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Universal bounds and blow-up estimates for a reaction-diffusion system

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Abstract

This paper is concerned with nonnegative solutions of the reaction-diffusion system:

$$u_t - \Delta u = v^p + \mu_1 u^r, \qquad v_t - \Delta v = u^q + \mu_2 v^s.$$

In a suitable range of parameters, we prove (initial and final) blow-up rates, as well as universal bounds for global solutions. This is done in connection with new Liouville-type theorems in a half-space, that we establish.

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1 Introduction

In this paper, we study (initial and final) blow-up rates, as well as universal bounds for global solutions, for a class of semilinear reaction-diffusion systems, in connection with Liouville-type theorems. Our study is motivated by [1], where Poláčik *et al.* developed a general method for obtaining universal initial and final blow-up rates for the scalar equation $u_t - \Delta u = u^p$ ($p > 1$), based on rescaling arguments and Liouville-type theorems, combined with a key doubling property. In this context, the Liouville-type theorem means the nonexistence of nontrivial, nonnegative and bounded solutions defined for all negative and positive times on the whole space $\mathbb{R}^n$, or on a half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_1 > 0\}$.

We here consider the system:

$$\begin{cases} u_t - \Delta u = v^p + \mu_1 u^r, \\ v_t - \Delta v = u^q + \mu_2 v^s, \end{cases} \tag{1}$$

where $p, q, r, s > 1$ and $\mu_1, \mu_2 \geq 0$. We use the following notation for the scaling exponents:

$$\alpha = \frac{p+1}{pq-1}, \qquad \beta = \frac{q+1}{pq-1}. \tag{2}$$

Let us recall that, even in the scalar case, the optimal exponent for the Liouville-type property is not presently known (see the monograph [2] and the recent work [3]). In the case of systems, as far as we know, the only nonexistence result in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_1 > 0\}$ is a consequence of the Fujita-type theorem in [4], Theorem 2, p.178. The latter asserts the stronger property of nonexistence in $\mathbb{R}_+ \times \mathbb{R}^n$ (instead of $\mathbb{R} \times \mathbb{R}^n$) for the Cauchy problem

$$\begin{cases} u_t - \triangle u = v^p, & t > 0, x \in \mathbb{R}^n, \\ v_t - \triangle v = u^q, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0, \quad v(0,x) = v_0, & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

where u_0, v_0 are nonnegative, continuous, and bounded functions, under the assumption $\max(\alpha, \beta) \geq n/2$ with α, β are given by (2). As a consequence, Cui [5] proved a Fujita-type theorem for the Cauchy problem associated with the system (1) for $\mu_1 = \mu_2 = 1$, assuming that at least one of the following five conditions (i)-(v) is satisfied:

$$\text{(i)} \quad r \leq 1 + 2/n, \qquad \text{(ii)} \quad s \leq 1 + 2/n, \qquad \text{(iii)} \quad \max(\alpha, \beta) \geq n/2, \tag{4}$$

$$\text{(iv)} \quad \max(\alpha, \beta) < n/2, \qquad p \leq 1 + 2/n \quad \text{and} \quad r < np/(np - 2), \tag{5}$$

$$\text{(v)} \quad \max(\alpha, \beta) < n/2, \qquad q \leq 1 + 2/n \quad \text{and} \quad s < nq/(nq - 2). \tag{6}$$

(Note that $\max(\alpha, \beta) < n/2$ implies $p, q > 2/n$.) See [5], Theorem 1.3, p.354. The conditions (i)-(v) are optimal; see [6]. It will turn out that problem (1) with $\mu_1 = \mu_2 = 1$ reveals a number of interesting, qualitatively new phenomena, in comparison with the unperturbed model system (3), such as the existence of non-simultaneous blowing-up solutions. See [7, 8].

There are also Fujita-type results for half-spaces (see [9]). However, their optimal exponents are always smaller than the corresponding exponents in the whole space, leading to more stringent conditions in the applications (to blow-up estimates). On the contrary, for Liouville-type results with scalar equations, it was shown in [1] that nonexistence in a half-space can be derived as a consequence of nonexistence in the whole space, *without requiring stronger restrictions on the exponents* (and actually the restriction becomes even weaker). This was done by adapting a monotonicity argument, based on moving planes, introduced by Dancer [10] for elliptic equations. One of our main concerns here is to extend the result from [1] to system (1). Namely, we establish the following theorem.

Theorem 1.1 *Let $p, q, r, s > 1$, $p \leq q$, and $\mu_1, \mu_2 \geq 0$ be such that at least one of the following five conditions* (a)-(e) *is satisfied*:

$$\text{(a)} \quad \beta \geq \frac{n-1}{2},$$

$$\text{(b)} \quad \mu_1 > 0 \quad \textit{and} \quad r \leq 1 + \frac{2}{n-1},$$

$$\text{(c)} \quad \mu_2 > 0 \quad \textit{and} \quad s \leq 1 + \frac{2}{n-1},$$

$$\text{(d)} \quad \beta < \frac{n-1}{2}, \qquad \mu_1 > 0, \qquad p \leq 1 + \frac{2}{n-1} \quad \textit{and} \quad r < \frac{(n-1)p}{(n-1)p - 2},$$

$$\text{(e)} \quad \beta < \frac{n-1}{2}, \qquad \mu_2 > 0, \qquad q \leq 1 + \frac{2}{n-1} \quad \textit{and} \quad s < \frac{(n-1)q}{(n-1)q - 2},$$

where β is given by (2). (*Note that $\beta < (n-1)/2$ implies $p,q > 2/(n-1)$.*) *Then the system*

$$\begin{cases} u_t - \triangle u = v^p + \mu_1 u^r, \quad t \in \mathbb{R}, x \in \mathbb{R}^n_+, \\ v_t - \triangle v = u^q + \mu_2 v^s, \quad t \in \mathbb{R}, x \in \mathbb{R}^n_+, \\ u = v = 0, \quad t \in \mathbb{R}, x \in \partial\mathbb{R}^n_+, \end{cases} \tag{7}$$

has no nontrivial, nonnegative, bounded, and classical solution.

By applying the method of [1], as an application of Theorem 1.1 and of the result of [5], we then obtain the following universal initial and final blow-up rates for system (1), as well as universal bound for global solutions.

