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Existence and n-multiplicity of positive periodic solutions for impulsive functional differential equations with two parameters

Qiong Meng* and Jurang Yan

*Correspondence:
mengqiong@qq.com
School of Mathematical Science, Shanxi University, Taiyuan, 030006, China

Abstract

In this paper, we employ the well-known Krasnoselskii fixed point theorem to study the existence and n-multiplicity of positive periodic solutions for the periodic impulsive functional differential equations with two parameters. The form including an impulsive term of the equations in this paper is rather general and incorporates as special cases various problems which have been studied extensively in the literature. Easily verifiable sufficient criteria are obtained for the existence and n-multiplicity of positive periodic solutions of the impulsive functional differential equations.

MSC: 34K20

Keywords: functional differential equation; impulse effect; Krasnoselskii fixed point theorem; positive periodic solution

1 Introduction

In this paper, we consider the following impulsive functional differential equations with two parameters:

$$\begin{cases} y'(t) = \operatorname{sgn}(-1)^i[h(t,y(t)) - \lambda f(t,y(t-\tau(t)))], & t \in \mathbf{R}\backslash\{t_k\}, \\ y(t_k^+) - y(t_k) = \mu I_k(t_k, y(t_k - \sigma(t_k))), & k \in \mathbf{Z}. \end{cases} \tag{E_i}$$

Throughout this paper, we use $i = 1, 2$, $k \in \mathbf{Z}$, where $\mathbf{Z}$ denotes the set of all integers, $\mathbf{R} = (-\infty,\infty)$ and $\mathbf{R}_+ = [0,\infty)$.

For system (E_i) we introduce the hypotheses:

(H_1) $\lambda \geq 0$ and $\mu \geq 0$ with $\lambda + \mu > 0$ are parameters.

(H_2) $f : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}_+$ satisfies the Caratheodory condition, that is, $f(t,y)$ is locally Lebesgue measurable ω-periodic ($\omega > 0$) function in t for each fixed y and continuous in y for each fixed t, τ, and $\sigma : \mathbf{R} \to \mathbf{R}$ are locally bounded Lebesgue measurable ω-periodic functions.

(H_3) There exist ω-periodic functions a_1 and $a_2 : \mathbf{R} \to \mathbf{R}$ which are locally bounded Lebesgue measurable so that $a_1(t)y \leq h(t,y) \leq a_2(t)y$ for all $y > 0$ and $\lim_{y\to 0^+} \frac{h(t,y)}{y}$ exists, $\int_0^\omega a_1(t)\,dt > 0$.

(H_4) There exists a positive integer ρ such that $t_{k+\rho} = t_k + \omega$.

(H_5) $I_k : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}_+$ satisfies the Caratheodory condition and are ω-periodic functions in t. $\{t_k\}$, $k \in \mathbf{Z}$, is an increasing sequence of real numbers with $\lim_{k\to\pm\infty} t_k = \pm\infty$. Moreover, $I_{k+\rho}(t_{k+\rho}, y) = I_k(t_k, y)$ for all k.

The periodic system (E_i) include many periodic mathematical ecological models with or without impulse effects. This type of equations has been proposed as models for a variety of physiological precesses and conditions including production of blood cells, respiration, and cardiac arrhythmias; see [1–5]. The study of positive periodic solutions for impulsive functional differential equations has attracted considerable attention, and research results emerge continuously; see [6–16].

The purpose of this paper is to obtain some weaker conditions for the global existence of positive periodic solutions of (E_i) and the number $n \geq 1$ of periodic solutions. Following the technique in [6, 17] and by using the well-known Krasnoselskii fixed point theorem, we show that, for $\lambda + \mu > 0$, the number $n \geq 1$ of positive ω-periodic solutions of (E_i) can be determined by the behaviors of the quotient of $\frac{f(t,y)}{y}$ at any point $y \in (0, \infty)$ and $y \to 0^+$, $y \to \infty$, $t \in \mathbf{R}$. In particular, for $\lambda = 0$ and $\mu > 0$, the global existence of positive ω-periodic solutions of (E_i) is caused completely by impulse effects. These results are new and they generalize and improve those in [6–8, 17]. For $n = 1$, the results of this paper also improve those in [7–10].

