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Local well-posedness of generalized BBM equations with generalized damping on 1D torus

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Abstract

We consider the periodic initial value problem associated to the generalized Benjamin-Bona-Mahony equation with generalized damping on the one dimensional torus. In contrast to the classical BBM equation, the main difference is that the generalized equation contains two nonlocal operators, and the main difficulty comes from two nonlocal operators. By the fixed point theorem, we prove that the periodic initial value problem is locally well-posed. We also prove that if the solution exists globally in time, it exhibits some asymptotic behavior.

MSC: 35A07; 35Q53

Keywords: generalized Benjamin-Bona-Mahony equation; nonlocal operator; generalized damping; local well-posedness

1 Introduction

The classical Benjamin-Bona-Mahony (BBM) equation

$$u_t - u_{xxt} + u_x + uu_x = 0 \tag{1.1}$$

was proposed in [1] as a model for propagation of long waves which incorporates nonlinear dispersive and dissipative effects. It has extensively been studied in the recent literature; see for example [1–10] on the existence and uniqueness of solutions and [11–17] on the global attractors and references therein.

In this paper, we consider the periodic initial value problem of generalized BBM equations with generalized damping on the 1D torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$:

$$\begin{cases} u_t + L_p u_t + u_x + uu_x + M_\alpha u = 0, & x \in \mathbb{T}, t \in (0, T), \\ u(0,x) = u_0(x), & x \in \mathbb{T}, \\ \int_{\mathbb{T}} u(t,x)\,dx = 0, \end{cases} \tag{1.2}$$

where the two nonlocal operators are defined by

$$\widehat{L_p u}(k) = |k|^{p+1}\widehat{u}(k), \quad p > 0;$$

$$\widehat{M_\alpha u}(k) = |k|^{2\alpha}\widehat{u}(k), \quad \alpha \in \left(0, \frac{p+1}{2}\right], \tag{1.3}$$

and $\widehat{u}(k)$ is the kth Fourier coefficient of $u(t,x)$ in x.

For $\alpha = 1$, the generalized damping becomes a parabolic damping,

$$M_1 u = -u_{xx} \quad \text{as } \widehat{M_1 u}(k) = |k|^2\widehat{u}(k).$$

For $\alpha = 0$, it is a weak damping,

$$M_0 u = u \quad \text{as } \widehat{M_0 u}(k) = \widehat{u}(k).$$

For example, Wang [18] considered the damped BBM equation $u_t - u_{txx} + \gamma(u - u_{xx}) + uu_x = f(x)$ (Introduction, p.134) and the BBM equation with different damping coefficients $u_t - u_{txx} + \gamma u - \nu u_{xx} + uu_x = f(x)$ (Remark 3.2, p.142).

In fact, one can consider more general damping terms. For example, Chehab *et al.* [19] studied the long-time behavior of the solution of a damped BBM equation

$$u_t - u_{xxt} + u_x + uu_x + M_\alpha u = 0, \quad x \in \mathbb{T}[0,L], t \in (0,T), \tag{1.4}$$

with

$$\widehat{M_\alpha u}(k) = \gamma_k \widehat{u}(k),$$

and $(\gamma_k)_{k\in\mathbb{Z}}$ are positive real numbers.

In the absence of fractional damping $M_\alpha u$, Carvajal and Panthee [20] proved that the Cauchy problem

$$\begin{cases} u_t + L_p u_t + u_x + (u^{k+1})_x = 0, & x \in \mathbb{R}, t \in (0,T), k \in \mathbb{Z}^+, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

is ill-posed for data with lower order Sobolev regularity and in a certain range of the Sobolev regularity, even if the solution exists globally in time, it fails to be smooth.

In this paper we study the generalized BBM equations with the fractional damping terms. In contrast to the classical BBM equation, the main difference is that equation (1.2) contains two nonlocal operators, its dissipation is weaker than the classical BBM equation. In the study of the periodic initial value problem (1.2), the main difficulty is that L_p and M_α are nonlocal operators. By the fixed point theorem and the Fourier analysis method, similar to [21–23], we prove the local well-posedness of the solution to the problem (1.2).

