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# On the two-point problem for implicit second-order ordinary differential equations

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**Abstract**

Given a nonempty set $Y \subseteq \mathbf{R}^n$ and a function $f : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \times Y \to \mathbf{R}$, we are interested in the problem of finding $u \in W^{2,p}([a,b], \mathbf{R}^n)$ such that

$$\begin{cases} f(t, u(t), u'(t), u''(t)) = 0 & \text{for a.e. } t \in [a,b], \\ u(a) = u(b) = 0_{\mathbf{R}^n}. \end{cases}$$

We prove an existence result where, for any fixed $(t,y) \in [a,b] \times Y$, the function $f(t,\cdot,\cdot,y)$ can be discontinuous even at all points $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$. The function $f(t,x,z,\cdot)$ is only assumed to be continuous and locally nonconstant.

We also show how the same approach can be applied to the implicit integral equation $f(t, \int_a^b g(t,z)u(z)\,dz, u(t)) = 0$. We prove an existence result (with $f(t,x,y)$ discontinuous in $x$ and continuous and locally nonconstant in $y$) which extends and improves in several directions some recent results in the field.

**Keywords:** implicit differential equations; two-point problem; discontinuity; discontinuous selections; implicit integral equations

## 1 Introduction

Let $[a,b]$ be a compact interval, and let $p \in [1,+\infty]$ and $n \in \mathbf{N}$. As usual, we denote by $W^{2,p}([a,b], \mathbf{R}^n)$ the set of all $u \in C^1([a,b], \mathbf{R}^n)$ such that $u'$ is absolutely continuous in $[a,b]$ and $u'' \in L^p([a,b], \mathbf{R}^n)$. Given a nonempty set $Y \subseteq \mathbf{R}^n$ and a function $f : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \times Y \to \mathbf{R}$, we are interested in the problem of finding $u \in W^{2,p}([a,b], \mathbf{R}^n)$ such that

$$\begin{cases} f(t, u(t), u'(t), u''(t)) = 0 & \text{for a.e. } t \in [a,b], \\ u(a) = u(b) = 0_{\mathbf{R}^n}. \end{cases} \tag{1}$$

To the best of our knowledge, there are not many existence results in the literature as regards problem (1). Some of them [1–3] deal with the special case where $n = 1$ and $f(t,x,z,y) = y - g(t,x,z,y)$, where $g$ satisfies some growth conditions with respect to $(x,z)$, and very strong conditions (such as Lipschitzianity) are imposed on $y$. In general, there is little known about the existence of solutions for boundary value problems associated to the equation

$$f\big(t, u(t), u'(t), \ldots, u^{(k)}(t)\big) = 0 \tag{2}$$

(even when $f$ is continuous), unless it is possible to solve the equation with respect to the highest derivative. In this case, a variety of existence results based on the Leray-Schauder theory is available. As regards boundary value problems, a global continuous solvability in $u^{(k)}$ is needed, which is rarely available (we refer to [4, 5] and to the references therein for more details).

One approach, used in [5, 6], consists to reduce (1) (or, more generally, (2)) to a differential inclusion of the form

$$u^{(k)}(t) \in \Phi\bigl(t, u(t), \dots, u^{(k-1)}(t)\bigr),$$

with a 'well-behaved' multifunction $\Phi$, and then to apply existence results for differential inclusions or selection arguments. Even in this case (see [5, 6]), the continuity of $f$ is assumed.

Our aim in this paper is to prove an existence result for problem (1), in the general vector case, where we assume only that the function $f(t,x,z,y)$, with respect to $y$, is continuous and locally nonconstant. Moreover, our assumptions do not imply any continuity with respect to $(x,z)$. In particular, a function $f$ satisfying our assumptions can be discontinuous at each point $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$, for all fixed $(t,y) \in [a,b] \times Y$.

Our approach is based on a selection theorem (Theorem 2.1 below) for multifunctions defined on subsets of product spaces. Such result, which refines a former version originally proved in [7], ensures the existence of a selection (for a given multifunction), whose set of discontinuities has a peculiar geometric property. That is, it is contained in the union of the set where the given multifunction fails to be lower semicontinuous, together with a finite family of sets, each of them with at least one projection of zero measure.

The selection theorem will be proved in Section 2, together with other preliminary results. In Section 3, our main result will be stated and proved, together with examples of application. Finally, in Section 4, we show how the same approach can be usefully employed to study an implicit integral equation of the form

$$f\left(t, \int_a^b g(t,z)u(z)\,dz, u(t)\right) = 0.$$

We obtain an existence result where, as above, the function $f(t,z,y)$ is only assumed, with respect to $y$, to be continuous and locally nonconstant. As before, our assumptions allow $f$ (with respect to the second variable) to be discontinuous at all points $x \in \mathbf{R}^n$. As showed in Section 4, our result extends and also improves in several directions some recent results.

## 2 Notations and preliminary results

In the following, given $n \in \mathbf{N}$, we denote by $\mathcal{A}_n$ the family of all subsets $U \subseteq \mathbf{R}^n$ such that, for every $i = 1, \dots, n$, the supremum and the infimum of the projection of $\overline{\text{conv}}(U)$ on the $i$th axis are both positive or both negative ('$\overline{\text{conv}}$' standing for 'closed convex hull').

Given any Lebesgue measurable set $V \subseteq \mathbf{R}^n$, we indicate by $\mathcal{L}(V)$ the family of all Lebesgue measurable subsets of $V$, and by $m_n$ the $n$-dimensional Lebesgue measure in $\mathbf{R}^n$. For all $i \in \{1, \dots, n\}$, we shall denote by $P_{n,i} : \mathbf{R}^n \to \mathbf{R}$ the projection over the $i$th axis. We also denote by $\mathcal{F}_n$ the family of all subsets $F \subseteq \mathbf{R}^n$ such that there exist sets $F_1, F_2, \dots, F_n \subseteq \mathbf{R}^n$, with $m_1(P_{n,i}(F_i)) = 0$ for all $i = 1, \dots, n$, such that $F = \bigcup_{i=1}^n F_i$. Of course, any set $F \in \mathcal{F}_n$ belongs to $\mathcal{L}(\mathbf{R}^n)$ and $m_n(F) = 0$.

The space $\mathbf{R}^n$ will be considered with its Euclidean norm $\|\cdot\|_n$. If $p \in [1,+\infty]$, the space $L^p([a,b],\mathbf{R}^n)$ will be considered with the usual norm

$$\|u\|_{L^p([a,b],\mathbf{R}^n)} := \left(\int_a^b \|u(t)\|_n^p \, dt\right)^{\frac{1}{p}} \quad \text{if } p < +\infty,$$

$$\|u\|_{L^\infty([a,b],\mathbf{R}^n)} := \operatorname*{esssup}_{t\in[a,b]} \|u(t)\|_n \quad \text{if } p = +\infty.$$

As usual, we put $L^p([a,b]) := L^p([a,b],\mathbf{R})$.

Let $X$ be a topological space. We shall denote by $\mathcal{B}(X)$ the Borel family of $X$. In the following, if $T$ is a Polish space (that is, a separable complete metric space) endowed with a positive regular Borel measure $\mu$, we shall denote by $\mathcal{T}_\mu$ the completion of the $\sigma$-algebra $\mathcal{B}(T)$ with respect to the measure $\mu$.

For the basic definitions and facts about multifunctions, we refer to [8]. Here, we only recall (see also [9]) that if $(S,\mathcal{D})$ is a measurable space and $X$ is a topological space, then a multifunction $F : S \to 2^X$ is said to be *$\mathcal{D}$-measurable* (resp., *$\mathcal{D}$-weakly measurable*) if for any closed (resp., open) set $A \subseteq X$ the set

$$F^-(A) := \{s \in S : F(s) \cap A \neq \emptyset\}$$

is measurable. Finally, if $x \in \mathbf{R}^n$ and $r > 0$, we put

$$\overline{B}_n(x,r) := \{v \in \mathbf{R}^n : \|v - x\|_n \leq r\}.$$

Our first goal in this section is to prove the following selection result, which is essential for our purposes (for definitions and basic properties as regards Souslin sets and spaces, the reader is referred to [10]).

**Theorem 2.1** *Let $T$ and $X_1, X_2, \ldots, X_k$ be Polish spaces, with $k \in \mathbf{N}$, and let $X := \prod_{j=1}^k X_j$ (endowed with the product topology). Let $\mu$, $\psi_1, \ldots, \psi_k$ be positive regular Borel measures over $T$, $X_1, X_2, \ldots, X_k$, respectively, with $\mu$ finite and $\psi_1, \ldots, \psi_k$ $\sigma$-finite.*

*Let $S$ be a separable metric space, $W \subseteq X$ a Souslin set, and let $F : T \times W \to 2^S$ be a multifunction with nonempty complete values. Let $E \subseteq W$ be a given set. Finally, for all $i \in \{1, \ldots, k\}$, let $P_{*,i} : X \to X_i$ be the projection over $X_i$. Assume that:*

(i) *the multifunction $F$ is $\mathcal{T}_\mu \otimes \mathcal{B}(W)$-weakly measurable;*
(ii) *for a.e. $t \in T$, one has*

$$\{x := (x_1, \ldots, x_k) \in W : F(t,\cdot) \text{ is not lower semicontinuous at } x\} \subseteq E. \tag{3}$$

*Then, there exist sets $Q_1, \ldots, Q_k$, with*

$$Q_i \in \mathcal{B}(X_i) \quad \text{and} \quad \psi_i(Q_i) = 0 \quad \text{for all } i = 1, \ldots, k,$$

*and a function $\phi : T \times W \to S$ such that:*

(a) *$\phi(t,x) \in F(t,x)$ for all $(t,x) \in T \times W$;*
(b) *for all $x := (x_1, x_2, \ldots, x_k) \in W \setminus [(\bigcup_{i=1}^k P_{*,i}^{-1}(Q_i)) \cup E]$, the function $\phi(\cdot,x)$ is $\mathcal{T}_\mu$-measurable over $T$;*

(c) *for a.e.* $t \in T$, *one has*

$$\begin{aligned}&\{x := (x_1, x_2, \dots, x_k) \in W : \phi(t,\cdot) \text{ is discontinuous at } x\} \\ &\quad \subseteq E \cup \left[ W \cap \left( \bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i) \right) \right].\end{aligned}$$

As noted in Section 1, Theorem 2.1 is a refinement of Theorem 2.2 of [7]. Although the proof is quite similar to the one of [7], for the reader's convenience we shall give it explicitly. First of all, we prove the two following lemmas.

**Lemma 2.2** *Let* $T$, $X$ *be two Polish spaces, and let* $\mu$ *be a finite positive regular Borel measure on* $T$. *Let* $W \subseteq X$ *be a Souslin set,* $f : T \times W \to \mathbf{R}$ *a given function,* $E \subseteq W$ *another set. Assume that*:

(i) $f$ *is* $\mathcal{T}_\mu \otimes \mathcal{B}(W)$-*measurable*;
(ii) $\inf_{T\times W} f > -\infty$;
(iii) *for all* $t \in T$, *one has*

$$\{x \in W : f(t,\cdot) \text{ is not lower semicontinuous at } x\} \subseteq E.$$

*Then, for each* $\varepsilon > 0$ *there exists a compact set* $K \subseteq T$ *such that* $\mu(T \setminus K) \le \varepsilon$ *and the function* $f|_{K\times W}$ *is lower semicontinuous at each point* $(t,x) \in K \times (W \setminus E)$.