The paper is organized as follows. In Section 2, we give some lemmas to prove the main results of this paper and several preliminaries are given. In Section 3, the existence theorems for the numbers 1, 2 and $n > 2$ of positive periodic solutions of (E_i) are proved by using the well-known fixed point theorem due to Krasnoselskii addressing the quotient of $\frac{f(t,y)}{y}$ and $\frac{I_k(t,y)}{y}$ at $y > 0$, $t \in \mathbf{R}$. An example is also given. In Section 4, we employ the results obtained in Section 3 to prove that the number 1 or 2 of positive periodic solutions of (E_i) can be determined by $\frac{f(t,y)}{y}$ and $\frac{I_k(t,y)}{y}$ when $y \to 0_+$ and $y \to \infty$, $t \in \mathbf{R}$.

2 Preliminaries

Throughout this paper, we will use the following notation:

$$\delta_i = e^{\int_0^\omega a_i(t)\,dt}, \quad i = 1, 2,$$

$$A_1 = \frac{1}{\delta_2 - 1}, \qquad B_1 = \frac{\delta_2}{\delta_1 - 1}, \qquad A_2 = \frac{\delta_2^{-1}}{1 - \delta_2^{-1}}, \qquad B_2 = \frac{1}{1 - \delta_1^{-1}}$$

and

$$\beta = \min\left\{\frac{A_1}{B_1}, \frac{A_2}{B_2}\right\}.$$

Let E be the Banach space defined by

$$\begin{aligned} E = \bigl\{&y(t) : \mathbf{R} \to \mathbf{R} \text{ is continuous in } (t_k, t_{k+1}), y\bigl(t_k^+\bigr), y\bigl(t_k^-\bigr) \text{ exists}, \\ &y\bigl(t_k^-\bigr) = y(t_k), k \in \mathbf{Z} \text{ and } y(t + \omega) = y(t), t \in \mathbf{R}\bigr\} \end{aligned}$$

with norm $\|y\| = \sup_{0\leq t\leq\omega} |y(t)|$. Define P to be a cone in E by

$$P = \bigl\{y \in E : y(t) \geq \beta\|y\|, t \in [0, \omega]\bigr\}.$$

For a positive constant r, we also define Ω_r by

$$\Omega_r = \{y \in P : \|y\| < r\} \quad \text{and} \quad \partial\Omega_r = \{y \in P : \|y\| = r\}.$$

Finally, we define two operators $T_i : P \to P$, $i = 1, 2$, as

$$\begin{aligned}(T_i y)(t) = {}& \lambda \int_t^{t+\omega} G_i(t,s) f\bigl(s, y\bigl(s - \tau(s)\bigr)\bigr)\, ds \\ &+ \mu \sum_{t \le t_k < t+\omega} G_i(t,t_k) I_k\bigl(t_k, y\bigl(t_k - \sigma(t_k)\bigr)\bigr),\end{aligned} \tag{2.1$_i$}$$

where

$$G_1(t,s) = \frac{e^{\int_t^s \frac{h(u,y(u))}{y(u)}\, du}}{e^{\int_0^\omega \frac{h(u,y(u))}{y(u)}\, du} - 1} \quad \text{and} \quad G_2(t,s) = \frac{e^{-\int_t^s \frac{h(u,y(u))}{y(u)}\, du}}{1 - e^{-\int_0^\omega \frac{h(u,y(u))}{y(u)}\, du}}. \tag{2.2}$$

Note that, from (H_3) and (2.2), we have

$$A_1 \le G_1(t,s) \le B_1 \quad \text{and} \quad A_2 \le G_2(t,s) \le B_2, \quad t,s \in \mathbf{R}.$$

A function $y : \mathbf{R} \to \mathbf{R}$ is said to be solution of (E_i) if the following conditions are satisfied:

(i) $y(t)$ is absolutely continuous on each (t_k, t_{k+1});
(ii) for each $k \in \mathbf{Z}$, $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k)$;
(iii) $y(t)$ satisfies the differential equation in (E_i) for almost everywhere on $\mathbf{R}$;
(iv) $y(t_k)$ satisfies the impulse condition in (E_i).