Theorem 1.2 *Let Ω be a* (*uniformly C^2*) *smooth domain of $\mathbb{R}^n$. Let $p,q,r,s > 1$, $p \leq q$, be such that*

$$r \leq \frac{p(q+1)}{p+1} \quad \text{if } \mu_1 > 0, \qquad s \leq \frac{q(p+1)}{q+1} \quad \text{if } \mu_2 > 0,$$

and one of the following five conditions is satisfied:

$$\begin{aligned} &\beta \geq \frac{n}{2}, \\ &\mu_1 > 0, \qquad r = \frac{p(q+1)}{p+1} \quad \text{and} \quad r \leq 1 + \frac{2}{n}, \\ &\mu_2 > 0, \qquad s = \frac{q(p+1)}{q+1} \quad \text{and} \quad s \leq 1 + \frac{2}{n}, \\ &\mu_1 > 0, \qquad r = \frac{p(q+1)}{p+1}, \qquad \beta < \frac{n}{2}, \qquad p \leq 1 + \frac{2}{n} \quad \text{and} \quad r < \frac{np}{np-2}, \\ &\mu_2 > 0, \qquad s = \frac{q(p+1)}{q+1}, \qquad \beta < \frac{n}{2}, \qquad q \leq 1 + \frac{2}{n} \quad \text{and} \quad s < \frac{nq}{nq-2}, \end{aligned}$$

where β is given by (2).

Then there exists a constant $C = C(p,q,r,s,\mu_1,\mu_2,\Omega) > 0$ such that, for any $T \in (0,\infty]$ and any nonnegative, classical solution (u,v) of

$$\begin{cases} u_t - \triangle u = v^p + \mu_1 u^r, \quad 0 < t < T, x \in \Omega, \\ v_t - \triangle v = u^q + \mu_2 v^s, \quad 0 < t < T, x \in \Omega, \\ u = v = 0, \quad 0 < t < T, x \in \partial\Omega, \end{cases} \tag{8}$$

we have

$$u(t,x) \leq C\bigl(1 + t^{-\alpha} + (T-t)^{-\alpha}\bigr), \quad 0 < t < T, x \in \Omega \tag{9}$$

and

$$v(t,x) \leq C\bigl(1 + t^{-\beta} + (T-t)^{-\beta}\bigr), \quad 0 < t < T, x \in \Omega. \tag{10}$$

(*Here $T - t := \infty$ in the case $T = \infty$.*)

Remark 1.1 Theorem 1.2 improves on previously known results (see [11–14]) in three directions:

(i) the constant C is independent of (u,v);

(ii) Ω can be an any (smooth) domain;

(iii) no monotonicity conditions are assumed either in space or in time.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1. Next, we prove Theorem 1.2, in Section 3.

2 Liouville-type theorem: proof of Theorem 1.1

In this section, we are concerned with the proof of Theorem 1.1. It is a consequence of [4], Theorem 2, p.178, [5], Theorem 1.3, p.354 and the following theorem.

Theorem 2.1 *Let* $p,q,r,s>1$, $\mu_1,\mu_2\geq 0$. *Then we have the following statements*:

(a) *The components of each positive, bounded, and classical solution* (u,v) *of* (7) *are increasing in* x_1:

$$\partial_{x_1}u(t,x)>0 \quad \text{and} \quad \partial_{x_1}v(t,x)>0, \quad t\in\mathbb{R}, x\in\mathbb{R}^n_+.$$

(b) *If there exists a positive, bounded, and classical solution of* (7), *then there exists a positive, bounded, and classical solution of*

$$\begin{cases} u_t-\triangle u=v^p+\mu_1u^r, & t\in\mathbb{R}, x\in\mathbb{R}^{n-1}, \\ v_t-\triangle v=u^q+\mu_2v^s, & t\in\mathbb{R}, x\in\mathbb{R}^{n-1}. \end{cases} \tag{11}$$

Proof *Part* (a). We put $f(u,v)=v^p+\mu_1u^r$ and $g(u,v)=u^q+\mu_2v^s$. For $\lambda>0$, let

$$\mathbb{T}_\lambda=\{x\in\mathbb{R}^n; 0<x_1<\lambda\}.$$

As in [1], for a function h defined on $\mathbb{R}^n_+$, let h^λ and $V_\lambda h$ be the functions defined on $\mathbb{T}_\lambda$ by

$$h^\lambda(x)=h(2\lambda-x_1,x'), \quad 0<x_1<\lambda, x'=(x_2,\dots,x_n)\in\mathbb{R}^{n-1},$$
$$V_\lambda h(x)=h^\lambda(x)-h(x), \quad x=(x_1,x')\in\mathbb{T}_\lambda.$$

Let (u,v) be a positive, bounded, and classical solution of (7). For each λ, $(V_\lambda u, V_\lambda v)$ satisfies the following system:

$$\begin{cases} V_\lambda u_t-\triangle V_\lambda u=C_1^\lambda(t,x)V_\lambda v+C_2^\lambda(t,x)V_\lambda u, & t\in\mathbb{R}, x\in\mathbb{T}_\lambda, \\ V_\lambda v_t-\triangle V_\lambda v=C_3^\lambda(t,x)V_\lambda u+C_4^\lambda(t,x)V_\lambda v, & t\in\mathbb{R}, x\in\mathbb{T}_\lambda, \\ V_\lambda u=V_\lambda v=0, & t\in\mathbb{R}, x_1=\lambda, x'\in\mathbb{R}^{n-1}, \\ V_\lambda u, V_\lambda v>0, & t\in\mathbb{R}, x_1=0, x'\in\mathbb{R}^{n-1}, \end{cases} \tag{12}$$

where

$$C_1^\lambda(t,x)=\int_0^1 f_v\bigl(0,v(t,x)+s\bigl(v^\lambda(t,x)-v(t,x)\bigr)\bigr)\,ds,$$

$$C_2^\lambda(t,x) = \int_0^1 f_u\bigl(u(t,x) + s\bigl(u^\lambda(t,x) - u(t,x)\bigr), 0\bigr)\,ds,$$

$$C_3^\lambda(t,x) = \int_0^1 g_u\bigl(u(t,x) + s\bigl(u^\lambda(t,x) - u(t,x)\bigr), 0\bigr)\,ds,$$

$$C_4^\lambda(t,x) = \int_0^1 g_v\bigl(0, v(t,x) + s\bigl(v^\lambda(t,x) - v(t,x)\bigr)\bigr)\,ds,$$

$$f_u = \mu_1 r u^{r-1}, \qquad f_v = p v^{p-1}, \qquad g_u = q u^{q-1}, \qquad g_v = \mu_2 s v^{s-1}.$$