2 Local well-posedness

We define a space

$$\dot{H}^\beta(\mathbb{T}) = \left\{ u \in L^2(\mathbb{T}) : \int_{\mathbb{T}} u\,dx = 0, \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\beta}\left|\widehat{u}(k)\right|^2 < +\infty \right\}$$

with the norm

$$|u|_\beta^2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\beta}\left|\widehat{u}(k)\right|^2.$$

Then we can obtain the local existence and uniqueness of the solution to the periodic initial value problem (1.2).

Theorem 2.1 *Assume* $\alpha \in (\frac{1}{2}, \frac{p+1}{2}]$. *If* $u_0(x) \in \dot{H}^\alpha(\mathbb{T})$, *there exist a constant* $T = C_0(|u_0|_\alpha) > 0$ *and a unique solution* $u(t,x) \in C([0,T], \dot{H}^\alpha(\mathbb{T}))$. *Moreover, for any constant* $M > 0$, $|u_0|_\alpha \leq M$, $|v_0|_\alpha \leq M$, *there exists a constant* $C_1 > 0$ *such that the solutions* $u(t,x)$, $v(t,x)$ *of the periodic initial value problem* (1.2) *with the initial data* $u_0(x) \in \dot{H}^\alpha(\mathbb{T})$ *and* $v_0(x) \in \dot{H}^\alpha(\mathbb{T})$, *respectively, satisfy*

$$\big|u(t) - v(t)\big|_\alpha \leq C_1|u_0 - v_0|_\alpha, \quad \forall t \leq \frac{1}{C_0 M}. \tag{2.1}$$

Proof We first write equation (1.2) in the following form:

$$\begin{aligned} u_t &= -(I+L_p)^{-1}\partial_x u - (I+L_p)^{-1}\partial_x\left(\frac{u^2}{2}\right) - (I+L_p)^{-1}M_\alpha u \\ &= -i\varphi(D_x)u - i\varphi(D_x)\left(\frac{u^2}{2}\right) - \varphi_\alpha(D_x)u \\ &= -\big[i\varphi(D_x) + \varphi_\alpha(D_x)\big]u - i\varphi(D_x)\left(\frac{u^2}{2}\right), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \widehat{\varphi(D_x)u}(k) &= \frac{k}{1+|k|^{p+1}}\widehat{u}(k), \\ \widehat{\varphi_\alpha(D_x)u}(k) &= \frac{|k|^{2\alpha}}{1+|k|^{p+1}}\widehat{u}(k), \quad k \in \mathbb{Z}. \end{aligned} \tag{2.3}$$

Then we get

$$u(t) = S_t u_0 - i\int_0^t S_{t-\tau}\varphi(D_x)\left(\frac{u^2}{2}\right)d\tau, \tag{2.4}$$

where

$$S_t u_0 = \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{ikx} e^{-\frac{ik+|k|^{2\alpha}}{1+|k|^{p+1}}t}\widehat{u_0}(k). \tag{2.5}$$

We define a map

$$\Phi\big(u(t)\big) = S_t u_0 - i\int_0^t S_{t-\tau}\varphi(D_x)\left(\frac{u^2}{2}\right)d\tau \tag{2.6}$$

and a closed ball

$$\overline{B}(\mathbb{T}) = \big\{u(t,x) \in C\big([0,T], \dot{H}^\alpha(\mathbb{T})\big) : \big|u(t) - u_0\big|_\alpha \leq 3|u_0|_\alpha\big\}. \tag{2.7}$$

We now prove that Φ has a unique fixed point in $\overline{B}(\mathbb{T})$.

Step one: Φ is onto, that is, for $u(t) \in \overline{B}(\mathbb{T})$ we have $\Phi(u(t)) \in \overline{B}(\mathbb{T})$.

According to the definition of the norm $|u(t)|_\alpha$, we get

$$\begin{aligned}|S_t u_0|_\alpha^2 &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}\big|\widehat{S_t u_0}(k)\big|^2\\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}\big|e^{-\frac{ik+|k|^{2\alpha}}{1+|k|^{p+1}}t}\widehat{u_0}(k)\big|^2\\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}e^{-\frac{2|k|^{2\alpha}}{1+|k|^{p+1}}t}\big|\widehat{u_0}(k)\big|^2\\ &\le \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}\big|\widehat{u_0}(k)\big|^2 = |u_0|_\alpha^2\end{aligned}$$