In the proofs of the main theorems of this paper we will use the following lemmas and theorem. The proofs of the lemmas are similar to those of the lemmas in [6]. We omit them.

Lemma 2.1 *Assume that* (H_1)-(H_5) *hold. Then $T : P \to P$ is well defined and is completely continuous.*

Lemma 2.2 *Assume that* (H_1)-(H_5) *hold. The existence of positive ω-periodic solutions of (E_i) is equivalent to that of non-zero fixed points of T in P.*

Theorem K [18, 19] *Let E be a Banach space and P be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let T be a completely continuous operator such that either*

(i) $\|Ty\| \ge \|y\|$, $y \in P \cap \partial\Omega_1$ *and* $\|Ty\| \le \|y\|$, $y \in P \cap \partial\Omega_2$

or

(ii) $\|Ty\| \le \|y\|$, $y \in P \cap \partial\Omega_1$ *and* $\|Ty\| \ge \|y\|$, $y \in P \cap \partial\Omega_2$,

then T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3 Existence of positive periodic solutions

In the sequel, we will use the following notations. For an ω-periodic integrable function $f(t) : \mathbf{R} \to \mathbf{R}_+$, let

$$\bar{f} = \frac{1}{\omega} \int_0^\omega f(t)\, dt, \qquad \hat{f} = \frac{1}{\omega} \sum_{0 \le t_k < \omega} f(t_k).$$

For a positive constant r and $y \in P$,

$$f_r^u(t) = \sup_{\|y\|=r, y\in P} \frac{f(t,y)}{y}, \qquad f_r^l(t) = \inf_{\|y\|=r, y\in P} \frac{f(t,y)}{y},$$
$$I_r^u(t) = \sup_{\substack{\|y\|=r, y\in P\\ k=1,2,\ldots,\rho}} \frac{I_k(t,y)}{y}, \qquad I_r^l(t) = \inf_{\substack{\|y\|=r, y\in P\\ k=1,2,\ldots,\rho}} \frac{I_k(t,y)}{y}.$$

Clearly from (H_2) and (H_5), f_r^u, f_r^l, I_r^u, and $I_r^l : \mathbf{R} \to \mathbf{R}_+$ are positive bounded Lebesgue measurable ω-periodic functions.

We are now in a position to state and prove our results of the existence of positive ω-periodic solution for (E_i).

Theorem 3.1 *Assume that* (H_1)-(H_5) *hold and there exist positive constants* r_1 *and* r_2 *with* $r_1 < r_2$ *such that*

$$B_i\omega\bigl(\lambda\bar{f}_{r_1}^u + \mu\hat{I}_{r_1}^u\bigr) \le 1 \tag{3.1}$$

and

$$A_i\omega\beta\bigl(\lambda\bar{f}_{r_2}^l + \mu\hat{I}_{r_2}^l\bigr) \ge 1, \tag{3.2}$$

then (E_i) *has a positive* ω*-periodic solution* $y(t)$ *with* $r_1 \le \|y\| \le r_2$.

