\phi_q(x(s))\right) ds$$

$$\geq \delta \|Tx\|$$

$$> \delta \omega_4 = \omega_2.$$

So, we have proved that $\psi(Tx) > \omega_2$ for all $x \in K(\psi, \omega_2, \omega_3)$ with $\|Tx\| > \omega_4$.

It now follows from Lemma 2.8 that problem (2.5) has (at least) three positive solutions $x_1, x_2, x_3 \in \bar{K}(\omega_3)$ satisfying (5.1).

It is observed from (2.2) that problem (1.5) has (at least) three positive solutions y_1, y_2, y_3 such that, for $i = 1, 2, 3$,

$$y_i(t) = \int_0^1 H_1(t, s) \phi_q(x_i(s)) \, ds, \quad t \in J \text{ and } \|y_i\| \leq \rho_2 \phi_q(\|x_i\|).$$

Moreover, since $x_i \in K$, we get for $t \in J$,

$$y_i(t) = \int_0^1 H_1(t, s) \phi_q(x_i(s)) \, ds$$

$$\geq \rho_1 \int_0^1 \phi_q(x_i(s)) \, ds$$

$$\geq \sigma \phi_q(\|x_i\|).$$

Then we get (5.2).

Further, since $x_2(t) > \omega_2$ for $t \in J$, we have, for $t \in J$,

$$y_2(t) = \int_0^1 H_1(t, s) \phi_q(x_2(s)) \, ds$$

$$\geq \rho_1 \int_0^1 \phi_q(x_2(s)) \, ds$$

$$\geq \rho_1 \phi_q(\omega_2).$$

Hence, we get (5.3). □

Remark 5.1 Leggett-Williams' fixed point theorem is a valid tool to show the existence of multiple positive solutions for various boundary value problems where the nonlinear terms F is of the form $F(t, y)$; see for example [25–27]. If F is of the form $F(t, y, y'')$, then it is very difficult to obtain the existence result of three positive solutions; see [20, 21] to name a few. In [28], using the five functionals fixed point theorem (which is a generalization of the Leggett-Williams' fixed point theorem), Avery *et al.* showed the existence of at least three positive solutions in the case that F is the form of $F(y, -y'')$.

6 Example

To illustrate how our main results can be used in practice, we present an example.

Let $a = 1$, $b = \frac{1}{2}$ and $g(t) = h(t) = \frac{1}{2}$. Consider the following fourth order p-Laplacian elasticity problem:

$$
\begin{cases}
(\phi_p(y''(t)))'' = F(t, y(t), y''(t)), & 0 < t < 1, \\
ay(0) - by'(0) = \int_0^1 g(s)y(s)\,ds, \\
ay(1) + by'(1) = \int_0^1 g(s)y(s)\,ds, \\
\phi_p(y''(0)) = \phi_p(y''(1)) = \int_0^1 h(s)\phi_p(y''(s))\,ds,
\end{cases}
\tag{6.1}
$$

and the nonlinear term F given by

$$
F(t, u, v) =
\begin{cases}
\frac{1}{2}(t^2 + 1)\frac{1}{5N_1}\omega_1 =: a_1(t), & t \in J, (u, v) \in D_1, \\
\frac{1}{5}(\frac{\omega_3}{N_1} + \frac{\omega_2}{N}) =: a_2, & t \in J, (u, v) \in D_2 \cup D_3 \cup D_4, \\
a_1(t) \cdot \frac{\rho_1\phi_q(\omega_2) - u}{\rho_1\phi_q(\omega_2) - \rho_2\phi_q(\omega_1)} + a_2 \cdot \frac{u - \rho_2\phi_q(\omega_1)}{\rho_1\phi_q(\omega_2) - \rho_2\phi_q(\omega_1)} =: a_5(t, u), & t \in J, (u, v) \in D_5, \\
a_1(t) \cdot \frac{\phi_q(\omega_2) + v}{\phi_q(\omega_2) - \phi_q(\omega_1)} + a_2 \cdot \frac{\phi_q(\omega_1) + v}{\phi_q(\omega_1) - \phi_q(\omega_2)} =: a_6(t, v), & t \in J, (u, v) \in D_6, \\
a_7(t, u, v), & t \in J, (u, v) \in D_7,
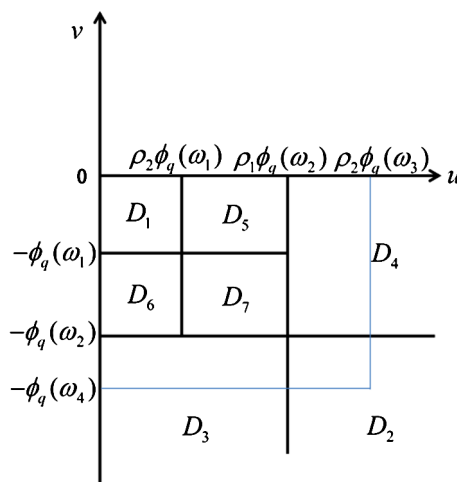\end{cases}
$$

