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The first and second expansion of large solutions for quasilinear elliptic equations with weight functions

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Abstract

By the *Karamata* regular variation theory and comparison principle, we establish the boundary behavior of positive weak solutions for the problem

$$\Delta_p u = b(x)f(u), \quad x \in \Omega, \qquad u|_{\partial\Omega} = \infty,$$

where Ω is a bounded domain with smooth boundary in R^N, the weight $b(x) \in C^\alpha(\overline{\Omega})$, which may be vanishing on the boundary and rapidly varying near the boundary, and the nonlinearity f may be rapidly varying at infinity. For the case $f(s) = s^m \pm f_1(s)$ with sufficiently large s, where $m > p - 1$ and f_1 is normalized regularly varying at infinity with index $m_1 \in (0, p - 1)$, we show the influence of the geometry of Ω on the boundary behavior of solutions. Finally, we prove the existence and uniqueness of the solution for the problem.

MSC: 35B40; 35H30; 35J70; 35J65

Keywords: quasilinear elliptic equation; large solution; blow-up rate; first and second expansion

1 Introduction and main results

In this paper, we consider the first and second expansions of positive weak large solutions near the boundary for the quasilinear elliptic problem of the form

$$\Delta_p u = b(x)f(u), \quad x \in \Omega, \qquad u|_{\partial\Omega} = \infty, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(p > 2)$, $\Omega \subset R^N$ $(N \geq 2)$ is a bounded domain with C^4-smooth boundary, $b(x)$ satisfies (b_1) and (b_2), and f satisfies (f_1), (f_2), and (f_3), where

(b_1) $b(x) \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, and $b(x)$ is positive in Ω;
(b_2) there exist $k \in \Lambda$ and $b_0 \in R$ such that

$$\lim_{d(x)\to 0} \frac{b(x)}{K^{p-2}(d(x))k^p(d(x))} = b_0,$$

where Λ denotes the set of all positive nondecreasing functions in $C^1(0,\delta_0)$ ($\delta_0 > 0$) such that

$$\lim_{t\to 0+}\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) := C_k \in [0,\infty), \quad K(t) = \int_0^t k(s)\,ds;$$

(f_1) $f \in C^1[0,\infty), f(0) = 0, f$ is increasing in $(0,\infty)$;
(f_2) $\int_1^\infty f^{-\frac{1}{p-1}}(v)\,dv < \infty$;
(f_3) there exists $C_f > 0$ such that $\lim_{s\to\infty} f'(s)\int_s^\infty \frac{dv}{f(v)} = C_f$.

Note that, some basic examples for $k \in \Lambda$ can be given as follows:
(1) $k(t) = t^{\frac{\alpha}{2}}$, $\alpha > 0$, $C_k = 2(2+\alpha)^{-1}$;
(2) $k(t) = e^{-t^{-\alpha}}$, $\alpha > 0$, $C_k = 0$;
(3) $k(t) = \frac{1}{(-\ln t)^\alpha}$, $\alpha > 0$, $C_k = 1$;
(4) $k(t) = e^{-e^{t^{-\alpha}}}$, $\alpha > 0$, $C_k = 0$;
(5) $k(t) = (\ln(1+t))^\alpha$, $\alpha > 0$, $C_k = (1+\alpha)^{-1}$.

p-Laplacian equations like (1.1) usually occur in the study of the generalized reaction-diffusion theory, non-Newtonian fluid theory, non-Newtonian filtration, and the turbulent flow of a gas in porous medium. In the non-Newtonian theory, the quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids, and those with $p < 2$ are called pseudoplastics. If $p = 2$, then they are Newtonian fluids. The p-Laplacian operator also appears in the study of torsional creep (elastic for $p = 2$ and plastic for $p < 2$; see [1]), flow through porous media ($p = \frac{3}{2}$; see [2]), and glacial sliding ($p \in (1, \frac{3}{4}]$; see [3]).

We are concerned with the positive weak large solutions for problem (1.1). By a solution for problem (1.1) we understand a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ that satisfies $\Delta_p u = b(x)f(u)$ in the weak sense and $u(x) \to \infty$ as $d(x) = \mathrm{dist}(x,\partial\Omega) \to 0$. Sometimes, the solution is also called a large solution, an explosive solution, or a boundary blow-up solution.

The study of large solutions started from the work of Bieberbach [4] for the case $b(x) = 1$, $f(u) = e^u$, $p = 2$, and $N = 2$, which plays an important role in the theories of Riemannian surfaces of negative constant curvatures and automorphic functions. More exactly, if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has a constant Gaussian curvature $-b^2$, then $\Delta u = b^2e^{2u}$. Rademacher [5] extended the results in [4] to the three-dimensional space. Later, C Bandle *et al.* discussed the existence, uniqueness, and accurate estimate of boundary behavior of large solutions for problem (1.1) with $p = 2$ and $b(x), f(u)$ satisfying some proper conditions and obtained some better results dealing with a gradient term (see [4–40] and the references therein). Also, some results in [4–40] have been extended to $p > 2$ (see [41–54]). Recently, boundary blow-up problems have been applied to Liouville theorems for logistic-like equations in R^N in [13], the analysis of blow-up for a parabolic equation with a nonlinear boundary condition in [14], and the characterization of the long-time behavior of positive solutions for the parabolic equations in [15, 16].

However, Cirstea and Rădulescu [17] first introduced *Karamata* regular variation theory approaching to study the uniqueness and asymptotic behavior of boundary blow-up solutions, which enables us to obtain some qualitative behavior of the boundary blow-up solutions in a general framework. The asymptotic behavior of the boundary blow-up solutions near the boundary has been investigated by many researchers (see [17–39] and their references). It is well known that the first-order asymptotic expansion of the solution $u(x)$ in terms of $d(x)$ is independent of the geometry of the domain, whereas the second-order asymptotic expansion of the solution $u(x)$ depends linearly on the mean curvature of the boundary of Ω. There have been many results about the first expansion of large solution for problem (1.1) with $p = 2$ and $b(x), f(u)$ satisfying some proper conditions under different regularity boundary conditions (see [21–28]) and the second expansion (see[29–34] and references therein). Bandle and Marcus [29] first studied the influence of the geometry of Ω on the boundary behavior of the unique radially symmetric solution for problem (1.1) in a ball or an annulus when $f(u)$ is of power form. Their results were extended by Bandle, Anedda, Porru *et al.* to more general boundary smooth domains, weights, and nonlinearities (see [30–34]). Specially, Cirstea *et al.* [17, 35–40] used the *Karamata* regular variation theory, nonlinear transformations, the perturbed method, the upper and lower solution method, and localization method to establish the first and second expansion of large solutions for problem (1.1) with $p = 2$, and Cirstea and Rădulescu [17] first introduced the set Λ. Recently, some results in [17, 35–40] have been extended to $p > 2$. For instance, Huang *et al.* [46–48] studied the existence and the first and second expansions of weak solutions when $b(x)$ and $f(u)$ satisfy some suitable conditions, which are different from the conditions in our paper. For more results about the p-Laplacian equations, we refer to [49–53] and references therein.