Proof Consider the Banach space E defined in Section 2 and P in E. Define two open sets Ω_{r_1} and Ω_{r_2} with $r_1 < r_2$. If $y \in P \cap \partial\Omega_{r_1}$, then $\beta\|y\| \le y(t) \le \|y\| = r_1$. From $(2.1)_i$ and (2.2) we have

$$\begin{aligned}(T_iy)(t) &\le B_i\biggl[\lambda\int_t^{t+\omega}\frac{f(s,y(s-\tau(s)))}{y(s-\tau(s))}y\bigl(s-\tau(s)\bigr)\,ds \\ &\quad + \mu\sum_{t\le t_k<t+\omega}\frac{I_k(t_k,y(t_k-\tau(t_k)))}{y(t_k-\tau(t_k))}y\bigl(t_k-\sigma(t_k)\bigr)\biggr] \\ &\le B_i\omega\bigl[\lambda\bar{f}_{r_1}^u\|y\| + \mu\hat{I}_{r_1}^u\|y\|\bigr].\end{aligned}$$

Hence from (3.1), we obtain $\|T_iy\| \le \|y\|$.

On the other hand, if $y \in P \cap \partial\Omega_{r_2}$, then $\beta\|y\| \le y(t) \le \|y\| = r_2$. From $(2.1)_i$ and (2.2) we have

$$\begin{aligned}(T_iy)(t) &\ge A_i\biggl[\lambda\int_t^{t+\omega}\frac{f(s,y(s-\tau(s)))}{y(s-\tau(s))}y\bigl(s-\tau(s)\bigr)\,ds \\ &\quad + \mu\sum_{t\le t_k<t+\omega}\frac{I_k(t_k,y(t_k-\tau(t_k)))}{y(t_k-\tau(t_k))}y\bigl(t_k-\sigma(t_k)\bigr)\biggr] \\ &\ge A_i\omega\bigl[\lambda\bar{f}_{r_2}^l\beta\|y\| + \mu\hat{I}_{r_2}^l\beta\|y\|\bigr].\end{aligned}$$

In view of (3.2), we obtain $\|T_iy\| \ge (T_iy)(t) \ge \|y\|$.

By Theorem K, T_i has a positive fixed point $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$. It follows from Lemma 2.2 that (E_i) has a positive ω-periodic solution with $r_1 \leq \|y\| \leq r_2$. The proof of Theorem 3.1 is complete. □

When $\lambda = 0$ or $\mu = 0$, from Theorem 3.1, we obtain immediately the following result.

Corollary 3.1 *Assume that* (H_1)-(H_5) *hold and there exist positive constants* r_1 *and* r_2 *with* $r_1 < r_2$;

(i) *if* $\lambda = 0$ *and*

$$\frac{1}{A_i \omega \beta \hat{I}^l_{r_2}} \leq \mu \leq \frac{1}{B_i \omega \hat{I}^u_{r_1}},$$

then (E_i) *has a positive* ω*-periodic solution* $y(t)$ *with* $r_1 \leq \|y\| \leq r_2$;

(ii) *if* $\mu = 0$ *and*

$$\frac{1}{A_i \omega \beta \bar{f}^l_{r_2}} \leq \lambda \leq \frac{1}{B_i \omega \bar{f}^u_{r_1}},$$

then (E_i) *has a positive* ω*-periodic solution* $y(t)$ *with* $r_1 \leq \|y\| \leq r_2$.

Theorem 3.2 *Assume that* (H_1)-(H_5) *hold and there exist positive constants* r_1 *and* r_2 *with* $r_1 < r_2$ *such that*

$$B_i \omega \left(\lambda \bar{f}^u_{r_2} + \mu \hat{I}^u_{r_2}\right) \leq 1$$

and

$$A_i \omega \beta \left(\lambda \bar{f}^l_{r_1} + \mu \hat{I}^l_{r_1}\right) \geq 1,$$

then (E_i) *has a positive* ω*-periodic solution* $y(t)$ *with* $r_1 \leq \|y\| \leq r_2$.