and

$$\begin{aligned}\left|S_{t-\tau}\varphi(D_x)\left(\frac{u^2}{2}\right)\right|_\alpha^2 &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}\big|S_{t-\tau}\widehat{\varphi(D_x)\big(u^2/2\big)}(k)\big|^2\\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}\left|e^{-\frac{ik+|k|^{2\alpha}}{1+|k|^{p+1}}(t-\tau)}\frac{k}{1+|k|^{p+1}}\widehat{\big(u^2/2\big)}(k)\right|^2\\ &\le \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}\big|\widehat{\big(u^2/2\big)}(k)\big|^2\\ &= \frac{1}{4}\sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}\big|\widehat{u^2}(k)\big|^2\\ &= \frac{1}{4}\big|u^2\big|_\alpha^2 \le C|u|_\alpha^4,\end{aligned}$$

the last inequality comes from the fact that $\dot{H}^\alpha(\mathbb{T})$ is an algebra for $\alpha > \frac{1}{2}$.

Putting the above two inequalities into (2.6) we have

$$\begin{aligned}\big|\Phi(u)\big|_\alpha &\le |S_t u_0|_\alpha + \int_0^t \left|S_{t-\tau}\varphi(D_x)\left(\frac{u^2}{2}\right)\right|_\alpha d\tau\\ &\le |u_0|_\alpha + C\int_0^t \big|u(\tau)\big|_\alpha^2\,d\tau\\ &\le |u_0|_\alpha + CT\sup_{t\in[0,T]}\big|u(t)\big|_\alpha^2.\end{aligned}\tag{2.8}$$

Since $u(t) \in \overline{B}(\mathbb{T})$ and

$$\big|u(t)\big|_\alpha - |u_0|_\alpha \le \big|u(t) - u_0\big|_\alpha \le 3|u_0|_\alpha,$$

we have $|u(t)|_\alpha \le 4|u_0|_\alpha$ and

$$\begin{aligned}\big|\Phi\big(u(t)\big) - u_0\big|_\alpha &\le \big|\Phi\big(u(t)\big)\big|_\alpha + |u_0|_\alpha \le 2|u_0|_\alpha + 16CT|u_0|_\alpha^2\\ &\le 3|u_0|_\alpha, \quad \text{if } 0 < T < \frac{1}{16C|u_0|_\alpha}.\end{aligned}$$

Therefore, for $T \in (0, \frac{1}{16C|u_0|_\alpha})$, we have $\Phi(u(t)) \in \overline{B}(\mathbb{T})$ for $u(t) \in \overline{B}(\mathbb{T})$.

Step two: Φ is a contractive mapping on $\overline{B}(\mathbb{T})$.

Let $u(t), v(t) \in \overline{B}(\mathbb{T})$. Since

$$\begin{aligned}
\left|\Phi\big(u(t)\big) - \Phi\big(v(t)\big)\right|_\alpha^2 &= \left|\int_0^t S_{t-\tau}\varphi(D_x)\left(\frac{u^2}{2} - \frac{v^2}{2}\right) d\tau\right|_\alpha^2 \\
&\leq \int_0^t \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha} \left| e^{-\frac{ik+|k|^{2\alpha}}{1+|k|^{p+1}}(t-\tau)} \frac{k}{1+|k|^{p+1}} \left(\widehat{\frac{u^2}{2} - \frac{v^2}{2}}\right)(k)\right|^2 d\tau \\
&\leq C\int_0^t \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha} \left|\big(\widehat{u^2 - v^2}\big)(k)\right|^2 d\tau \\
&= C\int_0^t \left|u^2 - v^2\right|_\alpha^2 d\tau \\
&\leq 2C\int_0^t \big(|u|_\alpha^2 + |v|_\alpha^2\big)|u-v|_\alpha^2 \, d\tau,
\end{aligned}$$

due to $u(t), v(t) \in \overline{B}(\mathbb{T})$, we have

$$\left|\Phi\big(u(t)\big) - \Phi\big(v(t)\big)\right|_\alpha^2 \leq 32C\big(|u_0|_\alpha^2 + |v_0|_\alpha^2\big)T \sup_{t\in[0,T]} \left|u(t) - v(t)\right|_\alpha^2, \tag{2.9}$$

that is,

$$\sup_{t\in[0,T]} \left|\Phi\big(u(t)\big) - \Phi\big(v(t)\big)\right|_\alpha \leq 4\sqrt{2CT\big(|u_0|_\alpha^2 + |v_0|_\alpha^2\big)} \sup_{t\in[0,T]} \left|u(t) - v(t)\right|_\alpha. \tag{2.10}$$