*Proof* Without loss of generality we can assume that $f \ge 0$ in $T \times W$. Let $d$ be the distance in $X$. Fix $n \in \mathbf{N}$, and let $f_n : T \times X \to [0,+\infty[$ be defined by putting, for all $(t,x) \in T \times X$,

$$f_n(t,x) := \inf_{y\in W} \bigl[ nd(x,y) + f(t,y) \bigr].$$

Let us observe the following facts.

(a) For all $x \in X$, the function $f_n(\cdot,x)$ is $\mathcal{T}_\mu$-measurable over $T$. To see this, fix $x \in X$. Our assumptions imply that the function

$$(t,y) \in T \times W \to nd(x,y) + f(t,y)$$

is $\mathcal{T}_\mu \otimes \mathcal{B}(W)$-measurable. Since $W$ (endowed with the relative topology) is a Souslin space, by Lemma III.39 of [11] our claim follows.

(b) For each fixed $t \in T$, the function $f_n(t,\cdot)$ is $n$-Lipschitzian over $X$ (the proof is straightforward).

(c) We have

$$f_n(t,x) \le f(t,x) \quad \text{for all } n \in \mathbf{N} \text{ and } (t,x) \in T \times W. \tag{4}$$

Now, let $\psi : T \times W \to [0,+\infty[$ be defined by putting

$$\psi(t,x) := \sup_{n\in\mathbf{N}} f_n(t,x).$$

In particular, by (4) we have

$$\psi(t,x) \leq f(t,x) \quad \text{for all } (t,x) \in T \times W. \tag{5}$$

We claim that

$$\psi(t,x) = f(t,x) \quad \text{for all } (t,x) \in T \times (W \setminus E). \tag{6}$$

To see this, fix $(t,x) \in T \times (W \setminus E)$ and assume that $\psi(t,x) < f(t,x)$. Let $\delta := f(t,x) - \psi(t,x) > 0$. Then, for each $n \in \mathbf{N}$ there exists $y_n \in W$ such that

$$nd(x,y_n) + f(t,y_n) < f(t,x) - \frac{\delta}{2},$$

hence

$$nd(x,y_n) < f(t,x) - \frac{\delta}{2} - f(t,y_n) \leq f(t,x) - \frac{\delta}{2}.$$

This implies that $\{y_n\} \to x$ in $X$ (hence, in particular, in $W$). By assumption (iii), we get

$$f(t,x) \leq \liminf_{n\to\infty} f(t,y_n) \leq \liminf_{n\to\infty}\left(f(t,x) - \frac{\delta}{2} - nd(x,y_n)\right) \leq f(t,x) - \frac{\delta}{2},$$

a contradiction. Thus, (6) is proved.

Now, in order to prove the conclusion, fix $\varepsilon > 0$. By Theorem 2 of [12], for each $n \in \mathbf{N}$ there exists a compact set $K_n \subseteq T$ such that $\mu(T \setminus K_n) < (\varepsilon/2^n)$ and the function $f_n|_{K_n\times X}$ is continuous. Let $K := \bigcap_{n\in\mathbf{N}} K_n$. Of course, $K$ is compact and $\mu(T \setminus K) < \varepsilon$. Moreover, for all $n \in \mathbf{N}$ the function $f_n|_{K\times W}$ is continuous.

Consequently, the function $\psi|_{K\times W}$ is l.s.c. Let us prove that $f|_{K\times W}$ is l.s.c. at each point $(t,x) \in K \times (W \setminus E)$. To this aim, fix $(t^*,x^*) \in K \times (W \setminus E)$ and $\gamma > 0$. Let $U$ be a neighborhood of $(x^*,y^*)$ in $K \times W$ such that

$$\psi(t,x) > \psi\left(t^*,x^*\right) - \gamma, \quad \forall (t,x) \in U$$

(such $U$ exists since $\psi|_{K\times W}$ is lower semicontinuous). For all $(t,x) \in U$, by (5) and (6) we get

$$f(t,x) \geq \psi(t,x) > \psi\left(t^*,x^*\right) - \gamma = f\left(t^*,x^*\right) - \gamma,$$

hence $f|_{K\times W}$ is l.s.c. at $(t^*,x^*)$, as desired. This concludes the proof. □

**Lemma 2.3** *Let $T$, $X$, $\mu$, $W$ and $E$ be as in Lemma 2.2. Let $S$ be a separable metric space, $F : T \times W \to 2^S$ a multifunction with nonempty values. Assume that*:

(i) *$F$ is $\mathcal{T}_\mu \otimes \mathcal{B}(W)$-weakly measurable*;

(ii) *for all $t \in T$, one has*

$$\left\{x \in W : F(t,\cdot) \text{ is not lower semicontinuous at } x\right\} \subseteq E.$$

*Then, for each $\varepsilon > 0$ there exists a compact set $K \subseteq T$ such that $\mu(T \setminus K) \leq \varepsilon$ and the multifunction $F|_{K\times W}$ is lower semicontinuous at each point $(t,x) \in K \times (W \setminus E)$.*

*Proof* Let $d_S$ be an equivalent distance over $S$ such that $d_S \leq 1$, and let $\{s_n\}$ be a dense sequence in $S$. Fix $\varepsilon > 0$. For each $n \in \mathbf{N}$, let $h_n : T \times W \to [0,+\infty[$ be defined by putting, for each $(t,x) \in T \times W$,

$$h_n(t,x) = -d_S\bigl(s_n, F(t,x)\bigr).$$

By assumption (i) and Theorem 3.3 of [9], the function $h_n$ is $\mathcal{T}_\mu \otimes \mathcal{B}(W)$-measurable. Moreover, by assumption (ii) and Lemma 3 of [13], for all $t \in T$ we have

$$\bigl\{x \in W : h_n(t,\cdot) \text{ is not lower semicontinuous at } x\bigr\} \subseteq E.$$

By Lemma 2.2, there exists a compact set $K_n \subseteq T$ such that $\mu(T \setminus K_n) < \varepsilon/2^n$ and $h_n|_{K_n\times W}$ is lower semicontinuous at each point $(t,x) \in K_n \times (W \setminus E)$. Let $K := \bigcap_{n\in\mathbf{N}} K_n$. Of course, $\mu(T \setminus K) \leq \varepsilon$ and, for all $n \in \mathbf{N}$, the function $h_n|_{K\times W}$ is lower semicontinuous at each point $(t,x) \in K \times (W \setminus E)$. Again by Lemma 3 of [13], this implies that the multifunction $F|_{K\times W}$ is lower semicontinuous at each point $(t,x) \in K \times (W \setminus E)$. This completes the proof. □

The following results are proved in [7]. We state them for easy reading (in the following, the space $\mathbf{N}^{\mathbf{N}}$ of all infinite sequences of integers is considered with the product topology; we recall that $\mathbf{N}^{\mathbf{N}}$ is Polish and zero-dimensional).

**Lemma 2.4** (Lemma 2.3 of [7]) *Let $p \in \mathbf{N}$, with $p \geq 2$, and let $Y_1, Y_2, \dots, Y_p$ be Polish spaces, endowed with $\sigma$-finite positive regular Borel measures $\mu_1, \mu_2, \dots, \mu_p$, respectively.*

*Then, there exist sets $Q_1, Q_2, \dots, Q_p$, with*

$$Q_i \in \mathcal{B}(Y_i) \quad \textit{and} \quad \mu_i(Q_i) = 0 \quad \textit{for all } i = 1, \dots, p,$$

*and two functions $\pi : \mathbf{N}^{\mathbf{N}} \to Y_1 \times Y_2 \times \cdots \times Y_p$ and $\sigma : Y_1 \times Y_2 \times \cdots \times Y_p \to \mathbf{N}^{\mathbf{N}}$, such that:*

(a) *the function $\pi$ is continuous and open;*
(b) *the function $\sigma$ is continuous at each point*

$$x := (x_1, x_2, \dots, x_p) \in (Y_1 \setminus Q_1) \times (Y_2 \setminus Q_2) \times \cdots \times (Y_p \setminus Q_p);$$

(c) *one has*

$$\pi\bigl(\sigma(x)\bigr) = x \quad \textit{for all } x \in Y_1 \times Y_2 \times \cdots \times Y_p.$$

**Lemma 2.5** (Lemma 2.4 of [7]) *Let $p \in \mathbf{N}$, with $p \geq 2$, and let $Y_1, Y_2, \dots, Y_p$ be Polish spaces, endowed with $\sigma$-finite positive regular Borel measures $\mu_1, \mu_2, \dots, \mu_p$, respectively. Let the sets $Q_1, Q_2, \dots, Q_p$ and the functions $\pi$ and $\sigma$ be as in the conclusion of Lemma* 2.3.

*Let $B \subseteq Y_1 \times \cdots \times Y_p$ a nonempty set, $V \subseteq B$ another given set, $E$ a metric space and $F : B \to 2^E$ a multifunction with nonempty complete values. Assume that $F$ is lower semicontinuous at each point $x = (x_1, \dots, x_p) \in B \setminus V$.*

*Then, there exists a function $g : B \to E$ such that:*

(a) *$g(x_1,\dots,x_p) \in F(x_1,\dots,x_p)$ for all $(x_1,\dots,x_p) \in B$;*
(b) *$g$ is continuous at each point*

$$(x_1,\dots,x_p) \in \big[\big[(Y_1 \setminus Q_1) \times (Y_2 \setminus Q_2) \times \cdots \times (Y_p \setminus Q_p)\big] \cap B\big] \setminus V.$$

*Proof of Theorem* 2.1 It is matter of routine to check that, without loss of generality, we can assume that (3) holds for all $t \in T$.