Inspired by the above works, in this paper, we introduce the constants C_f and C_k to get the asymptotic expansion of solutions for problem (1.1). In particular, when $b(x) = K^{p-2}(d(x))k(d(x))$ near the boundary and $f(s) = s^m \pm f_1(s)$ for sufficiently large s, where $m > p - 1$, and f_1 satisfies

(f_4) there exists $m_1 \in (0, p - 1)$ such that

$$\lim_{s\to\infty} \frac{sf_1'(s)}{f_1(s)} = m_1,$$

we show the influence of the geometry of Ω on the boundary behavior of solutions for problem (1.1). Finally, we prove the existence and uniqueness of the solution for problem (1.1). More precisely, we obtain the following results.

Theorem 1 *Let f satisfy* (f_1)-(f_3), *and b satisfy* (b_1)-(b_2). *Suppose that $1 < C_f < \frac{p-1}{p-2}$ in* (f_3) *and*

$$2C_f + C_k > 2 + \frac{p-2}{p-1} C_f C_k. \tag{1.2}$$

Then, for any solution u of problem (1.1), *we have*

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi(\gamma K^2(d(x)))} = 1, \tag{1.3}$$

where ϕ is uniquely defined by

$$\int_{\phi(t)}^{\infty} f^{-\frac{1}{p-1}}(\mu)\, d\mu = t, \quad \forall t > 0 \tag{1.4}$$

and

$$\gamma = \frac{1}{2}\left(\frac{b_0}{p-1}\cdot\frac{(p-1)(C_f-1)-C_f}{(2-C_k)(p-1)(C_f-1)-C_fC_k}\right)^{\frac{1}{p-1}}. \tag{1.5}$$

Remark 1 By (1.2) one can see that if $C_f > 1$, then C_k can be equal to zero and if $C_k > 0$, then C_f can be equal to 1.

Theorem 2 *Let f satisfy* (f_1), $f(s) = s^m \pm f_1(s)$ *for sufficiently large s*, $m > p-1$, f_1 *satisfy* (f_4), *and $b(x) = K^{p-2}(d(x))k(d(x))$ near the boundary, where k satisfies*

(k_1) $k \in C[0,a] \cap C^2(0,a]$ *for some* $a > 0$, $k(t) > 0$, $k'(t) > 0$, $\forall t \in (0,a]$, *and* $k(0) = 0$;
(k_2) $k \in \Lambda$ *with* $C_k > 0$;
(k_3) $\lim_{t\to 0} \frac{d^2}{dt^2}(\frac{K(t)}{k(t)}) = 0$.

The following two results hold:

(1) *if* $m + 1 - p > 2m_1$, *then, in a sufficiently small neighborhood of* $\partial\Omega$, *for any solution u of problem* (1.1), *we have*

$$u(x) = C_1\bigl(K\bigl(d(x)\bigr)\bigr)^{-\frac{2(p-1)}{m+1-p}}\left(1 + C_2(N-1)H(\overline{x})\frac{K(d(x))}{k(d(x))} + o\bigl(d(x)\bigr)\right), \tag{1.6}$$

where

$$C_1 = \left(\left(\frac{2(p-1)}{m+1-p}\right)^p\left(p-1+\frac{(m+1-p)C_k}{2}\right)\right)^{\frac{1}{m+1-p}},$$
$$C_2 = \frac{m+1-p}{2(p-1)(m-1)+(m+1)(m+1-p)C_k}.$$

(2) *if* $m + 1 - p \le 2m_1$ *and* $k(t) = t^{\frac{\theta}{2}}$ *with* $\theta > 0$ *such that* $\frac{\theta}{2+\theta} > \frac{2m_1-(m+p-1)}{m+1-p}$, *then* (1) *still holds.*

Remark 2 Some basic examples of k that satisfy (k_1)-(k_3) can be given as follows:

(1) $k(t) = t^{\frac{\alpha}{2}}$, $\alpha > 0$, where $C_k = \frac{2}{2+\alpha}$;
(2) $k(t) = e^{t^\alpha} - 1$, $\alpha > 1$, where $C_k = \frac{1}{1+\alpha}$;
(3) $k(t) = \ln(1 + t^\alpha)$, $\alpha > 1$, where $C_k = \frac{1}{1+\alpha}$.

Remark 3 If $k(t) = (\ln(1 + t))^\alpha$, $\alpha > 0$, then $\lim_{t\to 0+} \frac{d^2}{dt^2}(\frac{K(t)}{k(t)}) = \frac{\alpha}{2(1+\alpha)(2+\alpha)}$. In this case, $k(t)$ does not satisfy (k_3).

This paper is organized as follows. In Section 2, we present some notation and results in regular variation theory. Theorems 1 and 2 will be proved in Section 3. Finally, we prove the existence and uniqueness of the solution for problem (1.1) in Appendices A.1 and A.2.

2 Preliminary results

2.1 Properties of regularly varying function

Karamata regular variation theory was established by Karamata in 1930 and is a basic tool in stochastic process. In 1970, Haan improved the results, which have been applied in stochastic process, analytical function theory, integral functions, integral transform and asymptotic estimation of an integral sequence (see [55–57]).

In this section, we recall some basic definitions and qualities in regular variation theory.

Definition 1 A positive measurable function f defined on $[a,\infty)$ for some $a>0$ is called *regularly varying at infinity* with index ρ (written as $f\in RV_\rho$) if for each $\xi>0$ and some $\rho\in R$,

$$\lim_{t\to\infty}\frac{f(\xi t)}{f(t)}=\xi^\rho. \tag{2.1}$$

In particular, when $\rho=0$, f is called *slowly varying at infinity*. Clearly, if $f\in RV_\rho$, then $L(s):=\frac{f(s)}{s^\rho}$ is slowly varying at infinity.