Corollary 3.2 *Assume that* (H_1)-(H_5) *hold and there exist positive constants* r_1 *and* r_2 *with* $r_1 < r_2$;

(i) *if* $\lambda = 0$ *and*

$$\frac{1}{A_i \omega \beta \hat{I}^l_{r_1}} \leq \mu \leq \frac{1}{B_i \omega \hat{I}^u_{r_2}},$$

then (E_i) *has a positive* ω*-periodic solution* $y(t)$ *with* $r_1 \leq \|y\| \leq r_2$;

(ii) *if* $\mu = 0$ *and*

$$\frac{1}{A_i \omega \beta \bar{f}^l_{r_1}} \leq \lambda \leq \frac{1}{B_i \omega \bar{f}^u_{r_2}},$$

then (E_i) *has a positive* ω*-periodic solution* $y(t)$ *with* $r_1 \leq \|y\| \leq r_2$.

The proof of Theorem 3.2 will be omitted since it is similar to that of Theorem 3.1.

From Theorems 3.1 and 3.2, by using the same method, we can prove the following result.

Theorem 3.3 *Assume that* (H_1)-(H_5) *hold and there exist* $n+1$ *positive constants* r_m, $m = 1,2,\ldots,n+1$, *with* $r_1 < r_2 < \cdots < r_{n+1}$ *such that one of the following conditions is satisfied*:

$$\begin{aligned}
&B_i\omega\big(\lambda\bar{f}^u_{r_1} + \mu\hat{I}^u_{r_1}\big) \le 1,\\
&A_i\omega\beta\big(\lambda\bar{f}^l_{r_2} + \mu\hat{I}^l_{r_2}\big) \ge 1,\\
&B_i\omega\big(\lambda\bar{f}^u_{r_3} + \mu\hat{I}^u_{r_3}\big) \le 1,\\
&A_i\omega\beta\big(\lambda\bar{f}^l_{r_4} + \mu\hat{I}^l_{r_4}\big) \ge 1,\\
&\ldots,\\
&B_i\omega\big(\lambda\bar{f}^u_{r_{n+1}} + \mu\hat{I}^u_{r_{n+1}}\big) \le 1
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&A_i\omega\beta\big(\lambda\bar{f}^l_{r_1} + \mu\hat{I}^l_{r_1}\big) \ge 1,\\
&B_i\omega\beta\big(\lambda\bar{f}^u_{r_2} + \mu\hat{I}^u_{r_2}\big) \le 1,\\
&A_i\omega\beta\big(\lambda\bar{f}^l_{r_3} + \mu\hat{I}^l_{r_3}\big) \ge 1,\\
&B_i\omega\beta\big(\lambda\bar{f}^u_{r_4} + \mu\hat{I}^u_{r_4}\big) \le 1,\\
&\ldots,\\
&A_i\omega\beta\big(\lambda\bar{f}^l_{r_{n+1}} + \mu\hat{I}^l_{r_{n+1}}\big) \ge 1,
\end{aligned} \tag{3.4}$$

then (E_i) *has* n *positive* ω*-periodic solutions* $y_1, y_2, \ldots, y_n$ *with* $\|y_1\| \le \|y_2\| \le \cdots \le \|y_n\|$.

Remark 3.1 A simple example that satisfies conditions (3.3) or (3.4) is that functions f and I_k are ω-periodic functions in t and Ω-periodic ($\Omega > 0$) functions in y. Moreover, $r_{m+1} = r_m + \Omega$, $m = 1,2,\ldots,n$.

Example Consider the following impulse differential equation satisfying (H_1)-(H_5):

$$y'(t) = (-1)^i\big[h\big(t,y(t)\big) - f\big(t,y\big(t-\tau(t)\big)\big)\big], \qquad (3.5)'_i,\ i = 1,2$$

where

$$f(t,y) = \begin{cases} \xi p(t)y\sin y, & y \in [m\pi,(m+1)\pi),\\ p(t)y|\sin y|, & y \in [(m+1)\pi,(m+2)\pi), \end{cases}$$

$$I_k(t,y) = \begin{cases} \xi q(t)y\sin(y+2k\pi), & y \in [m\pi,(m+1)\pi),\\ q(t)y|\sin(y+2k\pi)|, & y \in [(m+1)\pi,(m+2)\pi), \end{cases}$$

where $m = 0,2,4,\ldots,n$, n is an even, and ξ is a constant. $p,q \in L(\mathbf{R},\mathbf{R}_+)$ are ω-periodic functions.