We claim that

$$V_\lambda u \geq 0 \quad \text{and} \quad V_\lambda v \geq 0 \quad \text{in } \mathbb{R} \times \mathbb{T}_\lambda \text{ for each } \lambda > 0. \tag{13}$$

With (13) at hand, by the maximum principle [2], Proposition 52.7, p.511 and (12), we obtain

$$V_\lambda u > 0 \quad \text{and} \quad V_\lambda v > 0 \quad \text{in } \mathbb{R} \times \mathbb{T}_\lambda \text{ for each } \lambda > 0.$$

Moreover, since $V_\lambda u(t,\lambda,x') = V_\lambda v(t,\lambda,x') = 0$, from the Hopf maximum principle, it follows that

$$\frac{\partial V_\lambda u(t,\lambda,x')}{\partial x_1} < 0 \quad \text{and} \quad \frac{\partial V_\lambda v(t,\lambda,x')}{\partial x_1} < 0.$$

Therefore, since

$$\frac{\partial}{\partial x_1} V_\lambda u\bigl(t,\lambda,x'\bigr) = -\frac{\partial u(t,2\lambda-\lambda,x')}{\partial x_1} - \frac{\partial u(t,\lambda,x')}{\partial x_1} = -2\frac{\partial u(t,\lambda,x')}{\partial x_1},$$

we obtain

$$\frac{\partial u(t,\lambda,x')}{\partial x_1} > 0 \quad \text{for each } \lambda > 0.$$

Similarly, we prove that $\frac{\partial v(t,\lambda,x')}{\partial x_1} > 0$, for each $\lambda > 0$. To complete the proof of Part (a), it is therefore sufficient to prove the claim (13). We recall first the following lemma of Dancer [10].

Lemma 2.1 *Given any positive constants l, λ satisfying $\lambda^2 l < \pi^2$, there exists a smooth function h on $\overline{\mathbb{T}}_\lambda$ such that*

$$\begin{cases} \triangle h + lh = 0, & x \in \mathbb{T}_\lambda, \\ h(x) > 0, & x \in \overline{\mathbb{T}}_\lambda, \\ h(x) \to \infty, & |x| \to \infty, \quad x \in \overline{\mathbb{T}}_\lambda. \end{cases} \tag{14}$$

We split the rest of the proof in two steps.

Step 1. *Proof of* (13) *for small* λ.

Let h be given by Lemma 2.1. Since h is a positive and smooth function on $\overline{\mathbb{T}}_\lambda$ such that $h(x) \to \infty$, as $|x| \to \infty$, there exists $\varepsilon > 0$ such that $h \geq \varepsilon$. Fix a positive constant γ and set

$$l := \sup_{t\in\mathbb{R}, x\in\mathbb{R}^n_+} \bigl(f_u\bigl(u(t,x),0\bigr) + f_v\bigl(0,v(t,x)\bigr) + g_u\bigl(u(t,x),0\bigr) + g_v\bigl(0,v(t,x)\bigr)\bigr) + \gamma, \tag{15}$$

which is finite, by the boundedness of (u,v). Define the function

$$(\overline{V}_\lambda u, \overline{V}_\lambda v) := \bigl(e^{\gamma t} V_\lambda u/h, e^{\gamma t} V_\lambda v/h\bigr),$$

where h is given by Lemma 2.1 for $\lambda > 0$ sufficiently small (so that $\lambda^2 l < \pi^2$). With (12) at hand, a simple computation shows that

$$\begin{cases} \overline{V}_\lambda u_t - \triangle \overline{V}_\lambda u - \frac{2\nabla h}{h} \cdot \nabla \overline{V}_\lambda u - (\gamma + C_2^\lambda(t,x) - l)\overline{V}_\lambda u - C_1^\lambda(t,x)\overline{V}_\lambda v \\ \quad = 0, \quad t \in \mathbb{R}, x \in \mathbb{T}_\lambda, \\ \overline{V}_\lambda v_t - \triangle \overline{V}_\lambda v - \frac{2\nabla h}{h} \cdot \nabla \overline{V}_\lambda v - (\gamma + C_4^\lambda(t,x) - l)\overline{V}_\lambda v - C_3^\lambda(t,x)\overline{V}_\lambda u \\ \quad = 0, \quad t \in \mathbb{R}, x \in \mathbb{T}_\lambda, \\ \overline{V}_\lambda u, \overline{V}_\lambda v \geq 0, \quad t \in \mathbb{R}, x \in \partial\mathbb{T}_\lambda, \\ \overline{V}_\lambda u(t,x), \overline{V}_\lambda v(t,x) \to 0, \quad t \in \mathbb{R}, x \in \overline{\mathbb{T}}_\lambda, |x| \to \infty. \end{cases} \tag{16}$$

Moreover, (15) implies that

$$\gamma + C_1^\lambda(t,x) + C_2^\lambda(t,x) - l \leq 0, \qquad \gamma + C_3^\lambda(t,x) + C_4^\lambda(t,x) - l \leq 0. \tag{17}$$

For $M > 0$ to be fixed later, we put

$$W := -\overline{V}_\lambda u - M, \qquad Z := -\overline{V}_\lambda v - M.$$

Using (17), we have

$$\begin{aligned} W_t - \triangle W &\leq C|\nabla W| + \bigl(\gamma + C_2^\lambda(t,x) - l\bigr)(W + M) + C_1^\lambda(t,x)(Z + M) \\ &\leq C|\nabla W| + \bigl(\gamma + C_2^\lambda(t,x) - l\bigr) W + C_1^\lambda(t,x) Z \end{aligned} \tag{18}$$

and

$$Z_t - \triangle Z \leq C|\nabla Z| + \bigl(\gamma + C_4^\lambda(t,x) - l\bigr) Z + C_3^\lambda(t,x) W. \tag{19}$$