Definition 2 A positive measurable function f defined on $[a,\infty)$ for some $a>0$ is called *rapidly varying at infinity* if for each $\rho>1$,

$$\lim_{s\to\infty}\frac{f(s)}{s^\rho}=\infty. \tag{2.2}$$

Some basic examples of slowly varying functions at infinity are listed as follows:

(1) every measurable function on $[a,\infty)$ which has a positive limit at infinity;
(2) $(\ln t)^s$ and $(\ln(\ln t))^s$, $s\in R$;
(3) $e^{(\ln t)^s}$, $0<s<1$.

Some basic examples of rapidly varying functions at infinity are given as follows:

(1) e^t and e^{e^t};
(2) $e^{e^{(\ln t)^s}}$, $e^{e^{t^s}}$ and $e^{e^{t^s}}$, $s>0$;
(3) $t^\gamma e^{(\ln t)^q}$ and $(\ln t)^\gamma e^{(\ln t)^q}$, $q>1$, $\gamma\in R$;
(4) $(\ln t)^\gamma e^{t^q}$ and $t^\gamma e^{t^\gamma}$, $q>0$, $\gamma\in R$.

We see that a positive measurable function h defined on $(0,a)$ for some $a>0$ is *regularly varying at zero* with index σ (written as $g\in RVZ_\sigma$) if $t\to g(\frac{1}{t})$ belongs to $RV_{-\sigma}$. Similarly, g is called *rapidly varying at zero* if $t\to g(\frac{1}{t})$ is rapidly varying at infinity.

Proposition 1 (Uniform convergence theorem) *If $f\in RV_\rho$, then* (2.1) *holds uniformly for $\xi\in[c_1,c_2]$ with $0<c_1<c_2$. Moreover, if $\rho<0$, then uniform convergence holds on intervals of the form (a_1,∞) with $a_1>0$; if $\rho>0$, then uniform convergence holds on intervals $(0,a_1]$, provided that f is bounded on $(0,a_1]$ for all $a_1>0$.*

Proposition 2 (Representation theorem) *A function L is slowly varying at infinity if and only if it can be written in the form*

$$L(s)=\varphi(s)\exp\left(\int_{a_1}^s\frac{y(t)}{t}\,dt\right),\quad s\ge a_1 \tag{2.3}$$

for some $a_1\ge a$, where the functions φ and y are measurable and as $s\to\infty$, $y(s)\to 0$ and $\varphi(s)\to c_0>0$.

We say that

$$\hat{L}(s)=c_0\exp\left(\int_{a_1}^s\frac{y(t)}{t}\,dt\right),\quad s\ge a_1, \tag{2.4}$$

is normalized slowly varying at infinity *and*

$$f(s) = c_0 s^{\rho} \hat{L}(s), \quad s \geq a_1, \tag{2.5}$$

is normalized regularly varying at infinity *with index* ρ (*written as* $f \in NRV_\rho$).

Similarly, g is called *normalized regularly varying at zero* with index σ (written as $g \in NRVZ_\sigma$) if $t \to g(\frac{1}{t})$ belongs to $NRV_{-\sigma}$. A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1, \infty) \quad \text{for some } a_1 > 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \rho. \tag{2.6}$$

Proposition 3 *If functions L_1, L_2 are slowly varying at infinity, then*

(1) *L^σ for every $\sigma \in R$, $c_1L + c_2L_1$ ($c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \circ L_1$ (if $L_1(t) \to +\infty$ as $t \to +\infty$) are also slowly varying at infinity*;

(2) *for every $\theta > 0$, $t^\theta L(t) \to +\infty$ and $t^{-\theta} L(t) \to 0$ as $t \to \infty$*;

(3) *for $\rho \in R$, $\frac{\ln(L(t))}{\ln t} \to 0$ and $\frac{\ln(t^\rho L(t))}{\ln t} \to \rho$ as $t \to +\infty$.*

Proposition 4 *If $f_1 \in RV_{\rho_1}, f_2 \in RV_{\rho_2}$ with $\lim_{t\to\infty} f_2(t) = \infty$, then $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$.*

Proposition 5 (Asymptotic behavior) *If a function L is slowly varying at infinity, then for $a \geq 0$ and $t \to \infty$, we have*

(1) $\int_a^t s^\beta L(s)\,ds \cong (\beta + 1)^{-1} t^{1+\beta} L(t)$ *for* $\beta > -1$;

(2) $\int_t^\infty s^\beta L(s)\,ds \cong (-\beta - 1)^{-1} t^{1+\beta} L(t)$ *for* $\beta < -1$.

Proposition 6 (Asymptotic behavior) *If a function H is slowly varying at infinity, then for $a > 0$ and $t \to 0^+$, we have*

(1) $\int_0^t s^\beta H(s)\,ds \cong (\beta + 1)^{-1} t^{1+\beta} H(t)$ *for* $\beta > -1$;

(2) $\int_t^\infty s^\beta H(s)\,ds \cong (-\beta - 1)^{-1} t^{1+\beta} H(t)$ *for* $\beta < -1$.

2.2 Auxiliary results

In this section, we give some auxiliary results, which will be used in Theorems 1 and 2.

Lemma 1 [38, 40]

(I) *If $k \in \Lambda$, then we have*:

  (1) $\lim_{t\to 0+} \frac{K(t)}{k(t)} = 0$;

  (2) $C_k \in [0, 1]$ *and* $\lim_{t\to 0+} \frac{K(t)k'(t)}{k^2(t)} = 1 - C_k$.

(II) (k_1)-(k_3) *implies that*

  (3) $\lim_{t\to 0+} (\frac{K(t)k'(t)}{k^2(t)} - (1 - C_k)) \frac{k(t)}{K(t)} = 0$.