Now, we show that $(3.5)'_i$ has n positive ω-periodic functions.

Let

$$r_1 = \frac{\pi}{2}, \qquad r_2 = \pi, \qquad r_3 = \frac{3\pi}{2}, \qquad \ldots, \qquad r_n = \frac{n}{2}\pi, \qquad r_{n+1} = \frac{n+1}{2}\pi.$$

Thus, for $r > 0$ and $y \in P$,

$$f_{r_1}^l(t) = \inf_{\|y\|=\frac{\pi}{2}, y\in P}\bigl\{\xi p(t)\sin y\bigr\} \geq \xi\delta p(t),$$

where $\delta = \inf_{\|y\|=\frac{\pi}{2}, y\in P}\{\sin y\}$. Clearly $\delta > 0$,

$$f_{r_2}^u(t) = \sup_{\|y\|=\pi, y\in P}\bigl\{\bigl|p(t)\bigr|\sin y\bigr\} \leq p(t),$$

$$I_{r_1}^l(t) = \inf_{\substack{\|y\|=r, y\in P\\ k=1,2,\ldots,\rho}}\bigl\{\xi q(t)\sin(y+2k\pi)\bigr\} \geq \xi\delta q(t),$$

$$I_{r_2}^u(t) = \sup_{\substack{\|y\|=r, y\in P\\ k=1,2,\ldots,\rho}}\bigl\{q(t)\bigl|\sin(y+2k\pi)\bigr|\bigr\} \leq q(t).$$

First, we choose $\lambda + \mu > 0$ enough small to satisfy

$$B_i\omega\bigl(\lambda\bar{f}_{r_2}^u + \mu\hat{I}_{r_2}^u\bigr) \leq B_i\omega(\lambda\bar{p} + \mu\hat{q}) \leq 1,$$

then we choose ξ sufficiently large such that

$$A_i\omega\beta\bigl(\lambda\bar{f}_{r_1}^l + \mu\hat{I}_{r_1}^l\bigr) \geq A_i\omega\beta\xi\delta(\lambda\bar{p} + \mu\hat{q}) \geq 1.$$

Similarly, we can obtain

$$B_i\omega\bigl(\lambda\bar{f}_{r_4}^u + \mu\hat{I}_{r_4}^u\bigr) \leq B_i\omega(\lambda\bar{p} + \mu\hat{q}) \leq 1$$

and

$$A_i\omega\beta\bigl(\lambda\bar{f}_{r_3}^l + \mu\hat{I}_{r_3}^l\bigr) \geq A_i\omega\beta\xi\delta(\lambda\bar{p} + \mu\hat{q}) \geq 1,$$

$$\ldots,$$

$$A_i\omega\beta\bigl(\lambda\bar{f}_{r_{n+1}}^l + \mu\hat{I}_{r_{n+1}}^l\bigr) \geq A_i\omega\beta\xi\delta(\lambda\bar{p} + \mu\hat{q}) \geq 1.$$

Therefore condition (3.4) of Theorem 3.3 is satisfied. By Theorem 3.3, $(3.5)_i'$ has n positive ω-periodic solutions $y_1, y_2, \ldots, y_n$ with $r_1 \leq \|y_1\| \leq r_2 \leq \|y_2\| \leq r_3 \leq \cdots \leq r_n \leq \|y_n\| \leq r_{n+1}$. The proof is completed.

Remark 3.2 Theorems 3.1 and 3.2 generalize and improve, respectively, Theorems 3.1 and 3.2 in [17].