Therefore, Φ is a contractive mapping on $\overline{B}(\mathbb{T})$ if $4\sqrt{2CT(|u_0|_\alpha^2 + |v_0|_\alpha^2)} < 1$, that is,

$$0 < T < \frac{1}{32C(|u_0|_\alpha^2 + |v_0|_\alpha^2)}.$$

Thanks to the Banach fixed point theorem, Φ has a unique fixed point $u(t)$ such that $u(t) = \Phi(u(t))$, that is, there exists a unique solution of the periodic initial value problem (1.2).

Step three: the continuity of solution with the initial data.

Let $u(t)$ and $v(t)$ be solutions of the periodic initial value problem (1.2) with the initial data u_0 and v_0, respectively, such that $|u_0|_\alpha \leq M$, $|v_0|_\alpha \leq M$. For $t \in [0,T]$, the Duhamel principle gives us the following formula:

$$u(t) - v(t) = S_t(u_0 - v_0) - i\int_0^t S_{t-\tau}\varphi(D_x)\left(\frac{u^2}{2} - \frac{v^2}{2}\right) d\tau, \tag{2.11}$$

hence

$$\begin{aligned}
\left|u(t) - v(t)\right|_\alpha &\leq \left|S_t(u_0 - v_0)\right|_\alpha + \int_0^t \left|S_{t-\tau}\varphi(D_x)\left(\frac{u^2}{2} - \frac{v^2}{2}\right)\right|_\alpha d\tau \\
&\leq |u_0 - v_0|_\alpha + C_0\big(\left|u(t)\right|_\alpha + \left|v(t)\right|_\alpha\big)T \sup_{t\in[0,T]} \left|u(t) - v(t)\right|_\alpha \\
&\leq |u_0 - v_0|_\alpha + 4C_0\big(|u_0|_\alpha + |v_0|_\alpha\big)T \sup_{t\in[0,T]} \left|u(t) - v(t)\right|_\alpha \\
&\leq |u_0 - v_0|_\alpha + 8C_0 MT \sup_{t\in[0,T]} \left|u(t) - v(t)\right|_\alpha,
\end{aligned}$$

if $T < \frac{1}{8C_0M}$, there exists a constant $C_1 = \frac{1}{1-8C_0MT} > 0$ such that

$$\left|u(t)-v(t)\right|_\alpha \le \sup_{t\in[0,T]} \left|u(t)-v(t)\right|_\alpha \le C_1|u_0-v_0|_\alpha. \tag{2.12}$$

The proof is complete. □

3 Asymptotic behavior of the solution

We first consider the corresponding problem with the linear equation

$$\begin{cases} u_t + L_p u_t + u_x + M_\alpha u = 0, & x\in\mathbb{T}, t\in(0,T), \\ u(0,x) = u_0(x), & x\in\mathbb{T}, \\ \int_{\mathbb{T}} u(t,x)\,dx = 0. \end{cases} \tag{3.1}$$

If $u(t)\in L^2(\mathbb{T})$, $\forall t>0$, then the kth Fourier coefficient $\widehat{u}_k(t)$ of $u(t,x)$ in x satisfies

$$\left(1+|k|^{p+1}\right)\widehat{u}_k'(t) + \left(ik+|k|^{2\alpha}\right)\widehat{u}_k(t) = 0, \quad k\in\mathbb{Z},$$

that is,

$$\widehat{u}_k(t) = e^{-\frac{ik+|k|^{2\alpha}}{1+|k|^{p+1}}t}\widehat{u}_k(0).$$

Therefore, we have

$$|u|^2_{\frac{p+1}{2}} = \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1}\left|\widehat{u}_k(t)\right|^2 = \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1}e^{-\frac{2|k|^{2\alpha}}{1+|k|^{p+1}}t}\left|\widehat{u}_k(0)\right|^2. \tag{3.2}$$

Theorem 3.1 *If $u_0(x)\in \dot{H}^{\frac{p+1}{2}}(\mathbb{T})$, then the unique solution $u(t,x)$ of the periodic initial value problem* (3.1) *satisfies*

$$\left|u(t)\right|^2_{\frac{p+1}{2}} \le |u_0|^2_{\frac{p+1}{2}}, \quad 0<\alpha\le\frac{p+1}{2}. \tag{3.3}$$