Lemma 2 [40] *Let f satisfy* (f_1), (f_2), *and* (f_3). *Then*

(1) $C_f \in [1, \infty)$;

(2) *there exists $S_0 > 0$ such that $\frac{f(s)}{s^q}$ is increasing in $[S_0, \infty)$, where $q \in (p - 1, \frac{C_f}{C_f - 1})$ for $1 < C_f < \frac{p-1}{p-2}$ and $q \in (p - 1, \infty)$ for $C_f = 1$*;

(3) *f satisfies the Keller-Osserman condition*

$$\int_1^\infty \frac{dt}{(p'F(t))^{\frac{1}{p}}} < \infty, \quad F(t) = \int_0^t f(s)\,ds, \tag{2.7}$$

where $\frac{1}{p'} + \frac{1}{p} = 1$;

(4) (f_3) *holds for* $C_f > 1$ *if and only if* $f \in NRV_{\frac{C_f}{C_f-1}}$;

(5) $C_f = 1$, *and* f *is rapidly varying at infinity.*

Lemma 3 *Let* f *satisfy* (f_1)-(f_3), *and* ϕ *be the solution for the problem*

$$\int_{\phi(t)}^{\infty} f^{-\frac{1}{p-1}}(s)\,ds = t, \quad \forall t > 0.$$

Then we have

(1) $-\phi'(t) = f^{\frac{1}{p-1}}(\phi(t))$, $\phi(t) > 0$, $t > 0$, $\phi(0) := \lim_{t\to 0+}\phi(t) = +\infty$, $\phi''(t) = \frac{1}{p-1} f^{\frac{2}{p-1}-1}(\phi(t))f'(\phi(t))$;

(2) $\lim_{t\to 0+}\frac{t\phi'(t)}{\phi(t)} = -\frac{(p-1)(C_f-1)}{C_f-(p-1)(C_f-1)}$, *i.e.*, $\phi \in NRVZ_{-\frac{(p-1)(C_f-1)}{C_f-(p-1)(C_f-1)}}$;

(3) $\lim_{t\to 0+}\frac{t\phi''(t)}{\phi'(t)} = -\frac{C_f}{C_f-(p-1)(C_f-1)}$, *i.e.*, $-\phi' \in NRVZ_{-\frac{C_f}{C_f-(p-1)(C_f-1)}}$;

(4) $\lim_{t\to 0+}\frac{\ln(\phi(t))}{-\ln t} = \frac{(p-1)(C_f-1)}{C_f-(p-1)(C_f-1)}$, $\lim_{t\to 0+}\frac{\ln(\phi'(t))}{-\ln t} = \frac{(p-1)C_f}{C_f-(p-1)(C_f-1)}$.

Proof (1) By the definition of ϕ and a direct calculation we can show (1).

(2) It follows from Proposition 5 that

$$\begin{aligned}
\lim_{t\to 0+}\frac{t\phi'(t)}{\phi(t)} &= -\lim_{t\to 0+}\frac{tf^{\frac{1}{p-1}}(\phi(t))}{\phi(t)} \\
&= -\lim_{u\to\infty}\frac{(f(u))^{\frac{1}{p-1}}}{u}\int_u^\infty \frac{dv}{f^{\frac{1}{p-1}}(v)} \\
&= -\lim_{u\to\infty} u^{\frac{C_f}{(p-1)(C_f-1)}-1}\hat{L}^{\frac{1}{p-1}}(u)\int_u^\infty v^{-\frac{C_f}{(p-1)(C_f-1)}}\hat{L}^{-\frac{1}{p-1}}(v)\,dv \\
&= -\left(\frac{C_f}{(p-1)(C_f-1)}-1\right)^{-1} \\
&\quad\cdot\lim_{u\to\infty} u^{\frac{C_f}{(p-1)(C_f-1)}-1}\hat{L}^{\frac{1}{p-1}}(u)u^{-\frac{C_f}{(p-1)(C_f-1)}+1}\hat{L}^{-\frac{1}{p-1}}(u) \\
&= -\frac{(p-1)(C_f-1)}{C_f-(p-1)(C_f-1)}.
\end{aligned}$$

(3) (f_3) implies that

$$\begin{aligned}
\lim_{t\to 0+}\frac{t\phi''(t)}{\phi'(t)} &= -\frac{1}{p-1}\lim_{t\to 0+} tf^{\frac{1}{p-1}-1}\big(\phi(t)\big)f'\big(\phi(t)\big) \\
&= -\frac{1}{p-1}\cdot\lim_{u\to\infty} f^{\frac{1}{p-1}-1}(u)f'(u)\int_u^\infty\frac{dv}{f^{\frac{1}{p-1}}(v)} \\
&= -\frac{1}{p-1}\cdot\lim_{u\to\infty}\frac{uf'(u)}{f(u)}\frac{f^{\frac{1}{p-1}}(u)}{u}\int_u^\infty\frac{dv}{f^{\frac{1}{p-1}}(v)} \\
&= -\frac{1}{p-1}\cdot\frac{(p-1)(C_f-1)}{C_f-(p-1)(C_f-1)}\cdot\frac{C_f}{C_f-1} \\
&= -\frac{C_f}{C_f-(p-1)(C_f-1)}.
\end{aligned}$$

The last result (4) follows from (2)-(3) and Proposition 3(3). □

Lemma 4 *Under the hypotheses in Theorem* 2, *let* $k \in \Lambda$ *and*

$$\Phi(t) = \big(K(t)\big)^{-\frac{2(p-1)}{m+1-p}}\big(1 + h(t)\big)$$

with $\lim_{t\to 0} h(t) = 0$. *The following two results hold*:

(1) *if* $m + 1 - p > 2m_1$, *then* $k(t)(K(t))^{\frac{(m+p-1)(p-1)}{m+1-p}} f_1(\Phi(t)) \to 0$ *as* $t \to 0$;

(2) *if* $m + 1 - p \le 2m_1$ *and* $k(t) = t^{\frac{\theta}{2}}$ *with* $\theta > 0$ *such that* $\frac{\theta}{2+\theta} > \frac{2m_1-(m+p-1)}{m+1-p}$, *then* (1) *still holds*.

Proof We know that $f_1 \in NRV_{m_1}$ by (f_4) with $m_1 \in (0, m)$ and $f_1(s) = c_0 s^{m_1}\hat{L}(s)$ for sufficiently large s, where $\hat{L}$ is normalized slowly varying at infinity, and $c_0 > 0$.