4 Applications of main results

Let $r \to 0$ or $r \to \infty$, $\bar{f}_0^u, \bar{f}_0^l, \hat{I}_0^u, \hat{I}_0^l, \bar{f}_\infty^u, \bar{f}_\infty^l, \hat{I}_\infty^u$, and $\hat{I}_\infty^l$ denote respectively the corresponding upper and lower limits of $\bar{f}_r^u, \bar{f}_r^l, \hat{I}_r^u$, and $\hat{f}_r^l$. In particular, let

$$\bar{f}_0 = \lim_{r\to 0}\bar{f}_r, \qquad \hat{I}_0 = \lim_{r\to 0}\hat{I}_r, \qquad \bar{f}_\infty = \lim_{r\to\infty}\bar{f}_r, \qquad \hat{I}_\infty = \lim_{r\to\infty}\hat{I}_r.$$

Theorem 4.1 *Assume that* (H_1)-(H_5) *hold and one of the following conditions is satisfied*:

$$\begin{aligned} &B_i\omega\big(\lambda\bar{f}_0^u + \mu\hat{I}_0^u\big) < 1, \\ &A_i\omega\beta\big(\lambda\bar{f}_\infty^l + \mu\hat{I}_\infty^l\big) > 1 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &B_i\omega\big(\lambda\bar{f}_\infty^u + \mu\hat{I}_\infty^u\big) < 1, \\ &A_i\omega\beta\big(\lambda\bar{f}_0^l + \mu\hat{I}_0^l\big) > 1, \end{aligned} \tag{4.2}$$

then (E_i) *has a positive* ω*-periodic solution.*

Proof From (4.1) we can choose positive constants r_1 and r_2 with $r_1 < r_2$ such that

$$B_i\omega\big(\lambda\bar{f}_{r_1}^u + \mu\hat{I}_{r_1}^u\big) \le 1$$

and

$$A_i\omega\beta\big(\lambda\bar{f}_{r_2}^l + \mu\hat{I}_{r_2}^l\big) \ge 1.$$

By Theorem 3.1, it follows that (E_i) has a positive ω-periodic solution.

Similarly, by Theorem 3.2, we can prove that if (4.2) holds, then (E_i) has a positive ω-periodic solution. The proof of Theorem 4.1 is complete. □

Theorem 4.2 *Assume that* (H_1)-(H_5) *hold and there exists* $r > 0$ *such that*

$$A_i\omega\beta\big(\lambda\bar{f}_r^l + \mu\hat{I}_r^l\big) \ge 1. \tag{4.3}$$

(i) *If* $\bar{f}_0 = \hat{I}_0 = 0$ *or* $\bar{f}_\infty = \hat{I}_\infty = 0$, *then* (E_i) *has a positive* ω*-periodic solution.*
(ii) *If* $\bar{f}_0 = \hat{I}_0 = \bar{f}_\infty = \hat{I}_\infty = 0$, *then* (E_i) *has two positive* ω*-periodic solutions.*

Proof From (2.1), for any $y \in P \cap \partial\Omega_r$ and λ, μ satisfying (4.3) we have

$$\begin{aligned} (T_iy)(t) &\ge A_i\left[\lambda\int_t^{t+\omega} f\big(s, y\big(s-\tau(s)\big)\big)\,ds + \mu\sum_{t\le t_k<t+\omega} I_k\big(t_k, y\big(t_k-\tau(t_k)\big)\big)\right] \\ &\ge A_i\left[\lambda\beta\|y\|\int_t^{t+\omega} f_r^l(s)\,ds + \mu\beta\|y\|\sum_{t\le t_k<t+\omega} I_k^l(t_k)\right] \\ &\ge A_i\omega\beta\|y\|\big(\lambda\bar{f}_r^l + \mu\hat{I}_r^l\big) \ge \|y\|. \end{aligned}$$

This yields

$$\|T_iy\| \ge \|y\| \quad \text{for } y \in P \cap \partial\Omega_r.$$

(i) If $\bar{f}_0 = \hat{I}_0 = 0$, then we choose r_1 with $0 < r_1 < r$ so that for λ and μ satisfying (4.3)

$$B_i\omega(\lambda\bar{f}_{r_1} + \mu\hat{I}_{r_1}) \le 1. \tag{4.4}$$

Thus from (2.1) we obtain

$$(T_i y)(t) \le B_i\omega(\lambda\bar{f}_{r_1} + \mu\hat{I}_{r_1})\|y\| \le \|y\|,$$

which implies that, for $y \in P \cap \partial\Omega_{r_1}$,

$$\|T_i y\| \le \|y\|.$$

It follows from Theorem 3.2 that (E_i) has a positive ω-periodic solution.