Furthermore, we have

$$\left|u(t)\right|^2_\alpha \le \frac{1}{et}|u_0|^2_{\frac{p+1}{2}}, \quad 0<\alpha<\frac{p+1}{2}, \forall t>0, \tag{3.4}$$

$$\left|u(t)\right|^2_\alpha \le e^{-t}|u_0|^2_{\frac{p+1}{2}}, \quad \alpha=\frac{p+1}{2}, \forall t>0. \tag{3.5}$$

Proof Equation (3.2) implies that

$$\begin{aligned} |u|^2_{\frac{p+1}{2}} &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1}e^{-\frac{2|k|^{2\alpha}}{1+|k|^{p+1}}t}\left|\widehat{u}_k(0)\right|^2 \\ &\le \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1}\left|\widehat{u}_k(0)\right|^2 \\ &= |u_0|^2_{\frac{p+1}{2}}, \quad 0<\alpha\le\frac{p+1}{2}. \end{aligned} \tag{3.6}$$

On the other hand, for $0 < \alpha < \frac{p+1}{2}$ we have

$$\begin{aligned} |u|_\alpha^2 &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha} e^{-\frac{2|k|^{2\alpha}}{1+|k|^{p+1}}t} \left|\widehat{u}_k(0)\right|^2 \\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1} \frac{1+|k|^{p+1}}{|k|^{p+1}} \frac{|k|^{2\alpha}}{1+|k|^{p+1}} e^{-\frac{2|k|^{2\alpha}}{1+|k|^{p+1}}t} \left|\widehat{u}_k(0)\right|^2 \\ &\leq \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1} \varphi_\alpha(k) e^{-\varphi_\alpha(k)t} \left|\widehat{u}_k(0)\right|^2, \end{aligned} \tag{3.7}$$

where

$$\varphi_\alpha(k) = \frac{2|k|^{2\alpha}}{1+|k|^{p+1}}, \quad 0 < \alpha < \frac{p+1}{2}, k \in \mathbb{Z} \setminus \{0\}. \tag{3.8}$$

Since the function xe^{-xt} is uniformly bounded by $\frac{1}{et}$, we have

$$|u|_\alpha^2 \leq \frac{1}{et} \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1} \left|\widehat{u}_k(0)\right|^2 = \frac{1}{et} |u_0|_{\frac{p+1}{2}}^2. \tag{3.9}$$

For $\alpha = \frac{p+1}{2}$, we have

$$\begin{aligned} |u|_{\frac{p+1}{2}}^2 &= \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1} e^{-\frac{2|k|^{p+1}}{1+|k|^{p+1}}t} \left|\widehat{u}_k(0)\right|^2 \\ &\leq \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1} e^{-t} \left|\widehat{u}_k(0)\right|^2 \\ &= e^{-t} \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1} \left|\widehat{u}_k(0)\right|^2 \\ &= e^{-t} |u_0|_{\frac{p+1}{2}}^2. \end{aligned} \tag{3.10}$$

The proof is complete. □

We now deal with the nonlinear equation (1.2), that is, $u_t + L_p u_t + u_x + uu_x + M_\alpha u = 0$. We can find similar kind of decreasing properties but less explicit than in the linear case.

Theorem 3.2 *If* $u_0(x) \in \dot{H}^{\frac{p+1}{2}}(\mathbb{T})$, *then the unique solution* $u(t,x)$ *of the periodic initial value problem* (1.2) *satisfies*

$$\lim_{t\to+\infty} \left|u(t)\right|_{\frac{p+1}{2}}^2 = 0. \tag{3.11}$$

Proof Since $\int_\mathbb{T} uu_x\,dx = 0$, $\int_\mathbb{T} u^2 u_x\,dx = 0$, and

$$\begin{aligned} \int_\mathbb{T} u(u_t + L_p u_t)\,dx &= \pi \frac{d}{dt}\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\widehat{u}_k(t)\right|^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{p+1} \left|\widehat{u}_k(t)\right|^2 \right) \\ &= \pi \frac{d}{dt} \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(1+|k|^{p+1}\right) \left|\widehat{u}_k(t)\right|^2, \end{aligned}$$

$$\int_{\mathbb{T}} uM_\alpha u\,dx = 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2\alpha}\left|\widehat{u}_k(t)\right|^2 = 2\pi\left|u(t)\right|_\alpha^2.$$

