Lu et al. *Boundary Value Problems* (2015) 2015:219
DOI 10.1186/s13661-015-0479-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Lower bounds for blow-up time of a nonlinear viscoelastic wave equation

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Abstract
This paper deals with the blow-up for a class of nonlinear viscoelastic wave equation. Under certain conditions on the data, we construct a lower bound for the blow-up time when blow-up occurs.

MSC: 60H15; 35L05; 35L70

Keywords: lower bounds; blow-up; viscoelastic wave equations

1 Introduction

In this paper, we study the blow-up solution for the following nonlinear viscoelastic wave equation:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\triangle u(s)\,ds + |u_t|^{q-2}u_t = |u|^{p-2}u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, g is a positive function satisfying some conditions to be specified later, and

$$2 < p, q \leq \begin{cases} \infty, & \text{if } n = 1, 2, \\ \frac{2n-2}{n-2}, & \text{if } n \geq 3. \end{cases} \tag{1.2}$$

The blow-up properties of the solution to (1.1) has been studied by many authors (see [1–5]). For instance, Messaoudi [2] studied (1.1) and proved a blow-up result for solutions with negative initial energy if $p > q \geq 2$ and a global result for $2 \leq p \leq q$. This result has been later improved by the same author in [3] to accommodate certain solutions with positive initial energy. In [4], Song and Zhong considered (1.1) for strong damping $-\Delta u_t$ and proved a blow-up result for solutions with positive initial energy by using the ideas of the 'potential well' theory introduced by Payne and Sattinger [6]. Wang [5] has investigated a sufficient conditions of the initial data with arbitrarily positive initial energy such that the corresponding solution of (1.1) with $q = 2$ blows up in finite time. For related results, we refer the reader to [7–10].

When blow-up occurs, the blow-up time T^* cannot usually be computed exactly. It is therefore of great importance in practice to determine lower and upper bounds for T^*.

The aim of this note is to derive a lower bound for T^* when blow-up occurs. We point out that it is, in general, very hard to obtain a lower bound estimate for viscoelastic wave equation problems, for the method to estimate the derivative of the control functional in parabolic cases is no longer effective and the memory part makes it difficult to estimate the energy. Our method is based on a first-order differential inequality technique for a suitably defined auxiliary function and makes use of some Sobolev-type inequality.

Before stating our main result, let us recall some results on the local existence, uniqueness, and blow-up of the solution

Theorem 1.1 (see [3]) *Let $(u_0(x), u_1(x)) \in H_0^1(\Omega) \times L^2(\Omega)$ and p, q satisfy condition* (1.2). *Let $g \in C^1[0,\infty)$ be a non-negative and non-increasing function satisfying*

$$1 - \int_0^\infty g(s)\,ds = l > 0. \tag{1.3}$$

Then problem (1.1) *has a unique local solution*

$$u \in C\bigl([0,T_m); H_0^1(\Omega)\bigr), \qquad u_t \in C\bigl([0,T_m); L^2(\Omega)\bigr) \cap L^q\bigl(\Omega \times (0,T_m)\bigr)$$

for some $T_m > 0$.

Remark 1.1 Condition (1.3) is necessary to guarantee the hyperbolicity and well-posedness of system (1.1).

Let λ be the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ and $\beta = \lambda / l^{\frac{1}{2}}$. We set

$$\alpha = \beta^{-\frac{p}{p-2}}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\alpha^2.$$

Define the energy functional $E(t)$ associated to our system (1.1),

$$E(t) = \frac{1}{2}\bigl\|u_t(t)\bigr\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\bigl\|\nabla u(t)\bigr\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p,$$

where

$$g \circ w(t) = \int_0^t g(t-s)\bigl\|w(s) - w(t)\bigr\|_2^2\,ds.$$

Moreover, we assume that g satisfies

$$\int_0^\infty g(s)\,ds < \frac{p^2 - 2p}{p^2 - 2p + 1}. \tag{1.4}$$

Then we have the following blow-up result.

Theorem 1.2 (see [3]) *Assume that p, q satisfy condition* (1.2) *and g satisfies* (1.3) *and* (1.4). *If $p > q$ and the initial data (u_0, u_1) satisfies*

$$E(0) > E_1, \qquad \|\nabla u_0\|_2 > \alpha,$$

then any solution of (1.1) *blows up in finite time.*

2 The main result

In this section, we switch to discuss the lower bound of the blow-up time for the blow-up solution of (1.1). Before we state and prove our main result, we need the following lemma.