Fix $n \in \mathbf{N}$. By Lemma 2.3 (applied to the multifunction $F : T \times W \to 2^S$, with $\varepsilon = 1/n$), there exists a compact set $\hat{K}_n \subseteq T$ such that

$$\mu(T \setminus \hat{K}_n) \le \frac{1}{n}$$

and the multifunction $F|_{\hat{K}_n \times W}$ is lower semicontinuous at each point $(t,x) \in \hat{K}_n \times (W \setminus E)$. Let us put

$$K_1 := \hat{K}_1, \qquad K_n := \hat{K}_n \Big\backslash \bigcup_{j=1}^{n-1} \hat{K}_j, \quad n \ge 2.$$

The sets $\{K_n\}$ are pairwise disjoint, the equality $\bigcup_{n\in\mathbf{N}} \hat{K}_n = \bigcup_{n\in\mathbf{N}} K_n$ holds, and for all $n \in \mathbf{N}$ the multifunction $F|_{K_n \times W}$ is lower semicontinuous at each point $(t,x) \in K_n \times (W \setminus E)$. Let

$$Y := T \setminus \bigcup_{n\in\mathbf{N}} K_n.$$

Hence $Y \in \mathcal{B}(T)$, and for all $j \in \mathbf{N}$ one has

$$\mu(Y) = \mu\left(T \setminus \bigcup_{n\in\mathbf{N}} K_n\right) = \mu\left(T \setminus \bigcup_{n\in\mathbf{N}} \hat{K}_n\right) = \mu\left(\bigcap_{n\in\mathbf{N}} (T \setminus \hat{K}_n)\right) \le \mu(T \setminus \hat{K}_j) \le \frac{1}{j},$$

hence $\mu(Y) = 0$. Now, let us apply Lemma 2.4 above to the $k+1$ spaces $T, X_1, \dots, X_k$. We see that there exist sets

$$Q_0 \in \mathcal{B}(T), \qquad Q_1 \in \mathcal{B}(X_1), \qquad \dots, \qquad Q_k \in \mathcal{B}(X_k)$$

and functions $\pi$ and $\sigma$ satisfying the conclusion of Lemma 2.4. For each $n \in \mathbf{N}$, let us apply Lemma 2.5, with

$$B = K_n \times W, \qquad V := K_n \times E.$$

We see that there exists a function $g_n : K_n \times W \to S$ such that

$$g_n(t,x) \in F(t,x), \quad \forall (t,x) \in K_n \times W$$

and $g_n : K_n \times W \to S$ is continuous at each point

$$\begin{aligned}(t,x) &\in \big(\big[(T \setminus Q_0) \times (X_1 \setminus Q_1) \times \cdots \times (X_k \setminus Q_k)\big] \cap [K_n \times W]\big) \setminus [K_n \times E] \\ &= \big[(K_n \setminus Q_0) \times \big(\big((X_1 \setminus Q_1) \times \cdots \times (X_k \setminus Q_k)\big) \cap W\big)\big] \setminus (K_n \times E)\end{aligned}$$

$$= (K_n \setminus Q_0) \times \bigl(\bigl[\bigl((X_1 \setminus Q_1) \times \cdots \times (X_k \setminus Q_k)\bigr) \cap W\bigr] \setminus E\bigr)$$
$$= (K_n \setminus Q_0) \times \bigl(\bigl[(X_1 \setminus Q_1) \times \cdots \times (X_k \setminus Q_k)\bigr] \cap (W \setminus E)\bigr).$$

For each $t \in Y$ and $x \in W$, choose any $h(t,x) \in F(t,x)$. Let $\phi : T \times W \to S$ be defined by

$$\phi(t,x) = \begin{cases} g_n(t,x) & \text{if } t \in K_n, \\ h(t,x) & \text{if } t \in Y. \end{cases} \tag{7}$$

It is immediate to check that $\phi(t,x) \in F(t,x)$ for all $(t,x) \in T \times W$.

Now, fix $t^* \in T \setminus (Q_0 \cup Y)$ (observe that $Q_0 \cup Y \in \mathcal{B}(T)$ and has null measure) and let $n \in \mathbf{N}$ such that $t^* \in K_n$. Hence, for all $x \in W$, we have $\phi(t^*,x) = g_n(t^*,x)$, that is, $\phi(t^*,\cdot) = g_n(t^*,\cdot)$. Consequently, we get

$$\bigl\{x \in W : \phi\bigl(t^*,\cdot\bigr) \text{ is discontinuous at } x\bigr\} \subseteq E \cup \left(W \Bigm\backslash \left[\prod_{i=1}^{k}(X_i \setminus Q_i)\right]\right)$$
$$= E \cup \left[W \cap \left(\bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i)\right)\right].$$

Finally, if we fix

$$x := (x_1, x_2, \ldots, x_k) \in W \Bigm\backslash \left[\left(\bigcup_{i=1}^{k} P_{*,i}^{-1}(Q_i)\right) \cup E\right]$$
$$= \bigl[\bigl((X_1 \setminus Q_1) \times \cdots \times (X_k \setminus Q_k)\bigr) \cap W\bigr] \setminus E,$$

by (7) we see that for all $n \in \mathbf{N}$ the function $\phi(\cdot,x)|_{K_n \setminus Q_0}$ is $\mathcal{B}(K_n \setminus Q_0)$-measurable (since $g_n(\cdot,x)|_{K_n \setminus Q_0}$ is continuous), and $\phi(\cdot,x)|_{K_n \cap Q_0}$ is $\mathcal{T}_\mu$-measurable (since $\mu(Q_0) = 0$ and $\mathcal{T}_\mu$ is complete). It follows that for all $n \in \mathbf{N}$ the function $\phi(\cdot,x)|_{K_n}$ is $\mathcal{T}_\mu$-measurable. Hence, the function $\phi(\cdot,x)$ is $\mathcal{T}_\mu$-measurable over $T$ (since $\mu(Y) = 0$). The proof is complete. □

Finally, we prove the following proposition.

**Proposition 2.6** *Let $\psi : [a,b] \times \mathbf{R}^n \to \mathbf{R}^k$ be a given function, $E \subseteq \mathbf{R}^n$ be a Lebesgue measurable set, with $m_n(E) = 0$, and let $D$ be a countable dense subset of $\mathbf{R}^n$, with $D \cap E = \emptyset$. Assume that*:

(i) *for all $t \in [a,b]$, the function $\psi(t,\cdot)$ is bounded*;

(ii) *for all $x \in D$, the function $\psi(\cdot,x)$ is $\mathcal{L}([a,b])$-measurable.*

*Let $G : [a,b] \times \mathbf{R}^n \to 2^{\mathbf{R}^k}$ be the multifunction defined by setting, for each $(t,x) \in [a,b] \times \mathbf{R}^n$,*

$$G(t,x) = \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}\Bigl(\bigcup_{\substack{y \in D \\ \|y-x\|_n \le \frac{1}{m}}} \bigl\{\psi(t,y)\bigr\}\Bigr)}.$$

*Then one has*:

(a) *$G$ has nonempty closed convex values*;

(b) *for all $x \in \mathbf{R}^n$, the multifunction $G(\cdot,x)$ is $\mathcal{L}([a,b])$-measurable*;

(c) *for all* $t \in [a,b]$, *the multifunction* $G(t,\cdot)$ *has closed graph*;
(d) *if* $t \in [a,b]$, *and* $\psi(t,\cdot)|_{\mathbf{R}^n\setminus E}$ *is continuous at* $x \in \mathbf{R}^n \setminus E$, *then one has*

$$G(t,x) = \{\psi(t,x)\}.$$

*Proof* Conclusions (a), (b), and (c) can be proved by arguing exactly as in the proof of Proposition 2 of [14]. To prove conclusion (d), fix $t \in [a,b]$, and let $x \in \mathbf{R}^n \setminus E$ such that $\psi(t,\cdot)|_{\mathbf{R}^n\setminus E}$ is continuous at $x$. Fix $\varepsilon > 0$. By continuity, there exists $\delta > 0$ such that

$$\|\psi(t,y) - \psi(t,x)\|_k \le \varepsilon, \quad \forall y \in \mathbf{R}^n \setminus E, \text{with } \|y-x\|_n < \delta.$$

Hence, if $m \in \mathbf{N}$, with $\frac{1}{m} < \delta$, taking into account that $E \cap D \neq \emptyset$, we get

$$\overline{\operatorname{conv}}\Bigl(\overline{\bigcup_{\substack{y\in D\\ \|y-x\|_n\le\frac{1}{m}}} \{\psi(t,y)\}}\Bigr) \subseteq \overline{B}_k\bigl(\psi(t,x),\varepsilon\bigr).$$

Consequently, we get $G(t,x) \subseteq \overline{B}_k(\psi(t,x),\varepsilon)$. Since $\varepsilon > 0$ was arbitrary and $G(t,x) \neq \emptyset$, we get the conclusion. □

## 3 The main result

The following is our main result.

**Theorem 3.1** *Let* $[a,b]$ *be a compact interval, and let* $Y \in \mathcal{A}_n$ *be a closed, connected and locally connected subset of* $\mathbf{R}^n$. *Let* $f : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \times Y \to \mathbf{R}$ *be a given function,* $\Sigma \subseteq Y \times Y$ *a countable set, dense in* $Y \times Y$, $D'$ *and* $D''$ *two dense subset of* $Y$. *Assume that there exist* $2n$ *sets*

$$V_1, V_2, \ldots, V_{2n} \in \mathcal{B}(\mathbf{R}),$$

*with* $m_1(V_i) = 0$ *for all* $i = 1,\ldots,2n$, *such that, if one puts*

$$\Omega := \prod_{i=1}^{2n}[\mathbf{R} \setminus V_i],$$

$$f^* : [a,b] \times \Omega \times Y \to \mathbf{R}, \qquad f^* := f|_{[a,b]\times\Omega\times Y},$$

*one has*:
(i) *for all* $(y_1,y_2) \in \Sigma$, *one has*

$$\bigl\{(t,x,z) \in [a,b] \times \Omega : f^*(t,x,z,y_1) < 0 < f^*(t,x,z,y_2)\bigr\} \in \mathcal{L}\bigl([a,b]\bigr) \otimes \mathcal{B}(\Omega);$$

(ii) *for a.e.* $t \in [a,b]$, *and for all* $y \in D'$, *the function* $f^*(t,\cdot,\cdot,y)$ *is l.s.c. over* $\Omega$;
(iii) *for a.e.* $t \in [a,b]$, *and for all* $y \in D''$, *the function* $f^*(t,\cdot,\cdot,y)$ *is u.s.c. over* $\Omega$;
(iv) *for a.e.* $t \in [a,b]$, *and for all* $(x,z) \in \Omega$, *the function* $f(t,x,z,\cdot)$ *is continuous over* $Y$,

$$0 \in \operatorname{int}_{\mathbf{R}}\bigl(f(t,x,z,Y)\bigr)$$

*and*

$$\operatorname{int}_Y\bigl(\{y \in Y : f(t,x,z,y) = 0\}\bigr) = \emptyset;$$

(v) *there exist* $p \in [1,+\infty]$ *and a positive function* $\beta \in L^p([a,b])$ *such that, for a.e.* $t \in [a,b]$*, and for all* $(x,z) \in \Omega$*, one has*

$$\{y \in Y : f(t,x,z,y) = 0\} \subseteq \overline{B}_n\bigl(0_{\mathbf{R}^n}, \beta(t)\bigr).$$

*Then, there exists* $u \in W^{2,p}([a,b],\mathbf{R}^n)$ *such that*

$$\begin{cases} f(t,u(t),u'(t),u''(t)) = 0 & \text{for a.e. } t \in [a,b], \\ u(a) = u(b) = 0_{\mathbf{R}^n}, \end{cases}$$

*and*

$$\bigl(u(t),u'(t)\bigr) \in \Omega \quad \text{for a.e. } t \in [a,b].$$

*Proof* From now on, we assume that assumptions (ii)-(v) are satisfied for all $t \in [a,b]$ (it is routine matter to check that it is not restrictive to do this). First of all, we assume $p < +\infty$. Fix $t \in [a,b]$. Let

$$V_t : \Omega \to 2^Y, \qquad E_t : \Omega \to 2^Y, \qquad Q_t : \Omega \to 2^Y$$

be the multifunctions defined by setting, for all $(x,z) \in \Omega$,

$$\begin{aligned} V_t(x,z) &:= \{y \in Y : f^*(t,x,z,y) = 0\}, \\ E_t(x,z) &:= \{y \in Y : y \text{ is a local extremum for } f^*(t,x,z,\cdot)\}, \\ Q_t(x,z) &:= V_t(x,z) \setminus E_t(x,z). \end{aligned}$$

By assumptions (ii), (iii), (iv), and Theorem 2.2 of [15], the multifunction $Q_t : \Omega \to 2^Y$ is lower semicontinuous in $\Omega$ with nonempty closed values (in $Y$, hence in $\mathbf{R}^n$ since $Y$ is closed). Now, let

$$Q : [a,b] \times \Omega \to 2^Y, \qquad Q(t,x,z) := Q_t(x,z).$$

By what precedes, the multifunction $Q$ has nonempty closed values and, for each fixed $t \in [a,b]$, the multifunction $Q(t,\cdot,\cdot)$ is lower semicontinuous in $\Omega$.