The equation and zero mean condition in (1.2) imply that

$$\frac{1}{2}\frac{d}{dt}\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(t)\right|^2 + \left|u(t)\right|_\alpha^2 = 0, \tag{3.12}$$

hence

$$\frac{d}{dt}\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(t)\right|^2 = -2\left|u(t)\right|_\alpha^2 \le 0. \tag{3.13}$$

It implies that $\sum_{k\in\mathbb{Z}\setminus\{0\}}(1+|k|^{p+1})|\widehat{u}_k(t)|^2$ is decreasing in t, so we have

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(t)\right|^2 \le \sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(0)\right|^2, \quad \forall t\ge 0. \tag{3.14}$$

Therefore,

$$\begin{aligned}\left|u(t)\right|_{\frac{p+1}{2}}^2 &= \sum_{k\in\mathbb{Z}\setminus\{0\}}|k|^{p+1}\left|\widehat{u}_k(t)\right|^2\\ &\le \sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(0)\right|^2\\ &\le 2|u_0|_{\frac{p+1}{2}}^2, \quad \forall t\ge 0.\end{aligned} \tag{3.15}$$

Equation (3.15) and $u_0(x)\in\dot{H}^{\frac{p+1}{2}}$ lead to $u(t)\in\dot{H}^{\frac{p+1}{2}}$ and then $u(t)\in\dot{H}^\alpha$ for $\frac{1}{2}<\alpha\le\frac{p+1}{2}$.

On the other hand, (3.13) implies that $\sum_{k\in\mathbb{Z}\setminus\{0\}}(1+|k|^{p+1})|\widehat{u}_k(t)|^2$ is decreasing in t and bounded below by zero, then the limit

$$\lim_{t\to+\infty}\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(t)\right|^2$$

exists, we denote it by A.

Denote

$$\lim_{t\to+\infty}\left|u(t)\right|_\alpha^2 = B.$$

If $B>0$, for large enough t, we have $|u(t)|_\alpha^2 > B/2$, then there is a constant $T>0$ such that

$$\frac{d}{dt}\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(t)\right|^2 < -B, \quad t>T,$$

hence

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(t)\right|^2 < \sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(T)\right|^2 - B(t-T),$$

and then

$$\lim_{t\to+\infty}\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(t)\right|^2\leq-\infty.$$

This contradiction leads to $\lim_{t\to+\infty}|u(t)|_\alpha^2 = B = 0$, that is,

$$\lim_{t\to+\infty}\sum_{k\in\mathbb{Z}\setminus\{0\}}|k|^{2\alpha}\left|\widehat{u}_k(t)\right|^2=0,$$

then $\forall k\in\mathbb{Z}$, $\lim_{t\to+\infty}|\widehat{u}_k(t)|^2=0$, therefore we have

$$\lim_{t\to+\infty}\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(1+|k|^{p+1}\right)\left|\widehat{u}_k(t)\right|^2=A=0$$

and

$$\lim_{t\to+\infty}\sum_{k\in\mathbb{Z}\setminus\{0\}}|k|^{p+1}\left|\widehat{u}_k(t)\right|^2=0,\quad i.e.\ \lim_{t\to+\infty}\left|u(t)\right|_{\frac{p+1}{2}}^2=0.$$

The proof is complete. □

Remark 3.1 This paper gives the local well-posedness for the subcritical index $\alpha>\frac{1}{2}$. The interesting case would be to consider the supercritical case $0<\alpha<\frac{1}{2}$ and the critical case $\alpha=\frac{1}{2}$. In the supercritical case $0<\alpha<\frac{1}{2}$, there will be less dissipation, so the dispersive part comes to play a principal role. In the cases $0<\alpha<\frac{1}{2}$ and $\alpha=\frac{1}{2}$, $\dot{H}^\alpha(\mathbb{T})$ is not an algebra, we must find another way to establish the estimates on the nonlinear term. We will consider the supercritical case in future work.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JK carried out the nonlocal operator, YG carried out the Benjamin-Bona-Mahony equation, and YT carried out the well-posedness. All authors read and approved the final manuscript.

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