Let

$$\Phi_1(t) = \big(K(t)\big)^{-\frac{2(p-1)}{m+1-p}}.$$

We see that $\hat{L}(\Phi_1(t))$ is also normalized slowly varying at zero, and by a similar argument as in Propositions 1 and 3(2), for every $\beta > 0$ and $t \to 0+$, we have

$$\big(\Phi_1(t)\big)^{\beta}\hat{L}\big(\Phi_1(t)\big) \to 0 \quad \text{and} \quad \hat{L}\big(\Phi(t)\big)\big(\hat{L}\big(\Phi_1(t)\big)\big)^{-1}\big(1 + h(t)\big)^{m_1} \to 0. \tag{2.8}$$

(1) If $m + 1 - p > 2m_1$, let $2\beta \in (0, m + p - 1 - 2m_1)$, then by (2.8) we have

$$\begin{aligned}
&k(t)\big(K(t)\big)^{\frac{(m+p-1)(p-1)}{m+1-p}} f_1\big(\Phi(t)\big)\\
&\quad = k(t)\big(K(t)\big)^{\frac{(m+p-1)(p-1)}{m+1-p}} c_0\big(\Phi(t)\big)^{m_1}\hat{L}\big(\Phi(t)\big)\\
&\quad = c_0 k(t)\big(K(t)\big)^{\frac{(m+p-1)(p-1)-2(p-1)m_1-2(p-1)\beta}{m+1-p}}\big(\Phi_1(t)\big)^{\beta}\hat{L}\big(\Phi_1(t)\big)\\
&\qquad \cdot \hat{L}\big(\Phi(t)\big)\big(\hat{L}\big(\Phi_1(t)\big)\big)^{-1}\big(1 + h(t)\big)^{m_1} \to 0 \quad \text{as } t \to 0.
\end{aligned}$$

(2) If $m + 1 - p \le 2m_1$ and $k(t) = t^{\frac{\theta}{2}}$ with $\theta > 0$ such that $\frac{\theta}{2+\theta} > \frac{2m_1-(m+p-1)}{m+1-p}$ and $\frac{2(p-1)m_1}{m+1-p} \in (0, \frac{\theta}{2+\theta} - \frac{2m_1-(m+p-1)}{m+1-p})$, then we get

$$\begin{aligned}
&k(t)\big(K(t)\big)^{\frac{(m+p-1)(p-1)}{m+1-p}} f_1\big(\Phi(t)\big)\\
&\quad = ct^{\sigma}\big(\Phi_1(t)\big)^{\beta}\hat{L}\big(\Phi_1(t)\big)\hat{L}\big(\Phi(t)\big)\big(\hat{L}\big(\Phi_1(t)\big)\big)^{-1}\big(1 + h(t)\big)^{m_1} \to 0 \quad \text{as } t \to 0,
\end{aligned}$$

where

$$\begin{aligned}
c &= c_0\left(\frac{\theta + 2}{2}\right)^{\frac{(p-1)(2m_1+2\beta-m+1-p)}{m+1-p}},\\
\sigma &= \frac{\theta(m + 1 - p) - (2 + \theta)(p - 1)(2m_1 + 2\beta - m + 1 - p)}{2(m + 1 - p)} > 0.
\end{aligned}$$

□

3 Proofs of main results

In this section, we mainly prove Theorem 1 and Theorem 2. For the proofs, we use the upper and lower solution method. One critical step is to set up the comparison principle. Thus, we first give the comparison principle in general form for quasilinear elliptic equations.

Lemma 5 [54] *Suppose that D is a bounded domain in R^N and $a(x)$ and $\beta(x)$ are continuous functions on D with $\|a\|_{L^\infty(D)} < \infty$, $\beta(x) \geq 0$, $\beta(x) \not\equiv 0$ for $x \in D$. Let $u_1, u_2 \in C^1(D)$ be positive in D and satisfy in the sense of distributions*

$$-\Delta_p u_1 - a(x)u_1^{p-1} + \beta(x)g(u_1) \geq 0 \geq -\Delta_p u_2 - a(x)u_2^{p-1} + \beta(x)g(u_2),$$
$$\overline{\lim_{d(x,\partial\Omega)\to 0}} \left(u_2^{p-1} - u_1^{p-1}\right) \leq 0,$$

where $g \in C^0([0,\infty))$, and $\frac{g(s)}{s^{p-1}}$ is increasing for

$$s \in \left(\inf_D\{u_1, u_2\}, \sup_D\{u_1, u_2\}\right).$$

Then $u_1 \geq u_2$ in D.

Next, fix $\epsilon > 0$. For all $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}$. Since Ω is smooth, there exists $\delta_0 > 0$ such that $d \in C^2(\Omega_{\delta_0})$ and

$$\left|\nabla d(x)\right| = 1, \qquad \Delta d(x) = -(N-1)H(\overline{x})d(x) + o(1), \quad \forall x \in \Omega_{\delta_0}. \tag{3.1}$$

Proof of Theorem 1 Let $\epsilon \in (0, \frac{b_0}{4})$ and $\gamma_1 = \gamma - \frac{2\epsilon\gamma}{b_0}$, $\gamma_2 = \gamma + \frac{2\epsilon\gamma}{b_0}$. We have

$$\frac{\gamma}{2} < \gamma_1 < \gamma < \gamma_2 < 2\gamma.$$

Set

$$d_1(x) = d(x) - \rho, \qquad d_2(x) = d(x) + \rho, \tag{3.2}$$
$$\overline{u}_\epsilon = \phi\left(\gamma_1 K^2\left(d_1(x)\right)\right), \quad x \in D_\rho^- \quad \text{and} \quad \underline{u}_\epsilon = \phi\left(\gamma_2 K^2\left(d_2(x)\right)\right), \quad x \in D_\rho^+. \tag{3.3}$$

Then, for $x \in D_\rho^-$,

$$\begin{aligned}
&\Delta_p \overline{u}_\epsilon - b(x)f(\overline{u}_\epsilon) \\
&\quad = (p-1)\left|2\gamma_1\phi'\left(\gamma_1 K^2\left(d_1(x)\right)\right)K\left(d_1(x)\right)k\left(d_1(x)\right)\right|^{p-2} \\
&\qquad \cdot \left(\phi''\left(\gamma_1 K^2\left(d_1(x)\right)\right)4\gamma_1^2 K^2\left(d_1(x)\right)k^2\left(d_1(x)\right)\right. \\
&\qquad + \phi'\left(\gamma_1 K^2\left(d_1(x)\right)\right)2\gamma_1 k^2\left(d_1(x)\right) + \phi'\left(\gamma_1 K^2\left(d_1(x)\right)\right)2\gamma_1 K\left(d_1(x)\right)k'\left(d_1(x)\right) \\
&\qquad + 2\gamma_1\phi'\left(\gamma_1 K^2\left(d_1(x)\right)\right)K\left(d_1(x)\right)k\left(d_1(x)\right)\Delta d(x)\Big) \\
&\qquad - b_0 K^{p-2}\left(d_1(x)\right)k^p\left(d_1(x)\right)f\left(\phi\left(\gamma_1 K^2\left(d_1(x)\right)\right)\right) \\
&\quad = (p-1)(2\gamma_1)^{p-2} f\left(\phi\left(\gamma_1 K^2\left(d_1(x)\right)\right)\right)K^{p-2}\left(d_1(x)\right)k^p\left(d_1(x)\right) \\
&\qquad \cdot \left(4\gamma_1\left(\frac{\gamma_1 K^2(d_1(x))\phi''(\gamma_1 K^2(d_1(x)))}{-\phi'(\gamma_1 K^2(d_1(x)))} - \frac{C_f}{C_f - (p-1)(C_f - 1)}\right)\right. \\
&\qquad - 2\gamma_1\left(\frac{K(d_1(x))k'(d_1(x))}{k^2(d_1(x))} - (1 - C_k)\right) - 2\gamma_1\frac{K(d_1(x))}{k(d_1(x))}\Delta d(x) \\
&\qquad - \left(\frac{1}{(2\gamma_1)^{p-2}(p-1)}\frac{b(x)}{K^{p-2}(d_1(x))k^p(d_1(x))} - \frac{b_0}{(2\gamma_1)^{p-2}(p-1)}\right) \\
&\qquad \left. - \frac{b_0}{(2\gamma_1)^{p-2}(p-1)} + \frac{4\gamma_1 C_f}{C_f - (p-1)(C_f - 1)} - 2\gamma_1(1 - C_k) - 2\gamma_1\right).
\end{aligned}$$