By the last two properties in (16), we have $W_+(t,\cdot) := \max(0,W)(t,\cdot), Z_+(t,\cdot) := \max(0,Z)(t,\cdot) \in H_0^1(\mathbb{T}_\lambda)$ for all $t \in \mathbb{R}$. Multiplying (18) with W_+, integrating by parts and using $C_1^\lambda \geq 0$, it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}_\lambda} W_+^2 &\leq -\int_{\mathbb{T}_\lambda} |\nabla W_+|^2 + C\int_{\mathbb{T}_\lambda} |\nabla W_+| W_+ \\ &\quad + \int_{\mathbb{T}_\lambda} \bigl(\gamma + C_2^\lambda(t,x) - l\bigr) W_+^2 + \int_{\mathbb{T}_\lambda} C_1^\lambda(t,x) Z_+ W_+ \leq K \int_{\mathbb{T}_\lambda} \bigl(W_+^2 + Z_+^2\bigr) \end{aligned}$$

for some constants $C, K > 0$. Arguing similarly on Z_+ and adding up, we obtain

$$\frac{d}{dt}\int_{\mathbb{T}_\lambda} \bigl(W_+^2 + Z_+^2\bigr) \leq 2K \int_{\mathbb{T}_\lambda} \bigl(W_+^2 + Z_+^2\bigr). \tag{20}$$

We now set $A := \varepsilon^{-1}\max(\|u\|_\infty, \|v\|_\infty)$, where $\varepsilon = \inf_{\mathbb{T}_\lambda} h > 0$, and, for any given $t_0 \in \mathbb{R}$, we choose

$$M = Ae^{\gamma t_0} \geq \max\Bigl(\sup_{x\in\mathbb{T}_\lambda} |\overline{V}_\lambda u|(t_0,x), \sup_{x\in\mathbb{T}_\lambda} |\overline{V}_\lambda v|(t_0,x)\Bigr).$$

Then $W_+(t_0,\cdot) \equiv Z_+(t_0,\cdot) \equiv 0$ and it follows from (20) that $W, Z \le 0$ in $(t_0,\infty) \times \mathbb{T}_\lambda$. Consequently, for all $t_0, t \in \mathbb{R}$ with $t_0 < t$, we have

$$\sup_{x\in\mathbb{T}_\lambda} \frac{-V_\lambda u(t,x)}{h(x)} \le Me^{-\gamma t} = Ae^{-\gamma(t-t_0)}.$$

Letting $t_0 \to -\infty$, we obtain $V_\lambda u \ge 0$ everywhere, and similarly $V_\lambda v \ge 0$. We conclude that (13) holds for λ small.

Step 2. *Proof of* (13) *for large* λ (*hence for all* λ).

Let

$$\lambda_0 = \sup\{\mu > 0 \mid (13) \text{ holds for all } \lambda \in (0,\mu)\}. \tag{21}$$

By Step 1, it follows that $\lambda_0 > 0$. We assume by contradiction that $\lambda_0 < \infty$. Then there exists a sequence $\lambda_k \ge \lambda_0$ such that $\lambda_k \to \lambda_0$ and the set

$$F_k := \{(t,x) \in \mathbb{R} \times \mathbb{T}_{\lambda_k} \mid \min(V_{\lambda_k}u(t,x), V_{\lambda_k}v(t,x)) < 0\}$$

is nonempty. Set

$$\begin{aligned} m_k := \sup\{&\max(u(t,y_1,x'), v(t,y_1,x')) \mid t \in \mathbb{R}, y_1 \in (0,2\lambda_k), x' \in \mathbb{R}^{n-1}, \\ &\exists x_1 \in (0,\lambda_k)/(t,x_1,x') \in F_k\}. \end{aligned}$$

We may assume that either:

(i) $m_k \ge \varepsilon_0$ for some $\varepsilon_0 > 0$ (by passing to a subsequence if necessary);

(ii) $m_k \to 0$.

If case (i) holds, there exist some sequences $t^k \in \mathbb{R}$, $x_1^k \in (0,\lambda_k)$, $y_1^k \in (0,2\lambda_k)$, and $\xi^k \in \mathbb{R}^{n-1}$ such that

$$\min(V_{\lambda_k}u(t^k,x_1^k,\xi^k), V_{\lambda_k}v(t^k,x_1^k,\xi^k)) < 0 \quad \text{and} \quad \max(u(t^k,y_1^k,\xi^k), v(t^k,y_1^k,\xi^k)) \ge \varepsilon_0.$$

By passing to a subsequence, we may assume that $x_1^k \to a$ and $y_1^k \to b$ for some $a, b \in [0,\lambda_0]$. Next, we consider the functions

$$u_k(t,x) := u(t+t^k, x_1, x'+\xi^k), \qquad v_k(t,x) := v(t+t^k, x_1, x'+\xi^k), \quad t \in \mathbb{R}, (x_1,x') \in \mathbb{R}^n.$$

For all k, (u_k, v_k) is a positive, bounded, and classical solution of (7) satisfying

$$\min(V_{\lambda_k}u_k(0,x_1^k,0), V_{\lambda_k}v_k(0,x_1^k,0)) = \min(V_{\lambda_k}u(t^k,x_1^k,\xi^k), V_{\lambda_k}v(t^k,x_1^k,\xi^k)) < 0$$

and

$$\max(u_k(0,y_1^k,0), v_k(0,y_1^k,0)) \ge \varepsilon_0.$$

Moreover, by definition of λ_0 it follows that $V_{\lambda_0}u_k, V_{\lambda_0}v_k \ge 0$ in $\mathbb{R} \times \mathbb{T}_{\lambda_0}$. Since u_k and v_k are uniformly bounded and by using parabolic estimates, it follows that some subsequence (still denoted (u_k, v_k)) converges in $C^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \overline{\mathbb{R}^n_+})$ to a nonnegative and bounded solution $(\overline{u}, \overline{v})$ of (7). The above properties of (u_k, v_k) imply that

$$\min\bigl(V_{\lambda_0}\overline{u}(0,a,0), V_{\lambda_0}\overline{v}(0,a,0)\bigr) \le 0, \qquad \max\bigl(\overline{u}(0,b,0), \overline{v}(0,b,0)\bigr) \ge \varepsilon_0$$

and $V_{\lambda_0}\overline{u}, V_{\lambda_0}\overline{v} \ge 0$ in $\mathbb{R} \times \mathbb{T}_{\lambda_0}$.