If $\bar{f}_\infty = \hat{I}_\infty = 0$, then we choose $r_2 > r$ so that for λ and μ satisfying (4.3)

$$B_i\omega(\lambda\bar{f}_{r_2} + \mu\hat{I}_{r_2}) \le 1. \tag{4.5}$$

Hence from (2.1) we have

$$(T_i y)(t) \le B_i\omega(\lambda\bar{f}_{r_2} + \mu\hat{I}_{r_2})\|y\| \le \|y\|,$$

which implies that, for $y \in P \cap \partial\Omega_{r_2}$,

$$\|T_i y\| \le \|y\|.$$

It follows from Theorem 3.2 that (E_i) has a positive ω-periodic solution.

(ii) If $\bar{f}_0 = \hat{I}_0 = \bar{f}_\infty = \hat{I}_\infty = 0$, then we can choose r_1 and r_2 with $r_1 < r < r_2$ so that (4.4) and (4.5) hold. Thus by Theorem 3.3 that (E_i) has respectively two positive ω-periodic solutions. The proof of Theorem 4.2 is complete. □

Similarly by using Theorems 3.2 and 3.3 we can obtain the following results.

Theorem 4.3 *Assume that* (H_1)-(H_5) *hold and there exists* $r > 0$ *such that*

$$B_i\omega\bigl(\lambda\bar{f}_r^u + \mu\hat{I}_r^l\bigr) \le 1.$$

(i) *If* $\bar{f}_0 = \hat{I}_0 = 0$ *or* $\bar{f}_\infty = \hat{I}_\infty = 0$, *then* (E_i) *has a positive* ω*-periodic solution.*
(ii) *If* $\bar{f}_0 = \hat{I}_0 = \bar{f}_\infty = \hat{I}_\infty = 0$, *then* (E_i) *has a positive* ω*-periodic solution.*

Remark 4.1 Theorems 4.1 and 4.2 generalize and improve, respectively, Corollaries 4.2 and 4.3 in [17], Theorems 3.1-3.3 in [6], and Theorem 1.3 in [17].

5 Discussion

In Sections 3 and 4 of this paper, by using the behaviors of the quotient of $\frac{f(t,y)}{y}$ at any point $y \in [0,\infty) + \{\infty\}$ we have proved the existence and n-multiplicity of positive ω-periodic solutions for impulsive functional differential equation (E_i) which are general enough to incorporate some periodic mathematical and ecological models. The method and technique are based on the application of the famous Krasnoselskii fixed point theorem on the cone of Banach space. In particular, when $\lambda = 0$ in (E_i), Corollaries 3.1 and 3.2 in Section 3, the existence of positive ω-periodic solutions is caused by impulsive effects. This is different from the corresponding continuous system. In fact, when $\lambda = 0$, (E_i) reduce to

$$\begin{cases} y'(t) = (-1)^i h(t, y(t)), \quad t \in \mathbf{R}\backslash\{t_k\}, i = 1, 2, \\ y(t_k^+) - y(t_k) = \mu I_k(t_k, y(t_k - \sigma(t_k))), \quad k \in \mathbf{Z}. \end{cases} \tag{$*$}$$

If the condition of Theorem 3.3 holds, then $(*)$ has n positive ω-periodic solutions by Theorem 3.3. But the equation $y'(t) = (-1)^i h(t, y)$ has no periodic solution.

The method of this paper on impulsive differential equations is not only restricted to scalar equations, but also it can be used for systems of impulsive functional equations and impulsive N-species competitive systems and impulsive neutral functional differential equations; see for example [11, 20].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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