By (b_1), (b_2), and Lemmas 1-3 we see that there exists $\delta_\epsilon \in (0, \frac{\delta_0}{2})$ sufficiently small such that

(r_1) $(b_0 - \epsilon)K^{p-2}(d(x) - \rho)k^p(d(x) - \rho) \leq (b_0 - \epsilon)K^{p-2}(d(x))k^p(d(x)) < b(x)$, $x \in D_\rho^- = \Omega_{2\delta_\epsilon}/\overline{\Omega}_\rho$; $b(x) < (b_0 + \epsilon)K^{p-2}(d(x))k^p(d(x)) \leq (b_0 + \epsilon)K^{p-2}(d(x) + \rho)k^p(d(x) + \rho)$, $x \in D_\rho^+ = \Omega_{2\delta_\epsilon - \rho}$, where $\rho \in (0, \delta_\epsilon)$,

(r_2)

$$\begin{aligned} &4\gamma\left|\gamma_1\frac{K^2(t)\phi''(\gamma_1K^2(t))}{\phi'(\gamma_1K^2(t))} - \frac{C_f}{C_f-(p-1)(C_f-1)}\right| + 2\gamma\left|\frac{K(t)k'(t)}{k^2(t)} - (1-C_k)\right| \\ &\quad + 2\gamma\frac{K(t)}{k(t)}\left|\Delta d(x)\right| < \epsilon, \quad \forall(x,t) \in \Omega_{2\delta_\epsilon} \times (0, 2\delta_\epsilon), \end{aligned}$$

and by the value of γ_1 in Theorem 1,

$$-\frac{b_0}{(2\gamma_1)^{p-2}(p-1)} + \frac{4\gamma_1C_f}{C_f-(p-1)(C_f-1)} - 2\gamma_1(1-C_k) - 2\gamma_1 = 0.$$

Then

$$\Delta_p\overline{u}_\epsilon - b(x)f(\overline{u}_\epsilon) \leq 0,$$

i.e., $\overline{u}_\epsilon$ is a supersolution of Eq. (1.1) in D_ρ^-.

Similarly, we can show that $\underline{u}_\epsilon$ is a subsolution of Eq. (1.1) in D_ρ^+.

Now let u be an arbitrary solution of problem (1.1) and $C_1(\delta_\epsilon) := \max_{d(x)\geq\delta_\epsilon} u(x)$. We see that

$$u \leq C_1(\delta_\epsilon) + \overline{u}_\epsilon \quad \text{on } \partial D_\rho^-.$$

Since ϕ is decreasing and $\gamma_1 > \gamma$, we have

$$\underline{u}_\epsilon \leq \phi\big(\gamma K(2\delta_\epsilon)\big) := C_2(\delta_\epsilon) \quad \text{whenever } d(x) = 2\delta_\epsilon - \rho$$

and

$$\underline{u}_\epsilon \leq u + C_2(\delta_\epsilon) \quad \text{on } \partial D_\rho^+.$$

It follows by (f_1) and Lemma 5 that

$$u \leq C_1(\delta_\epsilon) + \overline{u}_\epsilon, \quad x \in D_\rho^- \quad \text{and} \quad \underline{u}_\epsilon \leq u + C_2(\delta_\epsilon), \quad x \in D_\rho^+. \tag{3.4}$$

Hence, letting $\rho \to 0$, we have, for $x \in D_\rho^- \cap D_\rho^+$,

$$1 - \frac{C_2(\delta_\epsilon)}{\phi(\gamma_2K^2(d(x)))} \leq \frac{u(x)}{\phi(\gamma_2K^2(d(x)))}$$

and

$$\frac{u(x)}{\phi(\gamma_1K^2(d(x)))} \leq 1 + \frac{C_1(\delta_\epsilon)}{\phi(\gamma_1)K^2(d(x))}.$$

Consequently,

$$1 \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi(\gamma_2 K^2(d(x)))} \quad \text{and} \quad \limsup_{d(x)\to 0} \frac{u(x)}{\phi(\gamma_1 K^2(d(x)))} \le 1.$$

Thus, letting $\epsilon \to 0$, we have

$$1 \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi(\gamma K^2(d(x)))} \quad \text{and} \quad \limsup_{d(x)\to 0} \frac{u(x)}{\phi(\gamma K^2(d(x)))} \le 1,$$

that is,

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi(\gamma K^2(d(x)))} = 1.$$

The proof is complete. □

Proof of Theorem 2 Let $\epsilon \in (0,1)$ and

$$\begin{aligned}
\overline{u}_\epsilon &= C_1\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2(p-1)}{m+1-p}}\left(1 + C_2(N-1)\bigl(H(\overline{x}) + \epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right), \quad x \in D_\rho^-, \\
\underline{u}_\epsilon &= C_1\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2(p-1)}{m+1-p}}\left(1 + C_2(N-1)\bigl(H(\overline{x}) - \epsilon\bigr)\frac{K(d_2(x))}{k(d_2(x))}\right), \quad x \in D_\rho^+.
\end{aligned}$$