Now we prove that the multifunction $Q$ is $\mathcal{L}([a,b]) \times \mathcal{B}(\Omega)$-measurable. To this aim, let $A \subseteq Y$ be a nonempty open connected set, such that $Q^-(A) \neq \emptyset$. We claim that

$$Q^-(A) = \bigcup_{(y_1,y_2)\in(A\times A)\cap\Sigma} \bigl\{(t,x,z) \in [a,b] \times \Omega : f(t,x,z,y_1) < 0 < f(t,x,z,y_2)\bigr\}. \tag{8}$$

To this aim, fix any $(t_0,x_0,z_0) \in Q^-(A)$. Therefore, $(t_0,x_0,z_0) \in [a,b] \times \Omega$ and there exists $y_0 \in A \cap Q(t_0,x_0,z_0)$. That is, $y_0 \in A$, $f(t_0,x_0,z_0,y_0) = 0$ and $y_0$ is not a local extremum for the function $f(t_0,x_0,z_0,\cdot)$. This implies that there exist two points $y',y'' \in A$ such that

$$f(t_0,x_0,z_0,y') < 0 \quad \text{and} \quad f(t_0,x_0,z_0,y'') > 0.$$

By the continuity of the function $f(t_0,x_0,z_0,\cdot)$, there exist two open sets $B_1,B_2 \subseteq Y$ such that $y' \in B_1$, $y'' \in B_2$, and

$$f(t_0,x_0,z_0,y) < 0 \quad \text{for all } y \in B_1,$$
$$f(t_0,x_0,z_0,y) > 0 \quad \text{for all } y \in B_2.$$

Put $A_1 := B_1 \cap A$, $A_2 := B_2 \cap A$. Of course, both $A_1$ and $A_2$ are open in $Y$ and nonempty (since they contain $y'$ and $y''$, respectively).

Let $(y_1,y_2) \in (A_1 \times A_2) \cap \Sigma$ (this last set is nonempty since $\Sigma$ is dense in $Y \times Y$). We see that

$$(t_0,x_0,z_0) \in \{(t,x,z) \in [a,b] \times \Omega : f(t,x,z,y_1) < 0 < f(t,x,z,y_2)\},$$

hence $(t_0,x_0,z_0)$ belongs to the right-hand side of (8). Conversely, let $(t^*,x^*,z^*)$ belong to the right-hand side of (8). Thus, there exists $(y_1,y_2) \in (A \times A) \cap \Sigma$ such that

$$f(t^*,x^*,z^*,y_1) < 0 < f(t^*,x^*,z^*,y_2). \tag{9}$$

Since $A$ is connected and $f(t^*,x^*,z^*,\cdot)$ is continuous in $Y$, there exists $y_3 \in A$ such that $f(t^*,x^*,z^*,y_3) = 0$. We now distinguish two cases.

(1) If $y_3$ is not a local extremum for the function $f(t^*,x^*,z^*,\cdot)$, then we get $y_3 \in Q(t^*,x^*,z^*) \cap A$, hence $(t^*,x^*,z^*) \in Q^-(A)$, as desired.

(2) If $y_3$ is a local extremum for the function $f(t^*,x^*,z^*,\cdot)$ (not absolute by assumption (iv)), then $y_3$ is also a local extremum for the function $f(t^*,x^*,z^*,\cdot)|_A$ (not absolute by (9)). Moreover, since $A$ is open in $Y$, by assumption (v) we get

$$\mathrm{int}_A(\{y \in A : f(t^*,x^*,z^*,y) = 0\}) = \emptyset.$$

Consequently, by Lemma 2.1 of [15], there exists a point $y^* \in A$ such that $f(t^*,x^*,z^*,y^*) = 0$ and $y^*$ is not a local extremum for the function $f(t^*,x^*,z^*,\cdot)|_A$. This easily implies that $y^*$ is not a local extremum for the function $f(t^*,x^*,z^*,\cdot)$ (considered over the whole set $Y$). Therefore, we have $y^* \in Q(t^*,x^*,z^*) \cap A$, hence $(t^*,x^*,z^*) \in Q^-(A)$, as desired.

Thus, the equality (8) is proved, and therefore, by assumption (i), the set $Q^-(A)$ is $\mathcal{L}([a,b]) \otimes \mathcal{B}(\Omega)$-measurable.

Our assumptions on $Y$ imply that it has a countable base of connected open sets. Therefore, it follows that the multifunction $Q$ is $\mathcal{L}([a,b]) \times \mathcal{B}(\Omega)$-weakly measurable. By Theorem 3.5 of [9], the multifunction $Q$ is also $\mathcal{L}([a,b]) \otimes \mathcal{B}(\Omega)$-measurable.

By Corollary 6.6.7 of [10], the set $\Omega$ is a Souslin set since it belongs to $\mathcal{B}(\mathbf{R}^{2n})$. By Theorem 2.1 (where the spaces $[a,b]$ and $\mathbf{R}$ are considered with the usual one-dimensional Lebesgue measure $m_1$ over their Borel families), there exist $Q_1,\dots,Q_{2n} \in \mathcal{B}(\mathbf{R})$, with $m_1(Q_1) = 0$ for all $i = 1,\dots,2n$, a set $K_0 \in \mathcal{L}([a,b])$, with $m_1(K_0) = 0$, and a function $\phi : [a,b] \times \Omega \to 2^{\mathbf{R}^n}$, such that:

(a) $\phi(t,x,z) \in Q(t,x,z)$ for all $(t,x,z) \in [a,b] \times \Omega$ (hence, in particular, the function $\phi$ takes its values in $Y$);
(b) for all $(x,z) \in \Omega \setminus [\bigcup_{i=1}^{2n} P_{2n,i}^{-1}(Q_i)]$, the function $\phi(\cdot,x,z)$ is $\mathcal{L}([a,b])$-measurable;
(c) for all $t \in [a,b] \setminus K_0$, one has

$$\bigl\{(x,z) \in \Omega : \phi(t,\cdot,\cdot) \text{ is discontinuous at } (x,z)\bigr\} \subseteq \Omega \cap \left[\bigcup_{i=1}^{2n} P_{2n,i}^{-1}(Q_i)\right].$$

Now, let $\psi : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$\psi(t,x,z) = \begin{cases} \phi(t,x,z) & \text{if } (x,z) \in \Omega, \\ 0_{\mathbf{R}^n} & \text{if } (x,z) \notin \Omega. \end{cases}$$

Observe that $\mathbf{R}^{2n} \setminus \Omega = \bigcup_{i=1}^{2n} P_{2n,i}^{-1}(V_i)$, and also

$$\Omega \Big\backslash \left[\bigcup_{i=1}^{2n} P_{2n,i}^{-1}(Q_i)\right] = \mathbf{R}^{2n} \Big\backslash \left[\bigcup_{i=1}^{2n} P_{2n,i}^{-1}(V_i \cup Q_i)\right] = \prod_{i=1}^{2n} \bigl[\mathbf{R} \setminus (V_i \cup Q_i)\bigr].$$

Let $D$ be a countable subset of $\Omega \setminus [\bigcup_{i=1}^{2n} P_{2n,i}^{-1}(Q_i)]$, dense in $\mathbf{R}^{2n}$. Of course, such a set $D$ exists since

$$m_{2n}\left(\bigcup_{i=1}^{2n} P_{2n,i}^{-1}(V_i \cup Q_i)\right) = 0.$$

Now we want to apply Proposition 2.6 to the function $\psi$, choosing $E = \mathbf{R}^{2n} \setminus \Omega$. To this aim, observe that:

(1) for all $t \in [a,b]$, and all $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$, we have $\psi(t,x,z) \in \overline{B}_n(0_{\mathbf{R}^n}, \beta(t))$ (this follows by the construction of $\phi$ and $Q$ and by assumption (v)), hence for all $t \in [a,b]$ the function $\psi(t,\cdot,\cdot)$ is bounded;
(2) for all $(x,z) \in D$, the function $\psi(\cdot,x,z) = \phi(\cdot,x,z)$ is $\mathcal{L}([a,b])$-measurable.

Consequently, if $G : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ is the multifunction defined by setting, for all $(t,x,z) \in [a,b] \times \mathbf{R}^n \times \mathbf{R}^n$,

$$\begin{aligned} G(t,x,z) &= \bigcap_{m\in\mathbf{N}} \overline{\operatorname{conv}}\left(\bigcup_{\substack{(v,w)\in D \\ \|(v,w)-(x,z)\|_{2n} \le \frac{1}{m}}} \bigl\{\psi(t,v,w)\bigr\}\right) \\ &= \bigcap_{m\in\mathbf{N}} \overline{\operatorname{conv}}\left(\bigcup_{\substack{(v,w)\in D \\ \|(v,w)-(x,z)\|_{2n} \le \frac{1}{m}}} \bigl\{\phi(t,v,w)\bigr\}\right), \end{aligned}$$

by Proposition 2.6 we see that:

(a)′ $G$ has nonempty closed convex values;
(b)′ for all $(x,z) \in \mathbf{R}^n \times \mathbf{R}^n$, the multifunction $G(\cdot,x,z)$ is $\mathcal{L}([a,b])$-measurable;
(c)′ for all $t \in [a,b]$, the multifunction $G(t,\cdot,\cdot)$ has closed graph;

(d)′ if $t \in [a,b]$, and the function $\psi(t,\cdot,\cdot)|_{\Omega} = \phi(t,\cdot,\cdot)$ is continuous at $(x,z) \in \Omega$, then one has

$$G(t,x,z) = \{\psi(t,x,z)\} = \{\phi(t,x,z)\}.$$

Moreover, observe that by the above construction we see that

$$G(t,x,z) \subseteq \overline{B}_n(0_{\mathbf{R}^n}, \beta(t)) \cap \overline{\operatorname{conv}}(Y) \quad \text{for all } (t,x,z) \in [a,b] \times \mathbf{R}^n \times \mathbf{R}^n. \tag{10}$$

Consequently, by Theorem 3 of [16], there exist $u \in W^{2,p}([a,b],\mathbf{R}^n)$ and a set $K_1 \in \mathcal{L}([a,b])$, with $m_1(K_1) = 0$, such that

$$\begin{cases} u''(t) \in G(t,u(t),u'(t)) & \text{for all } t \in [a,b] \setminus K_1, \\ u(a) = u(b) = 0_{\mathbf{R}^n}. \end{cases} \tag{11}$$