Next we claim that $\overline{u}$ and $\overline{v}$ are positive everywhere in $\mathbb{R} \times \mathbb{T}_{\lambda_0}$. Indeed, assume for contradiction that $\overline{u}(t_1, x_0)$ or $\overline{v}(t_1, x_0)$ vanishes for some $t_1 \in \mathbb{R}$ and $x_0 \in \mathbb{T}_{\lambda_0}$. Due to the coupled structure of the system, by the strong maximum principle, it follows that $\overline{u} \equiv \overline{v} \equiv 0$ in $(-\infty, t_1] \times \mathbb{T}_{\lambda_0}$, hence $t_1 < 0$. But then, using the boundedness of $\overline{u}$ and $\overline{v}$ and the maximum principle again, we deduce $\overline{u} \equiv \overline{v} \equiv 0$ on $[t_1, \infty) \times \mathbb{T}_{\lambda_0}$: a contradiction.

Now, since $(V_{\lambda_0}\overline{u}, V_{\lambda_0}\overline{v})$ solves the corresponding problem (12) and $V_{\lambda_0}\overline{u}, V_{\lambda_0}\overline{v} \ge 0$, it follows that $V_{\lambda_0}\overline{u}, V_{\lambda_0}\overline{v} > 0$ in $\mathbb{R} \times \mathbb{T}_{\lambda_0}$. In particular we necessarily have $a = \lambda_0$. Moreover, by the Hopf maximum principle, it follows that

$$2\partial_{x_1}\overline{u}(0, \lambda_0, 0) = -\partial_{x_1} V_{\lambda_0}\overline{u}(0, \lambda_0, 0) > 0.$$

Similarly, we obtain $\partial_{x_1}\overline{v}(0, \lambda_0, 0) > 0$. Consequently, $\partial_{x_1}\overline{u}(0, x_1, 0)$ and $\partial_{x_1}\overline{u}(0, x_1, 0)$ are bounded below by a positive constant on an interval around λ_0 and this remains valid if $\overline{u}$ and $\overline{v}$ are replaced, respectively, by u_k and v_k for k sufficiently large. That is, there exists $\delta > 0$ such that

$$2\partial_{x_1} u\bigl(t^k, x_1, \xi^k\bigr) = -\partial_{x_1} u_k(0, x_1, 0) > 0, \quad x_1 \in [\lambda_0 - \delta, \lambda_0 + \delta]. \tag{22}$$

Similarly, we obtain $\partial_{x_1} v(t^k, x_1, \xi^k) > 0$, for $x_1 \in [\lambda_0 - \delta, \lambda_0 + \delta]$. However, since $2\lambda_k - x_1^k > x_1^k$ both belong to $[\lambda_0 - \delta, \lambda_0 + \delta]$ for large k, this contradicts the assumption that $\min(V_{\lambda_k} u(t^k, x_1^k, \xi^k), V_{\lambda_k} v(t^k, x_1^k, \xi^k)) < 0$. Therefore the assumption (i) leads to a contradiction.

Now consider case (ii). We go back to the problem (12) with $\lambda = \lambda_k$ and k sufficiently large. By assumption, $g_u(0,v), f_u(0,v), f_v(u,0), g_v(u,0) = 0$ and the definitions of $C_i^{\lambda_k}$, $i = 1, \ldots, 4$, and m_k imply that, for

$$\overline{l}_k := \max\Bigl(\sup_{(t,x)\in F_k} \bigl(C_1^{\lambda_k}(t,x) + C_2^{\lambda_k}(t,x)\bigr), \sup_{(t,x)\in F_k} \bigl(C_3^{\lambda_k}(t,x) + C_4^{\lambda_k}(t,x)\bigr)\Bigr),$$

we have

$$\limsup_{k\to\infty} \overline{l}_k \le 0.$$

Fix k so large that $l := \overline{l}_k + \gamma < \lambda_k^{-2}\pi^2$, where γ is any small positive constant. Set $\lambda = \lambda_k$, apply Lemma 2.1 and let h be the resulting function. As above, the function $(\overline{V}_\lambda u, \overline{V}_\lambda v) := (e^{\gamma t} V_\lambda u/h, e^{\gamma t} V_\lambda v/h)$ satisfies the problem (16). Since

$$\gamma + C_1^\lambda(t,x) + C_2^\lambda(t,x) - l \le 0, \qquad \gamma + C_3^\lambda(t,x) + C_4^\lambda(t,x) - l \le 0 \quad \text{on } F_k,$$

it follows that inequality (18) (resp., (19)) is satisfied on the set $\{(t,x) \in \mathbb{R} \times \mathbb{T}_\lambda, W(t,x) > 0\}$ (resp., $\{(t,x) \in \mathbb{R} \times \mathbb{T}_\lambda, Z(t,x) > 0\}$). Then the argument at the end of Step 1, after equation

(19), still applies and yields $V_{\lambda_k}u, V_{\lambda_k}v \geq 0$, contradicting the nonemptiness of F_k. Therefore the two possibilities (i) and (ii) lead to a contradiction. Thus, $\lambda_0 = \infty$. This completes the proof of the claim, hence the proof of Part (a).

Part (b). Let (u, v) be a positive, bounded, and classical solution of (7). For $k = 1, 2, \ldots$, we consider the functions

$$u_k(t, x_1, x') := u(t, x_1 + k, x'), \qquad v_k(t, x_1, x') := v(t, x_1 + k, x'), \quad (t, x_1, x') \in \Omega_k,$$

where $\Omega_k = \mathbb{R} \times (-k, \infty) \times \mathbb{R}^{n-1}$. Since (u, v) is a positive, bounded, and classical solution of (7), then for all k, (u_k, v_k) is a positive, bounded, and classical solution (on its domains) of the following system:

$$\begin{cases} \partial_t u_k - \triangle u_k = v_k^p + \mu_1 u_k^r, \\ \partial_t v_k - \triangle v_k = u_k^q + \mu_2 v_k^s. \end{cases}$$