Using Lemma 4 and a direct calculation we see that, for $x \in D_\rho^-$,

$$\begin{aligned}
&K^{p-2}\bigl(d_1(x)\bigr)k^p\bigl(d_1(x)\bigr)f\bigl(\overline{u}_\epsilon(x)\bigr) \\
&\quad = K^{p-2}\bigl(d_1(x)\bigr)k^p\bigl(d_1(x)\bigr)\bigl(\overline{u}_\epsilon^m(x) \pm c_0\overline{u}_\epsilon^{m_1}(x)\hat{L}\bigl(\overline{u}_\epsilon(x)\bigr)\bigr) \\
&\quad = K^{p-2}\bigl(d_1(x)\bigr)k^p\bigl(d_1(x)\bigr) \\
&\qquad \cdot \Biggl[C_1^m\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2m(p-1)}{m+1-p}}\left(1 + mC_2(N-1)\bigl(H(\overline{x}) + \epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right) \\
&\qquad \pm c_0 C_1^{m_1}\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2m_1(p-1)}{m+1-p}}\hat{L}\bigl(\overline{u}_\epsilon(x)\bigr) \\
&\qquad \cdot \left(1 + C_2(N-1)\bigl(H(\overline{x}) + \epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right)^{m_1}\Biggr] \\
&\quad = K^{p-2}\bigl(d_1(x)\bigr)k^{p-1}\bigl(d_1(x)\bigr)\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2m(p-1)}{m+1-p}+1} \\
&\qquad \cdot \Biggl[C_1^m\frac{k(d_1(x))}{K(d_1(x))} + mC_1^m C_2(N-1)\bigl(H(\overline{x}) + \epsilon\bigr) \\
&\qquad \pm \bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2(m_1+m)(p-1)}{m+1-p}-1}k(t)\hat{L}\bigl(\overline{u}_\epsilon(x)\bigr) \\
&\qquad \cdot \left(1 + C_2(N-1)\bigl(H(\overline{x}) + \epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right)^{m_1}\Biggr]
\end{aligned}$$

and

$$\begin{aligned}
|\nabla\overline{u}_\epsilon| &= \bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{m+p-1}{m+1-p}}k\bigl(d_1(x)\bigr)\nabla d(x) \\
&\quad \cdot \left(1 + C_2(N-1)(H\overline{x} + \epsilon)\frac{K(d_1(x))}{k(d_1(x))}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ C_1\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2(p-1)}{m+1-p}} C_2(N-1)\bigl(H(\overline{x}+\epsilon)\bigr)\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)} \nabla d(x), \\
\Delta_p \overline{u}_\epsilon ={}& (p-1)|\nabla \overline{u}_\epsilon|^{p-2} \times \Biggl[\frac{2(p-1)(m+p-1)}{(m+1-p)^2} \\
&\cdot \bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{m+p-1}{m+1-p}-1} k^2\bigl(d_1(x)\bigr)\left(1 + C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right) \\
&- \frac{2(p-1)C_1}{(m+1-p)}\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{m+p-1}{m+1-p}} k'\bigl(d_1(x)\bigr) \\
&\cdot \left(1 + C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right) \\
&- \frac{2(p-1)C_1}{(m+1-p)}\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{m+p-1}{m+1-p}} k\bigl(d_1(x)\bigr)\Delta d(x) \\
&\cdot \left(1 + C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right) \\
&- \frac{4(p-1)C_1}{(m+1-p)}\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{m+p-1}{m+1-p}} \\
&\cdot k\bigl(d_1(x)\bigr)C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)} \\
&+ C_1C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2(p-1)}{m+1-p}} \frac{d^2}{dt^2}\left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)} \\
&+ C_1C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{2(p-1)}{m+1-p}} \frac{d}{dt} \\
&\cdot \left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)} \Delta d(x)\Biggr] \\
={}& (p-1)\left(\frac{2(p-1)C_1}{m+1-p}\right)^{p-2} \bigl(K\bigl(d_1(x)\bigr)\bigr)^{-\frac{(p-1)(p-2)+mp}{m+1-p}} k^p\bigl(d_1(x)\bigr) \\
&\cdot \Biggl[\frac{2(p-1)(m+p-1)C_1}{(m+1-p)^2} \\
&+ \frac{2(p-1)(m+p-1)C_1C_2}{(m+1-p)^2}(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))} \\
&- \frac{2(p-1)C_1}{(m+1-p)}\left(\frac{K(d_1(x))k'(d_1(x))}{k^2(d_1(x))} - (1-C_k)\right) \\
&- \frac{2(p-1)C_1}{(m+1-p)}\left(\frac{K(d_1(x))k'(d_1(x))}{k^2(d_1(x))} - (1-C_k)\right) \\
&\cdot C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))} \\
&- \frac{2(p-1)C_1}{(m+1-p)}(1-C_k)\left(1 + C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right) \\
&- \frac{2(p-1)C_1}{(m+1-p)}\frac{K(d_1(x))}{k(d_1(x))}\Delta d(x)\left(1 + C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\right) \\
&- \frac{4(p-1)C_1C_2}{(m+1-p)}\frac{K(d_1(x))}{k(d_1(x))}(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\left(\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)} \right. \\
&\left. - C_k + C_k\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ C_1C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K^2(d_1(x))}{k^2(d_1(x))}\frac{d^2}{dt^2}\left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)} \\
&+ C_1C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\frac{d}{dt}\left(\frac{K^2(t)}{k^2(t)}\right)\bigg|_{t=d_1(x)}\Delta d(x)\biggr] \\
={}& (p-1)\left(\frac{2(p-1)C_1}{m+1-p}\right)^{p-2}\bigl(K(d_1(x))\bigr)^{-\frac{(p-1)(m+p-1)}{m+1-p}}k^{p-1}\bigl(d_1(x)\bigr)\biggl[\frac{k(d_1(x))}{K(d_1(x))} \\
&\cdot\left(\frac{2(p-1)(m+p-1)C_1}{(m+1-p)^2}-\frac{2(p-1)C_1}{(m+1-p)}(1-C_k)\right) \\
&+ C_1C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\biggl(\frac{2(p-1)(m+p-1)}{(m+1-p)^2}-\frac{2(p-1)}{(m+1-p)}(1-C_k) \\
&-\frac{4(p-1)C_k}{(m+1-p)}\biggr)-\frac{2(p-1)C_1}{(m+1-p)}\Delta d(x)-\frac{2(p-1)C_1}{(m+1-p)}\biggl(\frac{K(d_1(x))k'(d_1(x))}{k^2(d_1(x))} \\
&-(1-C_k)\biggr)\frac{k(d_1(x))}{K(d_1(x))}-\frac{2(p-1)C_1C_2}{(m+1-p)}\left(\frac{K(d_1(x))k'(d_1(x))}{k^2(d_1(x))}-(1-C_k)\right) \\
&\cdot(N-1)\bigl(H(\overline{x})+\epsilon\bigr)-\frac{4(p-1)C_1C_2}{(m+1-p)}(N-1)\bigl(H(\overline{x})+\epsilon\bigr) \\
&\cdot\left(\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)}-C_k\right) \\
&-\frac{2(p-1)C_1C_2}{(m+1-p)}\Delta d(x)(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))} \\
&+ C_1C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\frac{d^2}{dt^2}\left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)} \\
&+ C_1C_2(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(d_1(x))}{k(d_1(x))}\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right)\bigg|_{t=d_1(x)}\Delta d(x)\biggr].
\end{aligned}
$$