In particular, by (10) we get

$$u''(t) \in \overline{\operatorname{conv}}(Y) \quad \text{for all } t \in [a,b] \setminus K_1. \tag{12}$$

Now, fix $i \in \{1,\ldots,n\}$. Of course, (12) implies that

$$u_i''(t) \in [\inf P_{n,i}(\overline{\operatorname{conv}}(Y)), \sup P_{n,i}(\overline{\operatorname{conv}}(Y))] \quad \text{for all } t \in [a,b] \setminus K_1$$

(of course, we are denoting by $u_i(t)$ the $i$th component of $u(t)$). In particular, taking into account that $Y \in \mathcal{A}_n$, this implies that $u_i''(t)$ has constant sign for all $t \in [a,b] \setminus K_1$. Assume that

$$u_i''(t) > 0 \quad \text{for all } t \in [a,b] \setminus K_1.$$

Since $u_i'(t)$ is absolutely continuous, we have

$$u_i'(t) = u_i'(a) + \int_a^t u_i''(s)\,ds \quad \text{for all } t \in [a,b],$$

hence $u_i'(t)$ is strictly increasing in $[a,b]$. Consequently, by Theorem 2 of [17], the function

$$(u_i')^{-1} : u_i'([a,b]) \to [a,b]$$

is absolutely continuous. Moreover, since $u_i(a) = u_i(b) = 0$, there exists $c_i \in ]a,b[$ such that

$$u_i'(t) < 0 \quad \text{for all } t \in [a,c_i[$$

and

$$u_i'(t) > 0 \quad \text{for all } t \in ]c_i,b].$$

Again by Theorem 2 of [17], the functions

$$(u_i|_{[a,c_i]})^{-1} : u_i([a,c_i]) \to [a,c_i]$$

and

$$(u_i|_{[c_i,b]})^{-1} : u_i([c_i,b]) \to [c_i,b]$$

are absolutely continuous. Consequently, by Theorem 18.25 of [18], for each Lebesgue measurable $U \subseteq \mathbf{R}$, with $m_1(U) = 0$, the sets

$$(u_i')^{-1}(U) := \{t \in [a,b] : u_i'(t) \in U\}$$

and

$$\begin{aligned}(u_i)^{-1}(U) &:= \{t \in [a,b] : u_i(t) \in U\} \\ &= \{t \in [a,c_i] : u_i(t) \in U\} \cup \{t \in [c_i,b] : u_i(t) \in U\}\end{aligned}$$

have null Lebesgue measure.

If, conversely, one has

$$u_i''(t) < 0 \quad \text{for all } t \in [a,b] \setminus K_1,$$

by an analogous argument we can get the same conclusion.

Now, for each $i = 1,\dots,n$, let us put

$$S_i := u_i^{-1}(V_i \cup Q_i), \qquad W_i := (u_i')^{-1}(V_{n+i} \cup Q_{n+i}),$$

and

$$S := K_0 \cup K_1 \cup \left[\bigcup_{i=1}^{n}(S_i \cup W_i)\right].$$

For what precedes, we have $m_1(S) = 0$. Now, fix $t \in [a,b] \setminus S$. Since $t \notin \bigcup_{i=1}^n (S_i \cup W_i)$, for all $i = 1,\dots,n$ we get

$$u_i(t) \notin V_i \cup Q_i, \qquad u_i'(t) \notin V_{n+i} \cup Q_{n+i},$$

hence

$$(u(t),u'(t)) \in \prod_{i=1}^{2n}[\mathbf{R} \setminus (V_i \cup Q_i)] = \Omega \Big\backslash \left[\bigcup_{i=1}^{2n} P_{2n,i}^{-1}(Q_i)\right].$$

Since $t \notin K_0$, this implies that the function $\phi(t,\cdot,\cdot) : \Omega \to \mathbf{R}^n$ is continuous at $(u(t),u'(t))$. Hence, by the property (d)′ and the above construction, taking into account that $t \notin K_1$ and (11), we get

$$u''(t) \in G(t,u(t),u'(t)) = \{\phi(t,u(t),u'(t))\} \subseteq Q(t,u(t),u'(t)) \subseteq V_t(u(t),u'(t)).$$

Therefore, we get $u''(t) \in Y$ and

$$f\big(t, u(t), u'(t), u''(t)\big) = 0.$$

Thus, our conclusion is proved in the case $p < +\infty$.

Now, assume $p = +\infty$. Fix any $q \in [1, +\infty[$. Since $\beta \in L^q([a,b])$, by the first part of the proof there exist a function $u \in W^{2,q}([a,b], \mathbf{R}^n)$ and a set $U \in \mathcal{L}([a,b])$, with $m_1(U) = 0$, such that

$$\begin{cases} f(t, u(t), u'(t), u''(t)) = 0 & \text{for all } t \in [a,b] \setminus U, \\ u(a) = u(b) = 0_{\mathbf{R}^n}, \end{cases}$$

and

$$\big(u(t), u'(t)\big) \in \Omega \quad \text{for all } t \in [a,b] \setminus U.$$

By (v) we get

$$\big\{y \in Y : f\big(t, u(t), u'(t), y\big) = 0\big\} \subseteq \overline{B}_n\big(0_{\mathbf{R}^n}, \beta(t)\big) \quad \text{for all } t \in [a,b] \setminus U,$$

hence

$$\big\|u''(t)\big\|_n \leq \beta(t) \quad \text{for all } t \in [a,b] \setminus U.$$

This implies $u'' \in L^\infty([a,b], \mathbf{R}^n)$, hence $u \in W^{2,\infty}([a,b], \mathbf{R}^n)$. The proof is now complete. □

**Remark** Theorem 3.1 can be put in the following equivalent form.

**Corollary 3.2** *Let* $[a,b]$, $Y, f$, $\Sigma$, $D'$ *and* $D''$ *be as in the statement of Theorem* 3.1. *Assume that there exists a set* $F \in \mathcal{F}_{2n}$ *such that, if one puts* $\Omega := \mathbf{R}^{2n} \setminus F$ *and* $f^* := f|_{[a,b]\times\Omega\times Y}$, *then assumptions* (i)-(v) *of Theorem* 3.1 *are satisfied.*

*Then the same conclusion of Theorem* 3.1 *holds.*

*Proof* Let $F_1, F_2, \ldots, F_{2n} \subseteq \mathbf{R}^{2n}$, with $m_1(P_{2n,i}(F_i)) = 0$ for all $i = 1, \ldots, 2n$, be such that $F = \bigcup_{i=1}^{2n} F_i$. For each $i = 1, \ldots, 2n$, let $V_i \in \mathcal{B}(\mathbf{R})$ be such that $P_{2n,i}(F_i) \subseteq V_i$ and $m_1(V_i) = 0$. If one puts $\Omega' := \prod_{i=1}^{2n} (\mathbf{R} \setminus V_i)$, then $\Omega' \subseteq \Omega$, hence all the assumptions of Theorem 3.1 are satisfied. The conclusion follows at once. □

Now we give a simple example of application of Theorem 3.1.

**Example 1** Let $n = 1$, $Y = [1, +\infty[$, $p \in [1, +\infty]$, and let $\alpha \in L^p([0,1])$, with $\alpha(t) > 0$ for all $t \in [0,1]$. Let

$$E := \big\{(x,z) \in \mathbf{R}^2 : x \in \mathbf{Q} \text{ or } z \in \mathbf{Q}\big\} = (\mathbf{Q} \times \mathbf{R}) \cup (\mathbf{R} \times \mathbf{Q}),$$

and let $f : [0,1] \times \mathbf{R} \times \mathbf{R} \times Y \to \mathbf{R}$ be defined by putting

$$f(t,x,z,y) = \begin{cases} 2y - \alpha(t) - 10 & \text{if } (x,z) \in E, \\ 3\cos(y + \frac{\pi}{2} - 1) + y(2 + |\cos x|) - |\cos(x+z)| - \alpha(t) - 3 & \text{if } (x,z) \notin E. \end{cases}$$

It is immediate to check that for each fixed $(t,y) \in [0,1] \times Y$ one has

$$f(t,x,z,y) \geq 2y - \alpha(t) - 7 \quad \text{for all } (x,z) \in \mathbf{R}^2 \setminus E.$$

Consequently, for each fixed $(t,y) \in [0,1] \times Y$, the function $f(t,\cdot,\cdot,y)$ *is discontinuous at all points* $(x,z) \in \mathbf{R}^2$.

Now, let us observe that such a function $f$ satisfies the assumptions of Theorem 3.1. To this aim, choose $V_1 = V_2 = \mathbf{Q}$, $D' = D'' = Y$, and let $\Sigma$ be any countable dense subset of $Y \times Y$. In this case, we have

$$\Omega = (\mathbf{R} \setminus \mathbf{Q}) \times (\mathbf{R} \setminus \mathbf{Q}),$$

hence in particular we get $\Omega = \mathbf{R}^2 \setminus E$ and

$$f^*(t,x,z,y) = 3\cos\left(y + \frac{\pi}{2} - 1\right) + y\bigl(2 + |\cos x|\bigr) - \bigl|\cos(x+z)\bigr| - \alpha(t) - 3$$

for all $(t,x,z,y) \in [0,1] \times \Omega \times Y$. At this point, observe what follows.

(a) For all fixed $y \in Y$, the function $f^*(\cdot,\cdot,\cdot,y)$ is $\mathcal{L}([0,1]) \otimes \mathcal{B}(\Omega)$-measurable. Therefore, assumption (i) of Theorem 3.1 is satisfied.

(b) For all $(t,y) \in [0,1] \times Y$, the function $f^*(t,\cdot,\cdot,y)$ is continuous over $\Omega$.

(c) Let $t \in [0,1]$ and $(x,z) \in \Omega$ be fixed. We see that $f(t,x,z,\cdot) = f^*(t,x,z,\cdot)$ is continuous in $Y$, and also

$$f(t,x,z,1) \leq -\alpha(t) < 0.$$

Since

$$\lim_{y \to +\infty} f(t,x,z,y) = +\infty,$$

we get $0 \in \operatorname{int}_{\mathbf{R}}(f(t,x,z,Y))$. Moreover, if we put $s(\cdot) := f(t,x,z,\cdot)$, we get

$$s'(y) = -3\sin\left(y + \frac{\pi}{2} - 1\right) + 2 + |\cos x|, \quad \forall y \in Y.$$

Consequently, the set $\{y \in Y : s'(y) = 0\}$ is countable, hence $s'$ is never identically equal to 0 in any interval $I \subseteq Y$. This implies that there exists no interval $I \subseteq Y$ such that $s(\cdot)$ is constant on $I$. Therefore, assumption (iv) of Theorem 3.1 is also satisfied. We also remark that if $x \neq k\pi$ (with $k \in \mathbf{Z}$), then $s'$ takes both positive and negative values, hence $f(t,x,z,\cdot)$ is not monotone.

(d) Let $t \in [0,1]$ and $(x,z) \in \Omega$ be fixed, and let $y \in Y$ be such that $f(t,x,z,y) = 0$. By the definition of $Y$ and $f$ we get

$$1 \le y \le \frac{1}{2}\bigl(7 + \alpha(t)\bigr).$$

Since the function

$$\beta(t) = \frac{1}{2}\bigl(7 + \alpha(t)\bigr)$$

belongs to $L^p([0,1])$, assumption (v) of Theorem 3.1 is also satisfied.