where

$$
a_7(t, u, v) := \frac{a_2}{2} \cdot \left[\frac{\phi_q(\omega_1) + v}{\phi_q(\omega_1) - \phi_q(\omega_2)} + \frac{u - \rho_2\phi_q(\omega_1)}{\rho_1\phi_q(\omega_2) - \rho_2\phi_q(\omega_1)} \right]
$$
$$
+ \frac{a_5(t, u)}{2} \cdot \frac{\phi_q(\omega_2) + v}{\phi_q(\omega_2) - \phi_q(\omega_1)} + \frac{a_6(t, v)}{2} \cdot \frac{\rho_1\phi_q(\omega_2) - u}{\rho_1\phi_q(\omega_2) - \rho_2\phi_q(\omega_1)},
$$

and the sets D_i ($i = 1, 2, \ldots, 7$) are defined by

$$
D_1 = \left\{ (u, v) : (u, v) \in [0, \rho_2\phi_q(\omega_1)] \times [-\phi_q(\omega_1), 0] \right\},
$$
$$
D_2 = \left\{ (u, v) : (u, v) \in [\rho_1\phi_q(\omega_2), +\infty) \times (-\infty, -\phi_q(\omega_2)] \right\},
$$
$$
D_3 = \left\{ (u, v) : (u, v) \in [0, \rho_1\phi_q(\omega_2)] \times (-\infty, -\phi_q(\omega_2)] \right\},
$$
$$
D_4 = \left\{ (u, v) : (u, v) \in [\rho_1\phi_q(\omega_2), +\infty) \times [-\phi_q(\omega_2), 0] \right\},
$$
$$
D_5 = \left\{ (u, v) : (u, v) \in [\rho_2\phi_q(\omega_1), \rho_1\phi_q(\omega_2)] \times [-\phi_q(\omega_1), 0] \right\},
$$
$$
D_6 = \left\{ (u, v) : (u, v) \in [0, \rho_2\phi_q(\omega_1)] \times [-\phi_q(\omega_2), -\phi_q(\omega_1)] \right\},
$$
$$
D_7 = \left\{ (u, v) : (u, v) \in [\rho_2\phi_q(\omega_1), \rho_1\phi_q(\omega_2)] \times [-\phi_q(\omega_2), -\phi_q(\omega_1)] \right\},
$$

and $N = \rho\beta$, $N_1 = \eta$. For details, see Figure 1.

Figure 1 The relationship of D_i, $i = 1, 2, 3, 4, 5, 6, 7$.

By calculation we obtain

$$\mu = \int_0^1 g(s)\,ds = \frac{1}{2}, \qquad v = \int_0^1 h(s)\,ds = \frac{1}{2},$$

$$\gamma_1 = \frac{1}{a - \mu} = 2, \qquad \gamma_2 = \frac{1}{1 - v} = 2,$$

$$\rho = \frac{\int_0^1 G_2(s,s) h(s)\,ds}{1 - v} = \frac{1}{6}, \qquad \eta = \frac{1}{4}\gamma_2 = \frac{1}{2},$$

$$\rho_1 = \frac{b^2 \gamma_1}{a + 2b} = \frac{1}{4}, \qquad \rho_2 = \frac{(a + b)^2 \gamma_1}{a + 2b} = \frac{9}{4}.$$