Moreover, $u_k = v_k = 0$ on $\partial\Omega_k$. From L^m parabolic estimates, there exists a subsequence $((u_k, v_k))_{k\geq 0}$ converges in $C^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n)$ to a nonnegative, bounded pair of functions $(\overline{u}, \overline{v})$. Moreover, $(\overline{u}, \overline{v})$ is a classical solution of

$$\begin{cases} \overline{u}_t - \triangle \overline{u} = \overline{v}^p + \mu_1 \overline{u}^r, \\ \overline{v}_t - \triangle \overline{v} = \overline{u}^q + \mu_2 \overline{v}^s. \end{cases}$$

Since u_k, v_k are increasing in x_1 (by Part (a)), then

$$u_k(t, x_1, x') = u(t, x_1 + k, x') \geq u(t, x_1 + k_0, x') > 0 \quad \text{for all } k \geq k_0,$$

with $x_1 + k_0 > 0$. Therefore, by passing to limits, we obtain $\overline{u}(t, x_1, x') \geq u(t, x_1 + k_0, x') > 0$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Similarly, we obtain $\overline{v} > 0$. Therefore also by Part (a), $\overline{u}$, $\overline{v}$ are increasing in x_1. Moreover, let $x_1 < x_2 \in \mathbb{R}$, we have $u_k(t, x_1, x') < u_k(t, x_2, x')$, $x_1 + k > 0$, passing to limits, we obtain $\overline{u}(t, x_1, x') \leq \overline{u}(t, x_2, x')$. Moreover, $u_{k+E(x_2)}(t, x_1, x') > u_k(t, x_2, x')$, passing to limits, we obtain $\overline{u}(t, x_1, x') \geq \overline{u}(t, x_2, x')$. Therefore, $\overline{u}(t, x_1, x') = \overline{u}(t, x_2, x')$ for all $x_1, x_2 \in \mathbb{R}$, which means that $\overline{u}$ is independent of x_1. Similarly, we prove that $\overline{v}$ is independent of x_1. This completes the proof of Theorem 2.1. □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1 We assume by contradiction that the system (7) has a positive, bounded, and classical solution (u, v). By Theorem 2.1(b), there exists a positive, bounded, and classical solution of (11) on $\mathbb{R} \times \mathbb{R}^{n-1}$; a contradiction with [4], Theorem 2, p.178 or [5], Theorem 1.3, p.354. □

3 Blow-up rates via Fujita and Liouville-type theorems: proof of Theorem 1.2

In this section, we are concerned with the proof of Theorem 1.2. We will use the following key doubling lemma from [15].

Lemma 3.1 *Let (X, d) be a complete metric space and let $\emptyset \neq D \subset \Sigma \subset X$, with Σ closed. Set $\Gamma = \Sigma \setminus D$. Finally let $M : D \to (0, \infty)$ be bounded on compact subsets of D and fix a* *real* $k > 0$. *If there exists* $y \in D$ *such that*

$$M(y)\,\mathrm{dist}(y, \Gamma) > 2k,$$

then there exists $x \in D$ *such that*

$$M(x)\,\mathrm{dist}(x, \Gamma) > 2k, \qquad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \text{for all } z \in D \cap \overline{B}_X\bigl(x, kM^{-1}(x)\bigr).$$

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2 We assume by contradiction that the theorem fails. Then there exist sequences $T_k \in (0, \infty)$, (u_k, v_k) being a solution of (8) in $(0, T_k) \times \Omega$, $y_k \in \Omega$, and $\sigma_k \in (0, T_k)$ such that the functions

$$M_k := u_k^{1/2\alpha} + v_k^{1/2\beta}, \quad k = 1, 2, \ldots, \tag{23}$$

satisfy

$$M_k(\sigma_k, y_k) > 2k\bigl(1 + d_k^{-1}(\sigma_k)\bigr), \tag{24}$$

where $d_k(t) = (\min(t, T_k - t))^{1/2}$. Then

$$M_k(\sigma_k, y_k) > 2kd_k^{-1}(\sigma_k) \tag{25}$$

and

$$M_k(\sigma_k, y_k) > 2k. \tag{26}$$

We will use Lemma 3.1 with $X = \mathbb{R}^{n+1}$ equipped with the parabolic distance d_P, defined by

$$d_P\bigl((t, x), (\sigma, y)\bigr) = |t - \sigma|^{1/2} + |x - y| \quad \text{for all } (t, x), (\sigma, y) \in \mathbb{R}^{n+1},$$

$\Sigma = \Sigma_k = [0, T_k] \times \overline{\Omega}$, $D = D_k = (0, T_k) \times \Omega$, and $\Gamma = \Gamma_k = \{0, T_k\} \times \overline{\Omega}$. Let us mention that

$$d_k(t) = d_P\bigl((t, x), \Gamma_k\bigr), \quad (t, x) \in \Sigma_k.$$

Indeed, let $(t, x) \in \Sigma_k$, we have

$$\begin{aligned} d_P\bigl((t, x), \Gamma_k\bigr) &= \inf_{(\sigma, y) \in \Gamma_k}\bigl(|t - \sigma|^{1/2} + |x - y|\bigr) \\ &= |x - x| + \inf\bigl((t - 0)^{1/2}, (T_k - t)^{1/2}\bigr) \\ &= \bigl(\min(t, T_k - t)\bigr)^{1/2} = d_k(t). \end{aligned}$$

As mentioned above by Lemma 3.1 with $X = \mathbb{R}^{n+1}$ equipped with the parabolic distance d_P and (26), it follows that there exist $x_k \in \Omega$, $t_k \in (0, T_k)$ such that

$$M_k(t_k, x_k) > 2kd_k^{-1}(t_k), \tag{27}$$

$$M_k(t_k, x_k) \geq M_k(\sigma_k, y_k) > 2k, \tag{28}$$

and

$$M_k(t,x) \leq 2M_k(t_k,x_k) \quad \text{for all } (t,x) \in D_k \cap \overline{B}_k\big((t_k,x_k), kM^{-1}(t_k,x_k)\big), \tag{29}$$

where

$$\overline{B}_k\big((t_k,x_k), kM^{-1}(t_k,x_k)\big) = \big\{(t,x) \in \mathbb{R}^{n+1}; |x - x_k| + |t - t_k|^{1/2} \leq kM^{-1}(t_k,x_k)\big\}.$$