Therefore, all the assumptions of Theorem 3.1 are fulfilled. Consequently, there exists $u \in W^{2,p}([0,1])$ such that

$$\begin{cases} f(t,u(t),u'(t),u''(t)) = 0 & \text{for a.e. } t \in [0,1], \\ u(0) = u(1) = 0 \end{cases} \tag{13}$$

and also

$$\bigl(u(t),u'(t)\bigr) \in (\mathbf{R} \setminus \mathbf{Q}) \times (\mathbf{R} \setminus \mathbf{Q}) \quad \text{for a.e. } t \in [0,1].$$

Finally, we observe that problem (13) does not admit the trivial solution $u(t) \equiv 0$.

The following result is an immediate consequence of Theorem 3.1.

**Theorem 3.3** *Let $[a,b]$ be a compact interval, and let $Y \in \mathcal{A}_n$ be a closed, connected and locally connected subset of $\mathbf{R}^n$. Let $\psi : [a,b] \times Y \to \mathbf{R}$ and $g : [a,b] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be two given functions, $D \subseteq Y$ a countable set, dense in $Y$. Assume that for all $y \in D$, the function $\psi(\cdot,y)$ is $\mathcal{L}([a,b])$-measurable, and for a.e. $t \in [a,b]$, the function $\psi(t,\cdot)$ is continuous in $Y$. Moreover, assume that there exist $2n$ sets*

$$V_1, V_2, \dots, V_{2n} \in \mathcal{B}(\mathbf{R}),$$

*with $m_1(V_i) = 0$ for all $i = 1,\dots,2n$, such that, if one puts*

$$\Omega := \prod_{i=1}^{2n} [\mathbf{R} \setminus V_i],$$

*one has*:

(i) *for all $t \in [a,b]$, the function $g(t,\cdot,\cdot)|_\Omega$ is continuous over $\Omega$*;
(ii) *for all $(x,z) \in \Omega$, the function $g(\cdot,x,z)$ is $\mathcal{L}([a,b])$-measurable*;
(iii) *for a.e. $t \in [a,b]$, one has $g(t,\Omega) \subseteq \operatorname{int}_{\mathbf{R}}(\psi(t,Y))$*;
(iv) *for a.e. $t \in [a,b]$ and for all $v \in \operatorname{int}_{\mathbf{R}}(\psi(t,Y))$, one has $\operatorname{int}_Y(\{y \in Y : \psi(t,y) = v\}) = \emptyset$*;
(v) *there exist $p \in [1,+\infty]$ and a positive function $\beta \in L^p([a,b])$ such that, for a.e. $t \in [a,b]$, and for all $(x,z) \in \Omega$, one has*

$$\bigl\{y \in Y : \psi(t,y) = g(t,x,z)\bigr\} \subseteq \overline{B}_n\bigl(0_{\mathbf{R}^n}, \beta(t)\bigr).$$

*Then, there exists* $u \in W^{2,p}([a,b],\mathbf{R}^n)$ *such that*

$$\begin{cases} \psi(t,u''(t)) = g(t,u(t),u'(t)) & \text{for a.e. } t \in [a,b], \\ u(a) = u(b) = 0_{\mathbf{R}^n}, \end{cases}$$

*and*

$$\big(u(t),u'(t)\big) \in \Omega \quad \text{for a.e. } t \in [a,b].$$

*Proof* Let us apply Theorem 3.1, with $f(t,x,z,y) = \psi(t,y) - g(t,x,z)$, $D' = D'' = Y$ and $\Sigma = D \times D$. Observe that by assumptions (i) and (ii) and by the lemma at p.198 of [12], the function $g|_{[a,b]\times\Omega}$ is $\mathcal{L}([a,b]) \times \mathcal{B}(\Omega)$-measurable. Consequently, assumption (i) of Theorem 3.1 is satisfied. It is a matter of routine to check that all the other assumptions of Theorem 3.1 are satisfied. Therefore, the conclusion follows at once. □

Arguing exactly as for Theorem 3.1, it can be easily seen that Theorem 3.3 can be put in the following equivalent form.

**Corollary 3.4** *Let* $[a,b]$, $Y$, $\psi$, $g$ *and* $D$ *be as in the statement of Theorem* 3.3. *Assume that there exists a set* $F \in \mathcal{F}_{2n}$ *such that, if one puts* $\Omega := \mathbf{R}^{2n} \setminus F$, *then assumptions* (i)-(v) *of Theorem* 3.3 *are satisfied.*

*Then, the same conclusion of Theorem* 3.3 *holds.*

**Remark** Theorem 3.3 (or, equivalently, Corollary 3.4) should be compared with Theorem 2.2 of [6] (valid for the case where $\psi$ does not depend on $t$ and the set $Y$ is bounded), where the function $g$ is assumed to be continuous in all variables. We now give an example of an application of Theorem 3.3 in the vector case $n = 2$.

**Example 2** Let $n = 2$. We now denote vectors of $\mathbf{R}^2$ by

$$x = (x_1,x_2), \qquad z = (z_1,z_2), \qquad y = (y_1,y_2).$$

Let $p \in [1,+\infty]$ and $\alpha \in L^p([0,1])$ be fixed, with $\alpha(t) \geq 3$ for all $t \in [0,1]$. Let

$$E := \big\{(x,z) \in \mathbf{R}^4 : \text{at least one of } x_1,x_2,z_1,z_2 \text{ is rational}\big\},$$

and let $h : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ and $g : [0,1] \times \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$h(t,y_1,y_2) = \sin\big[t(y_1 - 1)\big] + y_1 + y_2,$$

$$g(t,x_1,x_2,z_1,z_2) = \begin{cases} \alpha(t) + \cos^2(t\,x_1\,x_2\,z_1\,z_2) & \text{if } (x,z) \notin E, \\ \alpha(t) - 2 & \text{if } (x,z) \in E. \end{cases}$$

Of course, for all $t \in [0,1]$ the function $g(t,\cdot,\cdot)$ *is discontinuous at all points* $(x,z) \in \mathbf{R}^4$. However, Theorem 3.3 can be easily applied, by choosing

$$V_1 = V_2 = V_3 = V_4 = \mathbf{Q}, \qquad Y := [1,+\infty[\times[1,+\infty[, \qquad \psi := h|_{[0,1]\times Y},$$

and taking as $D$ any countable dense subset of $Y$.

In particular, we observe that in this case we have $\Omega = \mathbf{R}^4 \setminus E$ and

$$g(t,x_1,x_2,z_1,z_2) = \alpha(t) + \cos^2(t\,x_1\,x_2\,z_1\,z_2) \quad \text{for all } (x,z) \in \Omega.$$

Consequently, assumptions (i) and (ii) of Theorem 3.3 are satisfied. If we fix $t \in [0,1]$, then $\psi(t,1,1) = 2$ and $\psi(t,\cdot,\cdot)$ is upperly unbounded, hence

$$g(t,\Omega) \subseteq [3,+\infty[ \subseteq ]2,+\infty[ \subseteq \psi(t,Y).$$

Hence, assumption (iii) of Theorem 3.3 is also satisfied. We now prove that assumption (iv) is also satisfied (although this fact is quite intuitive, we provide a direct proof).

To this aim, fix $t \in [0,1]$ and $v^* \in \operatorname{int}(\psi(t,Y))$. Let $y^* := (y_1^*, y_2^*) \in Y$ be such that $\psi(t,y_1^*,y_2^*) = v^*$, and let $\Sigma$ be any open set in $Y$ such that $(y_1^*,y_2^*) \in \Sigma$. This implies that there exists $\delta > 0$ such that

$$\bigl(]y_1^* - \delta, y_1^* + \delta[ \times ]y_2^* - \delta, y_2^* + \delta[\bigr) \cap Y \subseteq \Sigma.$$

If we choose any $y_2 \in ]y_2^*, y_2^* + \delta[$, we get $(y_1^*, y_2) \in \Sigma$ and

$$\psi\bigl(t,y_1^*,y_2\bigr) > \psi\bigl(t,y_1^*,y_2^*\bigr) = v^*.$$

Hence, the set $\{y \in Y : \psi(t,y_1,y_2) = v^*\}$ has empty interior in $Y$, as desired.

Finally, let $t \in [0,1]$ and $(x,z) \in \Omega$. Take $y \in Y$ such that $\psi(t,y) = g(t,x,z)$. By the definitions of $Y$, $\psi$ and $g$ we get

$$\|y\|_2 \leq \alpha(t) + 2,$$

hence assumption (v) of Theorem 3.3 is also satisfied. Consequently, there exists $u \in W^{2,p}([0,1],\mathbf{R}^2)$ such that

$$\begin{cases} h(t,u''(t)) = g(t,u(t),u'(t)) & \text{for a.e. } t \in [0,1], \\ u(a) = u(b) = 0_{\mathbf{R}^2}, \end{cases} \tag{14}$$

and also

$$u(t) \in (\mathbf{R} \setminus \mathbf{Q}) \times (\mathbf{R} \setminus \mathbf{Q}), \qquad u'(t) \in (\mathbf{R} \setminus \mathbf{Q}) \times (\mathbf{R} \setminus \mathbf{Q}), \qquad u''(t) \in [1,+\infty[ \times [1,+\infty[$$

for a.e. $t \in [0,1]$. We also observe that problem (14) does not admit the trivial solution $u(t) \equiv 0_{\mathbf{R}^2}$.

## 4 Implicit integral equations

As announced in Section 1, our aim in this section it to apply the same approach as before to the study of implicit integral equations. From now on, let $n \in \mathbf{N}$ be fixed. In what follows, for simpler notation, we shall put $P_i := P_{n,i}$. The following theorem is the main result of this section.