Therefore, it follows from the definitions F, g and, h that (H$_1$)-(H$_2$) hold.
Choosing $\omega_1 = \phi_p(\frac{\delta \rho_1}{2 \rho_2}) \omega_2$, $\omega_3 = \frac{5 N_1}{N} \omega_2$, then ω_i ($i = 1, 2, 3$) satisfy

$$0 < \omega_1 < \phi_p\left(\frac{\rho_1}{\rho_2}\right) \omega_2 < \omega_2 < \frac{\omega_2}{\delta} \leq \omega_3.$$

Then, for $t \in J$, $(u, v) \in D_1$, we have

$$F(t, u, v) = a_1(t) \leq \frac{\omega_1}{5 N_1} < \frac{\omega_3}{5 N_1} < \frac{\omega_3}{N_1}. \tag{6.2}$$

For $t \in J$, $(u, v) \in D_2 \cup D_3 \cup D_4$, we have

$$F(t, u, v) = a_2 = \frac{1}{5}\left(\frac{\omega_3}{N_1} + \frac{\omega_2}{N}\right) = \frac{1}{5}\left(\frac{5 \omega_2}{N} + \frac{\omega_2}{N}\right) > \frac{\omega_2}{N},$$

$$F(t, u, v) = a_2 = \frac{1}{5}\left(\frac{\omega_3}{N_1} + \frac{\omega_2}{N}\right) = \frac{1}{5}\left(\frac{\omega_3}{N_1} + \frac{\omega_3}{5 N_1}\right) = \frac{6}{25 N_1} \omega_3 < \frac{1}{N_1} \omega_3. \tag{6.3}$$

For $t \in J$, $(u, v) \in D_5$, we have

$$F(t, u, v) = a_5(u) \leq a_1(t) + a_2 < \frac{\omega_3}{5 N_1} + \frac{6}{25} \cdot \frac{\omega_3}{N_1} = \frac{11}{25} \cdot \frac{\omega_3}{N_1} < \frac{\omega_3}{N_1}. \tag{6.4}$$

For $t \in J$, $(u, v) \in D_6$, we have

$$F(t, u, v) = a_6(v) \le a_1(t) + a_2 < \frac{\omega_3}{N_1}. \tag{6.5}$$

For $t \in J$, $(u, v) \in D_7$, we have

$$F(t, u, v) = a_7(u, v) \le a_1(t) + 2a_2 < \frac{\omega_3}{5N_1} + 2 \cdot \frac{6}{25} \frac{\omega_3}{N_1} = \frac{17}{25} \frac{\omega_3}{N_1} < \frac{\omega_3}{N_1}. \tag{6.6}$$

We shall check the conditions of Theorem 5.1. First, condition (A_3) is obviously satisfied by (6.2). Next, from (6.3)-(6.6), it follows that

$$F(t, u, v) < \frac{\omega_3}{N_1}, \quad t \in J, (u, v) \in \left[0, \rho_2 \phi_q(\omega_3)\right] \times \left[-\phi_q(\omega_3), 0\right].$$

Hence, condition (A_1) holds. Finally, by (6.3), we see that

$$F(t, u, v) = a_2 > \frac{\omega_2}{N},$$

$$t \in J, (u, v) \in \left[\rho_1 \phi_q(\delta \omega_2), \rho_2 \phi_q\left(\frac{1}{\delta}\omega_2\right)\right] \times \left[-\phi_q\left(\frac{1}{\delta}\omega_2\right), -\phi_q(\omega_2)\right] \subset D_2.$$

Condition (A_2) is satisfied.

It follows from Theorem 5.1 that problem (6.1) has (at least) three positive solutions y_1, y_2, y_3 such that

$$\begin{cases} \|y_1\| < \rho_2 \phi_q(\omega_1); \\ y_2(t) > \rho_1 \phi_q(\omega_2), \quad t \in J; \\ \|y_3\| > \sigma \phi_q(\omega_1). \end{cases} \qquad \square$$

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