In the rest of the proof, we use the notation

$$\lambda_k := M_k^{-1}(t_k, x_k).$$

By (28), we obtain

$$\lambda_k < \frac{1}{2k} \underset{k\to\infty}{\to} 0. \tag{30}$$

Moreover, we observe that

$$\left(t_k - \frac{k^2\lambda_k^2}{4}, t_k + \frac{k^2\lambda_k^2}{4}\right) \times \left(\Omega \cap \left\{|x - x_k| < \frac{k\lambda_k}{2}\right\}\right) \subset D_k \cap \overline{B}_k. \tag{31}$$

Indeed, we have $(\Omega \cap \{|x - x_k| < \frac{k\lambda_k}{2}\}) \subset \Omega$. Also by (27), we obtain

$$|t - t_k| < \frac{k^2\lambda_k^2}{4} < k^2\lambda_k^2 < d_k^2(t_k) = \min\big(t_k, (T_k - t_k)\big),$$

hence $t \in (0, T_k)$. Finally,

$$|x - x_k| + |t - t_k|^{1/2} \leq \frac{k\lambda_k}{2} + \frac{k\lambda_k}{2} = k\lambda_k.$$

Therefore, $(t,x) \subset \overline{B}_k$. Now, we rescale the functions u_k, v_k by setting

$$w_k(\sigma,y) := \lambda_k^{2\alpha} u_k\big(t_k + \lambda_k^2\sigma, x_k + \lambda_k y\big), \qquad z_k(\sigma,y) := \lambda_k^{2\beta} v_k\big(t_k + \lambda_k^2\sigma, x_k + \lambda_k y\big),$$

where $(t_k + \lambda_k^2\sigma, x_k + \lambda_k y) \in (t_k - \frac{k^2\lambda_k^2}{4}, t_k + \frac{k^2\lambda_k^2}{4}) \times (\Omega \cap \{|x - x_k| < \frac{k\lambda_k}{2}\})$, which imply that $(\sigma, y) \in \widetilde{D}_k := (-\frac{k^2}{4}, \frac{k^2}{4}) \times (\lambda_k^{-1}(\Omega - x_k) \cap \{|y| < \frac{k}{2}\})$.

The pair of functions (w_k, z_k) solves the system

$$\begin{cases} \partial_\sigma w_k - \triangle w_k = z_k^p + \lambda_k^{2\alpha(1-r)+2} w_k^r, & (\sigma,y) \in \widetilde{D}_k, \\ \partial_\sigma z_k - \triangle z_k = w_k^q + \lambda_k^{2\beta(1-s)+2} z_k^s, & (\sigma,y) \in \widetilde{D}_k, \\ w_k = z_k = 0, \quad y \in \lambda_k^{-1}(\partial\Omega - x_k), |y| < \frac{k}{2}, |s| < \frac{k^2}{4}. \end{cases} \tag{32}$$

Moreover,

$$w_k^{1/2\alpha}(0,0) + z_k^{1/2\beta}(0,0) = \lambda_k u_k^{1/2\alpha}(t_k,x_k) + \lambda_k v_k^{1/2\beta}(t_k,x_k) = \lambda_k\lambda_k^{-1} = 1. \tag{33}$$

By (29), we obtain

$$\begin{aligned}\big[w_k^{1/2\alpha} + z_k^{1/2\beta}\big](\sigma,y) &= \lambda_k u_k^{1/2\alpha}\big(t_k+\lambda_k^2\sigma, x_k+\lambda_k y\big) + \lambda_k v_k^{1/2\beta}\big(t_k+\lambda_k^2\sigma, x_k+\lambda_k y\big)\\ &= \lambda_k M_k\big(t_k+\lambda_k^2\sigma, x_k+\lambda_k y\big)\\ &\le 2\lambda_k M_k(t_k,x_k) = 2 \quad \text{for all } (\sigma,y)\in\widetilde{D}_k.\end{aligned}$$

Therefore, since $w_k, z_k \geq 0$, we obtain

$$0 \le w_k^q(\sigma,y) \le 2^{2\alpha q}, \qquad 0 \le z_k^p(\sigma,y) \le 2^{2\beta p} \quad \text{for all } (\sigma,y)\in\widetilde{D}_k. \tag{34}$$

Also, since $r \le p(q+1)/(p+1)$, $s \le q(p+1)/(q+1)$, and $\lambda_k \to 0$, we obtain

$$0 \le \lambda_k^{2\alpha(1-r)+2} w_k^r(\sigma,y) \le 2^{2\alpha r}, \qquad 0 \le \lambda_k^{2\beta(1-s)+2} z_k^s(\sigma,y) \le 2^{2\beta s} \tag{35}$$

for all, $(\sigma,y)\in\widetilde{D}_k$. Let $\rho_k := \operatorname{dist}(x_k,\partial\Omega)$. Then either the sequence $(\rho_k/\lambda_k)_k$ is bounded or unbounded. By passing to a subsequence, we may assume that either:

$$\text{(a)}\quad \frac{\rho_k}{\lambda_k} \to \infty \quad \text{or} \quad \text{(b)}\quad \frac{\rho_k}{\lambda_k} \to c \ge 0.$$

If case (a) holds, since $u_k, v_k \in C^{1,2}((0,T_k)\times\Omega)\cap C((0,T_k)\times\overline{\Omega})$, we obtain

$$w_k, z_k \in C^{1,2}(\widetilde{D}_k)\cap C\left(\left(-\frac{k^2}{4},\frac{k^2}{4}\right)\times\left(\lambda_k^{-1}(\overline{\Omega}-x_k)\cap\left\{|y|<\frac{k}{2}\right\}\right)\right).$$

Moreover,

$$w_k, z_k = 0 \quad \text{on } \left(-\frac{k^2}{4},\frac{k^2}{4}\right)\times\left(\lambda_k^{-1}(\partial\Omega-x_k)\cap\left\{|y|<\frac{k}{2}\right\}\right).$$

Set $\delta_1 := 1$ if $r = p(q+1)/(p+1)$, $\delta_1 := 0$ if $r < p(q+1)/(p+1)$, and $\delta_2 := 1$ if $s = q(p+1)/(q+1)$, $\delta_2 := 0$ if $s < q(p+1)/(q+1)$. By using interior L^m parabolic estimates, it follows that there exists a subsequence (w_k, z_k) that converges in $C^{\alpha}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^n)$, $0<\alpha<1$, to a pair of functions (w,z) a nonnegative, bounded, and classical solution of the problem

$$\begin{cases} w_\sigma - \triangle w = z^p + \mu_1\delta_1 w^r, & (\sigma,y)\in\mathbb{R}\times\mathbb{R}^n,\\ z_\sigma - \triangle z = w^q + \mu_2\delta_2 z^s, & (\sigma,y)\in\mathbb{R}\times\mathbb{R}^n.\end{cases}$$

Moreover, $w^{1/2\alpha}(0,0) + z^{1/2\beta}(0,0) = 1$; a contradiction with [4], Theorem 2, p.178 or [5], Theorem 1.3, p.354. (Note that these results remain valid with any positive coefficients in front of the terms w^r and z^s, instead 1.)