Lemma 2.1 *Suppose that* (1.2), (1.3), *and* (1.4) *hold. Let u be a solution of* (1.1). *Then energy functional $E(t)$ is non-increasing, that is, $E'(t) \le 0$.*

Proof By multiplying (1.1) by u_t and integrating over Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p\right) - \int_0^t g(t-s)\int_\Omega \nabla u(s)\cdot\nabla u_t(t)\,dx\,ds \\ &\quad = -\int_0^t \big\|u_t(s)\big\|_q^q \end{aligned} \tag{2.1}$$

for any regular solution. This result remains valid for weak solutions by a simple density argument. For the last term on the left side of (2.1), we have

$$\begin{aligned} &\int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\nabla u(s)\,dx\,ds \\ &\quad = \int_0^t g(t-s)\int_\Omega \nabla u_t(t)\big(\nabla u(s)-\nabla u(t)\big)\,dx\,ds + \int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\nabla u(t)\,dx\,ds \\ &\quad = -\frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_\Omega \big|\nabla u(s)-\nabla u(t)\big|^2\,dx\,ds + \frac{1}{2}\int_0^t g(s)\frac{d}{dt}\int_\Omega \big|\nabla u(t)\big|^2\,dx\,ds \\ &\quad = \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)\,ds\big\|\nabla u(t)\big\|_2^2 - (g\circ\nabla u)(t)\right) + \frac{1}{2}\big(g'\circ\nabla u\big)(t) - \frac{1}{2}g(t)\big\|\nabla u(t)\big\|_2^2. \end{aligned} \tag{2.2}$$

Inserting (2.2) into (2.1), we get

$$E'(t) = -\int_0^t \big\|u_t(s)\big\|_q^q + \frac{1}{2}\big(g'\circ\nabla u\big)(t) - \frac{1}{2}g(t)\big\|\nabla u(t)\big\|_2^2 \le 0,$$

where we also use g being non-negative and non-increasing function. □

Theorem 2.2 *Assume that the conditions in Theorem* 1.2 *hold. Let $u(x,t)$ be the solution of problem* (1.1), *which blows up at a finite time T^*. Then*

$$T^* \ge \int_{F(0)}^\infty \frac{1}{C_3 y^k + y + C_4}\,dy,$$

where the constants C_3, C_4, and the exponent k will be defined in (2.9), *and $F(0) = \int_\Omega |u_0|^p\,dx$.*

Proof Define $F(t) = \int_\Omega |u(t)|^p\,dx$. Then

$$F'(t) = p\int_\Omega |u|^{p-2}uu_t\,dx \le \frac{p}{2}\left(\int_\Omega |u|^{2p-2}\,dx + \int_\Omega |u_t|^2\,dx\right). \tag{2.3}$$

To estimate the first term on the right side of inequality (2.3), we consider the following two cases.

Case 1. $2 < p \leq \frac{2n}{n-1}$. Let $\gamma = 2p-2$, $\mu = n(p-2)$, $2^* = \frac{2n}{n-2}$. Applying Hölder's inequality and the embedding inequality, we have

$$\int_\Omega |u|^\gamma\,dx = \int_\Omega |u|^{\gamma\theta}|u|^{\gamma(1-\theta)}\,dx \leq \left(\int_\Omega |u|^\mu\,dx\right)^{\frac{\gamma\theta}{\mu}}\left(\int_\Omega |u|^{2^*}\,dx\right)^{\frac{\gamma(1-\theta)}{2^*}},$$

where θ satisfies

$$\frac{\gamma\theta}{\mu} + \frac{\gamma(1-\theta)}{2^*} = 1.$$

A straightforward computation shows

$$\theta = \frac{1-\frac{\gamma}{2^*}}{\frac{\gamma}{\mu}-\frac{\gamma}{2^*}} = \frac{\mu(2^*-\gamma)}{\gamma(2^*-\mu)},$$
$$\frac{\gamma\theta}{\mu} = \frac{2^*-\gamma}{2^*-\mu} = \frac{2}{n}, \qquad \frac{\gamma-\theta\gamma}{2^*} = 1-\frac{2}{n},$$

and then we have

$$\begin{aligned}
\|u\|_\gamma^\gamma &\leq \|u\|_\mu^{\gamma\theta}\|u\|_{2^*}^{\gamma(1-\theta)} = \|u\|_\mu^{\frac{2\mu}{n}}\|u\|_{2^*}^2\\
&\leq C_*^2\bigl(1+|\Omega|^{\frac{2(p-\mu)}{np}}\bigr)\|u\|_p^{\frac{2\mu}{n}}\|\nabla u\|_2^2\\
&\leq C_*^2\bigl(1+|\Omega|^{\frac{2(p-\mu)}{np}}\bigr)\bigl(\|u\|_p^{\frac{2\mu}{n}\cdot s}+\|\nabla u\|_2^{2t}\bigr)\\
&\leq C_1\bigl(\|u\|_p^p+\|\nabla u\|_2^2\bigr)^{k_1},
\end{aligned} \tag{2.4}$$

where we have used the Hölder inequality,

$$\|u\|_\mu^{\frac{2\mu}{n}} \leq |\Omega|^{\frac{2(p-\mu)}{np}}\|u\|_p^{\frac{2\mu}{n}} \leq \bigl(1+|\Omega|^{\frac{2(p-\mu)}{np}}\bigr)\|u\|_p^{\frac{2\mu}{n}},$$

and

$$\|u\|_{2^*} \leq C_*\|\nabla u\|_2,$$

here C_* is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$; $\frac{1}{s}+\frac{1}{t}=1$, letting $t = \frac{2\mu}{pn}s$, from which we can deduce $k_1 = \frac{3p-4}{p}$, $C_1 = C_*^2(1+|\Omega|^{\frac{2(p-\mu)}{np}})$.