**Theorem 4.1** *Let $[a,b]$ be a compact interval, and let $Y \in \mathcal{A}_n$ be a closed, connected and locally connected subset of $\mathbf{R}^n$. Let $f : [a,b] \times \mathbf{R}^n \times Y \to \mathbf{R}$ and $g : [a,b] \times [a,b] \to [0,+\infty[$ be two given functions, $\Sigma \subseteq Y \times Y$ a countable set, dense in $Y \times Y$, $D'$ and $D''$ two dense subset of $Y$. Moreover, let $p \in ]1,+\infty]$, $\phi_0 \in L^j([a,b])$, with $j > 1$ and $j \geq p'$ (the conjugate exponent of $p$), $\phi_1 \in L^{p'}([a,b])$. Assume that there exists a set $F \in \mathcal{F}_n$ such that:*

(i) *for all $(y_1,y_2) \in \Sigma$, one has*

$$\bigl\{(t,x) \in [a,b] \times \bigl(\mathbf{R}^n \setminus F\bigr) : f(t,x,y_1) < 0 < f(t,x,y_2)\bigr\} \in \mathcal{L}\bigl([a,b]\bigr) \otimes \mathcal{B}\bigl(\mathbf{R}^n \setminus F\bigr);$$

(ii) *for a.e. $t \in [a,b]$, and for all $y \in D'$, the function $f(t,\cdot,y)|_{\mathbf{R}^n\setminus F}$ is l.s.c.;*

(iii) *for a.e. $t \in [a,b]$, and for all $y \in D''$, the function $f(t,\cdot,y)|_{\mathbf{R}^n\setminus F}$ is u.s.c.;*

(iv) *for a.e. $t \in [a,b]$, and for all $x \in \mathbf{R}^n \setminus F$, the function $f(t,x,\cdot)$ is continuous over $Y$,*

$$0 \in \operatorname{int}_{\mathbf{R}}\bigl(f(t,x,Y)\bigr)$$

*and*

$$\operatorname{int}_Y\bigl(\bigl\{y \in Y : f(t,x,y) = 0\bigr\}\bigr) = \emptyset;$$

(v) *there exists a positive function $\beta \in L^p([a,b])$ such that, for a.e. $t \in [a,b]$, and for all $x \in \mathbf{R}^n \setminus F$, one has*

$$\bigl\{y \in Y : f(t,x,y) = 0\bigr\} \subseteq \overline{B}_n\bigl(0_{\mathbf{R}^n}, \beta(t)\bigr).$$

*Moreover, assume that:*

(vi) *for all $t \in [a,b]$, the function $g(t,\cdot)$ is measurable;*

(vii) *for a.e. $z \in [a,b]$, the function $g(\cdot,z)$ is continuous in $[a,b]$, differentiable in $]a,b[$ and*

$$g(t,z) \leq \phi_0(z), \qquad 0 < \frac{\partial g}{\partial t}(t,z) \leq \phi_1(z) \quad \textit{for all } t \in ]a,b[.$$

*Then, there exists $u \in L^p([a,b],\mathbf{R}^n)$ such that*

$$f\left(t, \int_a^b g(t,z)u(z)\,dz, u(t)\right) = 0 \quad \textit{for a.e. } t \in [a,b],$$

$$\bigl\|u(t)\bigr\|_n \leq \beta(t) \quad \textit{and} \quad \int_a^b g(t,z)u(z)\,dz \in \mathbf{R}^n \setminus F \quad \textit{for a.e. } t \in [a,b].$$

*Proof* Without loss of generality we can assume that assumptions (ii)-(v) are satisfied for all $t \in [a,b]$. Moreover, taking into account that $p > 1$, it is not restrictive to assume $j < +\infty$.

Let $F_1, F_2, \ldots, F_n \subseteq \mathbf{R}^n$, with $m_1(P_i(F_i)) = 0$ for all $i = 1,\ldots,n$, be such that $F = \bigcup_{i=1}^n F_i$. For all $i = 1,\ldots,n$, let $B_i \in \mathcal{B}(\mathbf{R})$ such that $P_i(F_i) \subseteq B_i$ and $m_1(B_i) = 0$. Let $C := \bigcup_{i=1}^n P_i^{-1}(B_i)$, and

$$\Omega := \mathbf{R}^n \setminus C = \prod_{i=1}^n (\mathbf{R} \setminus B_i) \in \mathcal{B}\bigl(\mathbf{R}^n\bigr).$$

Clearly, we have $F \subseteq C$ and thus $\Omega \subseteq \mathbf{R}^n \setminus F$. Let

$$V : [a,b] \times \Omega \to 2^Y, \qquad E : [a,b] \times \Omega \to 2^Y, \qquad Q : [a,b] \times \Omega \to 2^Y$$

be the multifunctions defined by setting, for all $(t,x) \in [a,b] \times \Omega$,

$$\begin{aligned} V(t,x) &:= \{y \in Y : f(t,x,y) = 0\}, \\ E(t,x) &:= \{y \in Y : y \text{ is a local extremum for } f(t,x,\cdot)\}, \\ Q(t,x) &:= V(t.x) \setminus E(t,x). \end{aligned}$$

By assumptions (ii), (iii), (iv), and Theorem 2.2. of [15], it is immediately seen that $Q$ has nonempty closed values (in $Y$, hence in $\mathbf{R}^n$) and for all $t \in [a,b]$ the multifunction $Q(t,\cdot)$ is lower semicontinuous in $\Omega$.

Arguing exactly as in the proof of Theorem 3.1, it can be checked that the multifunction $Q$ is $\mathcal{L}([a,b]) \times \mathcal{B}(\Omega)$-measurable. By Theorem 2.1 (applied with $T = [a,b]$, $X_i = \mathbf{R}$ for all $i = 1,\dots,n$, where all the spaces are considered with the usual one-dimensional Lebesgue measure $m_1$ over their Borel families), there exist $Q_1,\dots,Q_n \in \mathcal{B}(\mathbf{R})$, with $m_1(Q_i) = 0$ for all $i = 1,\dots,n$, a set $K_0 \in \mathcal{L}([a,b])$, with $m_1(K_0) = 0$, and a function $\phi : [a,b] \times \Omega \to 2^{\mathbf{R}^n}$, such that:

(a) $\phi(t,x) \in Q(t,x)$ for all $(t,x) \in [a,b] \times \Omega$ (hence the function $\phi$ takes its values in $Y$);
(b) for all $x \in \Omega \setminus [\bigcup_{i=1}^n P_i^{-1}(Q_i)]$, the function $\phi(\cdot,x)$ is $\mathcal{L}([a,b])$-measurable;
(c) for all $t \in [a,b] \setminus K_0$, one has

$$\{x \in \Omega : \phi(t,\cdot) \text{ is discontinuous at } x\} \subseteq \Omega \cap \left[\bigcup_{i=1}^n P_i^{-1}(Q_i)\right].$$

Let $\psi : [a,b] \times \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$\psi(t,x) = \begin{cases} \phi(t,x) & \text{if } x \in \Omega, \\ 0_{\mathbf{R}^n} & \text{if } x \notin \Omega. \end{cases}$$

Observe that

$$\Omega \setminus \left[\bigcup_{i=1}^n P_i^{-1}(Q_i)\right] = \mathbf{R}^n \setminus \left[\bigcup_{i=1}^n P_i^{-1}(B_i \cup Q_i)\right] = \prod_{i=1}^n \left[\mathbf{R} \setminus (B_i \cup Q_i)\right],$$

and let $D$ be any countable subset of $\Omega \setminus [\bigcup_{i=1}^n P_i^{-1}(Q_i)]$, dense in $\mathbf{R}^n$. Of course, such a set $D$ exists since

$$m_n\left(\bigcup_{i=1}^n P_i^{-1}(B_i \cup Q_i)\right) = 0.$$

Let $G : [a,b] \times \mathbf{R}^n \to 2^{\mathbf{R}^n}$ be the multifunction defined by setting, for all $(t,x) \in [a,b] \times \mathbf{R}^n$,

$$G(t,x) := \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}}\left(\overline{\bigcup_{\substack{v \in D \\ \|v-x\|_n \le \frac{1}{m}}} \{\psi(t,v)\}}\right) = \bigcap_{m \in \mathbf{N}} \overline{\operatorname{conv}}\left(\overline{\bigcup_{\substack{v \in D \\ \|v-x\|_n \le \frac{1}{m}}} \{\phi(t,v)\}}\right).$$

Applying Proposition 2.6, with $E = \mathbf{R}^n \setminus \Omega$, and taking into account (b) and assumption (v), we see that:

(a)′ $G$ has nonempty closed convex values;
(b)′ for all $x \in \mathbf{R}^n$, the multifunction $G(\cdot, x)$ is $\mathcal{L}([a,b])$-measurable;
(c)′ for all $t \in [a,b]$, the multifunction $G(t,\cdot)$ has closed graph;
(d)′ if $t \in [a,b]$, and the function $\psi(t,\cdot)|_\Omega = \phi(t,\cdot)$ is continuous at $x \in \Omega$, then one has

$$G(t,x) = \{\psi(t,x)\} = \{\phi(t,x)\}.$$

Moreover, observe that by the above construction and by assumption (v) we see that

$$G(t,x) \subseteq \overline{B}_n\big(0_{\mathbf{R}^n}, \beta(t)\big) \cap \overline{\operatorname{conv}}(Y) \quad \text{for all } (t,x) \in [a,b] \times \mathbf{R}^n. \tag{15}$$

Now we apply Theorem 1 of [16], choosing $T = [a,b]$, $X = Y = \mathbf{R}^n$, $s = p$, $q = j'$, $V = L^p(I, \mathbf{R}^n)$, $\Psi(u) = u$, $r = \|\beta\|_{L^p([a,b],\mathbf{R}^n)}$, $\varphi \equiv +\infty$, $F = G$ and

$$\Phi(u)(t) = \int_a^b g(t,z)u(z)\,dz.$$

To this aim, we can argue as in [19]. In particular, observe that:

(a) $\Phi(L^p(I,\mathbf{R}^n)) \subseteq C^0(I,\mathbf{R}^n)$. This follows easily from our assumptions (vi) and (vii) and the classical Lebesgue dominated convergence theorem.

(b) If $v \in L^p(I,\mathbf{R}^n)$ and $\{v^k\}$ is a sequence in $L^p(I,\mathbf{R}^n)$, weakly convergent to $v$ in $L^{j'}(I,\mathbf{R}^n)$, then the sequence $\{\Phi(v^k)\}$ converges to $\Phi(v)$ strongly in $L^1(I,\mathbf{R}^n)$. This follows by Theorem 2 at p.359 of [20], since $g$ is $j$th power summable in $[a,b] \times [a,b]$ (note that $g$ is measurable on $[a,b] \times [a,b]$ by the classical Scorza-Dragoni theorem; see [21] or also [9]).

(c) By (15), the function

$$\sigma : t \in [a,b] \to \sup_{x \in \mathbf{R}^n} \inf_{v \in G(t,x)} \|v\|_n$$

belongs to $L^p([a,b])$ and $\|\sigma\|_{L^p([a,b])} \leq \|\beta\|_{L^p([a,b],\mathbf{R}^n)}$ (as regards the measurability of $\sigma$, we refer to [16]).

Therefore, all the assumptions of Theorem 1 of [16] are satisfied. Consequently, there exist a function $\hat{u} \in L^p([a,b],\mathbf{R}^n)$ and a set $K_1 \in \mathcal{L}([a,b])$, with $m_1(K_1) = 0$, such that

$$\hat{u}(t) \in G\big(t, \Phi(\hat{u})(t)\big) = G\left(t, \int_a^b g(t,z)\hat{u}(z)\,dz\right) \quad \text{for all } t \in [a,b] \setminus K_1. \tag{16}$$

Fix $i \in \{1,\dots,n\}$, and let $\delta_i : [a,b] \to \mathbf{R}$ be the function

$$\delta_i(t) := P_i\big(\Phi(\hat{u})(t)\big) = \int_a^b g(t,z)\hat{u}_i(z)\,dz$$

(as before, $\hat{u}_i(t)$ denotes the $i$th component of $\hat{u}(t)$). By (15) and (16), since $Y \in \mathcal{A}_n$, the function $\hat{u}_i$ has constant sign in $[a,b] \setminus K_1$. Assume that

$$\hat{u}_i(t) > 0 \quad \text{for all } t \in [a,b] \setminus K_1 \tag{17}$$

(if, conversely, one has $\hat{u}_i(t) < 0$ for all $t \in [a,b] \setminus K_1$, the argument is analogous). By (17) and assumptions (vi) and (vii) we see that $\delta_i$ is strictly increasing. Moreover, by Lemma 2.2 at p.226 of [22], we get

$$\delta_i'(t) = \int_a^b \frac{\partial g}{\partial t}(t,z)\hat{u}_i(z)\,dz > 0 \quad \text{for all } t \in ]a,b[.$$

Consequently, by Theorem 2 of [17], the function

$$\delta_i^{-1} : \delta_i\big([a,b]\big) \to [a,b]$$

is absolutely continuous.