If case (b) holds. Let $\overline{x}_k \in \partial\Omega$ be such that $\rho_k = |x_k - \overline{x}_k|$. As in, *e.g.*, [16], for any k we can choose a local coordinate $\theta = \theta^{(k)} = (\theta_1,\theta_2,\dots,\theta_n)$ in an ε-neighborhood U_k of $\overline{x}_k$ such that the image of the boundary $\partial\Omega$ will be contained in the hyperplane $\theta_1 = 0$, $\overline{x}_k$ becomes 0, x_k becomes $\theta_k := (\rho_k, 0, 0, \dots, 0)$ and the image of U_k will contain the set $\{\theta : |\theta| < \varepsilon'\}$ for some $\varepsilon' > 0$. We may assume that ε, ε' are independent of k and the local charts are uniformly bounded in C^2. In these new coordinates, the system for $\varphi = \varphi_k(t,\theta) = u_k(t,x)$ and $\phi := \phi_k(t,\theta) := v_k(t,x)$ becomes

$$\begin{cases} \varphi_t - \sum_{i,j} a_{i,j} \frac{\partial^2\varphi}{\partial\theta_i\,\partial\theta_j} - \sum_i b_i \frac{\partial\varphi}{\partial\theta_i} = \phi^p + \mu_1\varphi^r, & t > 0, |\theta| < \varepsilon, \theta_1 > 0, \\ \phi_t - \sum_{i,j} a_{i,j} \frac{\partial^2\phi}{\partial\theta_i\,\partial\theta_j} - \sum_i b_i \frac{\partial\phi}{\partial\theta_i} = \phi^q + \mu_2\varphi^s, & t > 0, |\theta| < \varepsilon, \theta_1 > 0, \\ \varphi = \phi = 0, \quad t > 0, |\theta| < \varepsilon, \theta_1 = 0, \end{cases} \tag{36}$$

where $a_{i,j}(t,\theta) = \sum_l \frac{\partial\theta_i}{\partial x_l}\frac{\partial\theta_j}{\partial x_l} = D \cdot {}^tD$, with $D = D_k = (\partial\theta_i/\partial x_j)_{i,j}$, and $b_i(t,\theta) = \triangle\theta_i$, hence $A = A_k := (a_{ij})_{i,j}$ are uniformly elliptic. Also, since $\partial\Omega$ is uniformly C^2, it follows that the a^k_{ij} are uniformly bounded and the b^k_i in L^∞. Moreover, since $D(0)$ is a Euclidean transformation, it follows that $A_j(0) = D(0) \cdot {}^tD(0) = \mathrm{Id}$. Then the rescaled functions w_k, z_k defined by

$$w_k(\sigma,y) := \lambda_k^{2\alpha}\varphi_k\bigl(t_k + \lambda_k^2\sigma, \theta_k + \lambda_k y\bigr), \qquad z_k(\sigma,y) := \lambda_k^{2\beta}\phi_k\bigl(t_k + \lambda_k^2\sigma, \theta_k + \lambda_k y\bigr),$$

where $(\sigma,y) \in \{(\sigma,y) : |\sigma| < k^2/4; |y - \frac{\theta_k}{\lambda_k}| < \frac{\varepsilon}{\lambda_k}, y_1 > -\frac{\rho_k}{\lambda_k}\}$. Then (w_k, z_k) solves the system

$$\begin{cases} \partial_\sigma w_k - \sum_{i,j} a^{i,j} \frac{\partial^2 w_k}{\partial y^i\,\partial y^j} - \lambda_k \sum_i b^i \frac{\partial w_k}{\partial y^i} = z_k^p + \mu_1\lambda_k^{2\alpha(1-r)+2} w_k^r, \\ \partial_\sigma z_k - \sum_{i,j} a^{i,j} \frac{\partial^2 z_k}{\partial y^i\,\partial y^j} - \lambda_k \sum_i b^i \frac{\partial z_k}{\partial y^i} = w_k^q + \mu_2\lambda_k^{2\alpha(1-r)+2} z_k^s, \end{cases}$$

where $(\sigma,y) \in \{(\sigma,y) : |\sigma| < k^2/4; |y - \frac{\theta_k}{\lambda_k}| < \frac{\varepsilon}{\lambda_k}, y_1 > -\frac{\rho_k}{\lambda_k}\}$. Also

$$w_k = z_k = 0 \quad \text{for } |\sigma| < k^2/4, \left| y - \frac{\theta_k}{\lambda_k} \right| < \frac{\varepsilon}{\lambda_k}, y_1 = -\frac{\rho_k}{\lambda_k}.$$

As in the first case, by using interior-bounded L^m parabolic estimates, we conclude that there exists a subsequence (w_k, z_k) that converges in C^α_{loc}, $0 < \alpha < 1$, to a nonnegative bounded and classical solution of the problem

$$\begin{cases} w_\sigma - \triangle w = z^p + \mu_1\delta_1 w^r, & (\sigma,y) \in \mathbb{R} \times H_c, \\ z_\sigma - \triangle z = w^q + \mu_2\delta_2 z^s, & (\sigma,y) \in \mathbb{R} \times H_c, \\ w = z = 0, \quad y_1 = c, \end{cases}$$

where $H_c := \{y \in \mathbb{R}^n; y_1 > -c\}$. Moreover, $w^{1/2\alpha}(0,0) + z^{1/2\beta}(0,0) = 1$; a contradiction with Theorem 1.1. This finishes the proof of Theorem 1.2. □

Competing interests
The author declares to have no competing interests.

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