Case 2. $\frac{2n}{n-1} < p \leq \frac{2(n-1)}{n-2}$. Following the lines of the proof of inequality (2.4), we have

$$\begin{aligned}
\|u\|_\gamma^\gamma \leq |\Omega|^{1-\frac{\gamma}{2^*}}\|u\|_{2^*}^\gamma &\leq C_*^\gamma\bigl(1+|\Omega|^{1-\frac{\gamma}{2^*}}\bigr)\|\nabla u\|_2^\gamma\\
&\leq C_*^\gamma\bigl(1+|\Omega|^{1-\frac{\gamma}{2^*}}\bigr)\bigl(\|\nabla u\|_2^\gamma+\|u\|_p^{p(p-1)}\bigr)\\
&\leq C_2\bigl(\|u\|_p^p+\|\nabla u\|_2^2\bigr)^{k_2},
\end{aligned} \tag{2.5}$$

with $k_2 = p-1$, $C_2 = C_*^\gamma(1+|\Omega|^{1-\frac{\gamma}{2^*}})$.

From Lemma 2.1, we have

$$E(t) \leq E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p}\|u_0\|_p^p = E_0, \quad t \in [0, T^*). \tag{2.6}$$

Recalling the definition of $E(t)$, (1.4), and (2.6), we have

$$\begin{aligned}
&\|u_t(t)\|_2^2 + \frac{1}{(p-1)^2}\|\nabla u(t)\|_2^2 + g \circ u(t) \\
&\quad \le \|u_t(t)\|_2^2 + \left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + g \circ u(t) \\
&\quad = \frac{2}{p}\|u(t)\|_p^p + 2E(t) \le \frac{2}{p}F(t) + 2E_0.
\end{aligned} \tag{2.7}$$

Combining (2.3)-(2.7), we get

$$\begin{aligned}
F'(t) &\le \frac{p}{2}\left(C_i\left(\|u\|_p^p + \|\nabla u\|_2^2\right)^{k_i} + \frac{2}{p}F(t) + 2E_0\right) \\
&\le \frac{p}{2}\left(C_i\left(F(t) + C_0\left(\frac{2}{p}F(t) + 2E_0\right)\right)^{k_i} + \frac{2}{p}F(t) + 2E_0\right) \\
&\le \frac{p}{2}\left(C_i\left(\left(1 + \frac{2C_0}{p}\right)F(t) + 2C_0E_0\right)^{k_i} + \frac{2}{p}F(t) + 2E_0\right) \\
&\le \frac{pC_i}{2}2^{k_i-1}\left(\left(1 + \frac{2C_0}{p}\right)^{k_i}F(t)^{k_i} + (2C_0E_0)^{k_i}\right) + F(t) + pE_0 \\
&= C_3F(t)^{k_i} + F(t) + C_4,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
C_0 &= \frac{1}{1 - \int_0^\infty g(s)\,ds}, \\
C_3 &= \frac{pC_i2^{k_i}}{4}\left(1 + \frac{2C_0}{p}\right)^{k_i}, \\
C_4 &= pE_0 + \frac{pC_i}{4}(4C_0E_0)^{k_i}, \quad i = 1,2.
\end{aligned} \tag{2.9}$$

Applying Theorem 1.2, we have

$$\lim_{t\to T^*}\int_\Omega |u|^p\,dx = +\infty. \tag{2.10}$$

According to (2.8) and (2.10), we obtain

$$\int_{F(0)}^\infty \frac{1}{C_3y^k + y + C_4}\,dy \le T^*. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of the three authors, who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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Acknowledgements
The authors are indebted to the referee for giving some important suggestions which improved the presentations of this paper. This work is supported in part by China NSF Grant No. 11501442, the China Postdoctoral Science Foundation Grant No. 2013M540767, the Shanxi Provincial Postdoctoral Science Foundation, the scientific research program funded by Shanxi Provincial education department No. 14JK1474, and the doctor scientific research start fund project of Xi An University of Science and Technology Grant No. 2014QDJ042.

Received: 17 July 2015 Accepted: 10 November 2015 Published online: 01 December 2015

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