Now, put

$$S := \bigcup_{i=1}^{n} \big[\delta_i^{-1}\big((B_i \cup Q_i) \cap \delta_i\big([a,b]\big)\big)\big] \cup K_0 \cup K_1.$$

Since all functions $\delta_i^{-1}$ are absolutely continuous, by Theorem 18.25 of [18] the set $S$ has null Lebesgue measure. Fix any $t \in [a,b] \setminus S$. Since $t \notin K_1$, by (16) we get

$$\hat{u}(t) \in G\big(t, \Phi(\hat{u})(t)\big).$$

Moreover, by the definition of $S$, we easily get

$$\Phi(\hat{u})(t) \in \Omega \Big\backslash \bigcup_{i=1}^{n} P_i^{-1}(Q_i),$$

hence (taking into account (c) and that $t \notin K_0$) the function $\phi(t,\cdot)$ is continuous at $\Phi(\hat{u})(t)$. By (d)$'$, this implies that

$$G\big(t, \Phi(\hat{u})(t)\big) = \big\{\phi\big(t, \Phi(\hat{u})(t)\big)\big\}.$$

Thus, we get

$$\hat{u}(t) \in G\big(t, \Phi(\hat{u})(t)\big) = \big\{\phi\big(t, \Phi(\hat{u})(t)\big)\big\} \subseteq Q\big(t, \Phi(\hat{u})(t)\big),$$

hence

$$f\big(t, \Phi(\hat{u})(t), \hat{u}(t)\big) = f\left(t, \int_a^b g(t,z)\hat{u}(z)\,dz, \hat{u}(t)\right) = 0.$$

In particular, the above construction shows that

$$\int_a^b g(t,z)\hat{u}(z)\,dz = \Phi(\hat{u})(t) \in \Omega \subseteq \mathbf{R}^n \setminus F \quad \text{for all } t \in [a,b] \setminus S.$$

Finally, by (15) and (16) we immediately get

$$\big\|\hat{u}(t)\big\|_n \le \beta(t) \quad \text{for all } t \in [a,b] \setminus K_1.$$

This completes the proof. □

Arguing exactly as in the proof of Theorem 3.3, we obtain the following special case of Theorem 4.1.

**Theorem 4.2** *Let $[a,b]$ be a compact interval, and let $Y \in \mathcal{A}_n$ be a closed, connected and locally connected subset of $\mathbf{R}^n$. Let $\psi : [a,b] \times Y \to \mathbf{R}$, $f : [a,b] \times \mathbf{R}^n \to \mathbf{R}$ and $g : [a,b] \times [a,b] \to [0,+\infty[$ be three given functions, $D \subseteq Y$ a countable set, dense in $Y$. Assume that for all $y \in D$, the function $\psi(\cdot,y)$ is $\mathcal{L}([a,b])$-measurable, and for a.e. $t \in [a,b]$ the function $\psi(t,\cdot)$ is continuous in $Y$. Moreover, let $p \in ]1,+\infty]$, $\phi_0 \in L^j([a,b])$, with $j > 1$ and $j \geq p'$, $\phi_1 \in L^{p'}([a,b])$. Assume that there exists a set $F \in \mathcal{F}_n$ such that*:

(i) *for all $t \in [a,b]$, the function $f(t,\cdot)|_{\mathbf{R}^n \setminus F}$ is continuous*;

(ii) *for all $x \in \mathbf{R}^n \setminus F$, the function $g(\cdot,x)$ is $\mathcal{L}([a,b])$-measurable*;

(iii) *for a.e. $t \in [a,b]$, one has $f(t,\mathbf{R}^n \setminus F) \subseteq \mathrm{int}_{\mathbf{R}}(\psi(t,Y))$*;

(iv) *for a.e. $t \in [a,b]$ and for all $v \in \mathrm{int}_{\mathbf{R}}(\psi(t,Y))$, one has $\mathrm{int}_Y(\{y \in Y : \psi(t,y) = v\}) = \emptyset$*;

(v) *there exists a positive function $\beta \in L^p([a,b])$ such that, for a.e. $t \in [a,b]$, and for all $x \in \mathbf{R}^n \setminus F$, one has*

$$\bigl\{y \in Y : \psi(t,y) = f(t,x)\bigr\} \subseteq \overline{B}_n\bigl(0_{\mathbf{R}^n}, \beta(t)\bigr).$$

*Moreover, assume that assumptions* (vi) *and* (vii) *of Theorem* 4.1 *are satisfied.*

*Then there exists $u \in L^p([a,b],\mathbf{R}^n)$ such that*

$$\psi\bigl(t,u(t)\bigr) = f\left(t, \int_a^b g(t,z)u(z)\,dz\right) \quad \text{for a.e. } t \in [a,b], \tag{18}$$

$$\bigl\|u(t)\bigr\|_n \leq \beta(t) \quad \text{and} \quad \int_a^b g(t,z)u(z)\,dz \in \mathbf{R}^n \setminus F \quad \text{for a.e. } t \in [a,b].$$

**Remark** As already done in Section 3, it is not difficult to construct examples of application of Theorem 4.2, where for all $t \in [a,b]$ the function $f(t,\cdot)$ is discontinuous at all points $x \in \mathbf{R}^n$. We observe that (18) has been intensively studied in the last years. A common assumption in the literature (see, for instance, [23–26], to which we also refer for motivations for studying (18)) is the continuity of the function $f$ with respect to the second variable. Very recently, some existence results have been obtained for some special cases of (18), which do not assume the continuity of $f$ with respect to the second variable (see [14, 19, 27–30]). It is not difficult to check that some of them [19, 27, 28, 30] can be obtained as special cases of Theorem 4.2, which, in turn, improves them in several directions. In particular, with respect to the main results of [19, 27, 30] (dealing with the case $n = 1$ or $\psi$ not depending on $t$ explicitly), Theorem 4.2 does not assume that the discontinuity set $F \in \mathcal{F}_n$ is closed. Moreover, with respect to the main result of [27] (valid for the case $n = 1$), Theorem 4.2 does not assume that for all $t \in [a,b]$ the sets

$$\bigl\{y \in Y : y \text{ is a local minimum for } \psi(t,\cdot)\bigr\},$$

$$\bigl\{y \in Y : y \text{ is a local maximum for } \psi(t,\cdot)\bigr\}$$

are closed.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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**References**

1. Fitzpatrick, PM: Existence results for equations involving noncompact perturbations of Fredholm mappings with applications to differential equations. J. Math. Anal. Appl. **66**, 151-177 (1978)
2. Petryshyn, WV: Solvability of various boundary value problems for the equation $x'' = f(t,x,x',x'') - y$. Pac. J. Math. **122**, 169-195 (1986)
3. Tineo, A: Existence of solutions for a class of boundary value problems for the equation $x'' = F(t,x,x',x'')$. Comment. Math. Univ. Carol. **9**, 285-291 (1988)
4. Carl, S, Heikkilä, S: On discontinuous first order implicit boundary value problems. J. Differ. Equ. **148**, 100-121 (1998)
5. Frigon, M, Kaczynski, T: Boundary value problems for systems of implicit differential equations. J. Math. Anal. Appl. **179**, 317-326 (1993)
6. Marano, SA: On a boundary value problem for the differential equation $f(x,u,u',u'') = 0$. J. Math. Anal. Appl. **182**, 309-319 (1994)
7. Cubiotti, P, Yao, JC: Two-point problem for vector differential inclusions with discontinuous right-hand side. Appl. Anal. **93**, 1811-1823 (2014)
8. Klein, E, Thompson, AC: Theory of Correspondences. Wiley, New York (1984)
9. Himmelberg, CJ: Measurable relations. Fundam. Math. **87**, 53-72 (1975)
10. Bogachev, VI: Measure Theory. Springer, Berlin (2007)
11. Castaing, C, Valadier, M: Convex Analysis and Measurable Multifunctions. Springer, Berlin (1977)
12. Kucia, A: Scorza Dragoni type theorems. Fundam. Math. **138**, 197-203 (1991)
13. Anello, G, Cubiotti, P: Parametrization of Riemann-measurable selections for multifunctions of two variables with application to differential inclusions. Ann. Pol. Math. **83**, 179-187 (2004)
14. Cubiotti, P: Non-autonomous vector integral equations with discontinuous right-hand side. Comment. Math. Univ. Carol. **42**, 319-329 (2001)
15. Ricceri, B: Sur la semi-continuitè infèrieure de certaines multifonctions. C. R. Acad. Sci. Paris **294**, 265-267 (1982)
16. Naselli Ricceri, O, Ricceri, B: An existence theorem for inclusions of the type $\Psi(u)(t) \in F(t, \Phi(u)(t))$ and application to a multivalued boundary value problem. Appl. Anal. **38**, 259-270 (1990)
17. Villani, A: On Lusin's condition for the inverse function. Rend. Circ. Mat. Palermo **33**, 331-335 (1984)
18. Hewitt, E, Stromberg, K: Real and Abstract Analysis. Springer, Berlin (1965)
19. Anello, G, Cubiotti, P: Non-autonomous implicit integral equations with discontinuous right-hand side. Comment. Math. Univ. Carol. **45**, 417-429 (2004)
20. Kantorovic, LV, Akilov, GP: Functional Analysis in Normed Spaces. Pergamon, Oxford (1964)
21. Scorza Dragoni, G: Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile. Rend. Semin. Mat. Univ. Padova **17**, 102-106 (1948)
22. Lang, S: Real and Functional Analysis. Springer, New York (1993)
23. Banas, J, Knap, Z: Integrable solutions of a functional-integral equation. Rev. Mat. Univ. Complut. Madr. **2**, 31-38 (1989)
24. Emmanuele, G: About the existence of integrable solutions of a functional-integral equation. Rev. Mat. Univ. Complut. Madr. **4**, 65-69 (1991)
25. Emmanuele, G: Integrable solutions of a functional-integral equation. J. Integral Equ. Appl. **4**, 89-94 (1992)
26. Fečkan, M: Nonnegative solutions of nonlinear integral equations. Comment. Math. Univ. Carol. **36**, 615-627 (1995)
27. Anello, G, Cubiotti, P: A note on non-autonomous implicit integral equations with discontinuous right-hand side. J. Integral Equ. Appl. **19**, 391-403 (2007)
28. Cammaroto, F, Cubiotti, P: Implicit integral equations with discontinuous right-hand side. Comment. Math. Univ. Carol. **38**, 241-246 (1997)
29. Cammaroto, F, Cubiotti, P: Vector integral equations with discontinuous right-hand side. Comment. Math. Univ. Carol. **40**, 483-490 (1999)
30. Cubiotti, P, Yao, JC: Implicit vector integral equations associated with discontinuous operators. Abstr. Appl. Anal. **2014**, Article ID 301675 (2014)