By the value of C_1, C_2 from Theorem 2 and (3.1) we know that

$$
\begin{aligned}
&\frac{2(p-1)(m+p-1)C_1}{(m+1-p)^2}-\frac{2(p-1)C_1(1-C_k)}{(m+1-p)}=\frac{C_1^m}{p-1}\left(\frac{2(p-1)C_1}{m+1-p}\right)^{-(p-2)}, \\
&\frac{2(p-1)(m+p-1)C_1C_2}{(m+1-p)^2}-\frac{2(p-1)C_1C_2(1-C_k)}{(m+1-p)}-\frac{4(p-1)C_1C_2C_k}{(m+1-p)} \\
&\quad+\frac{2(p-1)C_1}{(m+1-p)}=\frac{mC_1^mC_2}{p-1}\left(\frac{2(p-1)C_1}{m+1-p}\right)^{-(p-2)},
\end{aligned}
$$

and by Lemmas 1 and 4 we get that there exists $\delta_\epsilon\in(0,\frac{\delta}{2})$ sufficiently small such that, for $(x,t)\in\Omega_{2\delta_\epsilon}\times(0,2\delta_\epsilon)$,

$$
\begin{aligned}
&-\frac{2(p-1)C_1}{(m+1-p)}\left(\frac{K(t)k'(t)}{k^2(t)}-(1-C_k)\right)\frac{k(t)}{K(t)} \\
&\quad-\frac{2(p-1)C_1C_2}{(m+1-p)}\left(\frac{K(t)k'(t)}{k^2(t)}-(1-C_k)\right) \\
&\quad\cdot(N-1)\bigl(H(\overline{x})+\epsilon\bigr)-\frac{4(p-1)C_1C_2}{(m+1-p)}(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\left(\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right)-C_k\right) \\
&\quad-\frac{2(p-1)C_1C_2}{(m+1-p)}\Delta d(x)(N-1)\bigl(H(\overline{x})+\epsilon\bigr)\frac{K(t)}{k(t)}
\end{aligned}
$$

$$
\begin{aligned}
&+ C_1C_2(N-1)\big(H(\overline{x})+\epsilon\big)\frac{K(t)}{k(t)}\frac{d^2}{dt^2}\left(\frac{K(t)}{k(t)}\right)\\
&+ C_1C_2(N-1)\big(H(\overline{x})+\epsilon\big)\frac{K(t)}{k(t)}\frac{d}{dt}\left(\frac{K(t)}{k(t)}\right)\Delta d(x) \mp \frac{c_0C_1^m}{p-1}\left(\frac{2(p-1)C_1}{m+1-p}\right)^{-(p-2)}\\
&\cdot \big(K(t)\big)^{-\frac{2(m_1+m)(p-1)}{m+1-p}-1}k(t)\hat{L}\big(\Phi(t)\big)\left(1+C_2(N-1)\big(H(\overline{x})+\epsilon\big)\frac{K(t)}{k(t)}\right)^{m_1}\\
\le{}& \frac{C_1(N-1)}{m+1-p}\epsilon,
\end{aligned}
$$

where $\Phi(t)$ is given in Lemma 4 with $h(t) = \pm C_2(N-1)(H(\overline{x})+1)\frac{K(t)}{k(t)}$.

Thus, for $x \in D_\rho^-$, we have

$$
\begin{aligned}
&\Delta_p\overline{u}_\epsilon(x) - K^{p-2}\big(d_1(x)\big)k^p\big(d_1(x)\big)f\big(\overline{u}_\epsilon(x)\big)\\
&\le (p-1)\left(\frac{2(p-1)C_1}{m+1-p}\right)^{p-2}\big(K\big(d_1(x)\big)\big)^{-\frac{(p-1)(m+p-1)}{m+1-p}}\\
&\quad\cdot k^{p-1}\big(d_1(x)\big)\left(-\frac{2C_1(N-1)}{m+1-p}\epsilon + \frac{C_1(N-1)}{m+1-p}\epsilon\right)\\
&\le 0,
\end{aligned}
$$

i.e., $\overline{u}_\epsilon(x)$ is a supersolution of Eq. (1.1) in D_ρ^-.

Similarly, we can show that $\underline{u}_\epsilon$ is a subsolution of Eq. (1.1) in D_ρ^+.

By (3.4) and letting $\rho \to 0$, we have that, for $x \in D_\rho^- \cap D_\rho^+$,

$$
C_1\big(K\big(d(x)\big)\big)^{-\frac{2(p-1)}{m+1-p}}\left(1+C_2(N-1)\big(H(\overline{x})+\epsilon\big)\frac{K(d(x))}{k(d(x))}\right) + C_1(\delta_\epsilon) \ge u(x), \tag{3.5}
$$

$$
C_1\big(K\big(d(x)\big)\big)^{-\frac{2(p-1)}{m+1-p}}\left(1+C_2(N-1)\big(H(\overline{x})-\epsilon\big)\frac{K(d(x))}{k(d(x))}\right) - C_2(\delta_\epsilon) \le u(x). \tag{3.6}
$$

The proof is complete. □

Appendix

In this appendix, we prove the existence and uniqueness of the solution for problem (1.1).

A.1 The existence of solutions for problem (1.1)

In the first part, we give the existence of solutions for problem (1.1).

Theorem 3 *Let f satisfy* (f_1) *and the Keller-*Osserman *condition* (2.7), *and b satisfy* (b_1). *Then problem* (1.1) *has at least one solution $u \in W^{1,p}_{\mathrm{loc}}(\overline{\Omega})$ satisfying*

$$
u(x) \ge \psi\big(\overline{v}(x)\big), \quad \forall x \in \Omega. \tag{A.1}
$$

Furthermore, if f satisfies $\int_0^1 f^{-\frac{1}{p-1}}(s)\,ds = \infty$, then

$$
u > 0, \quad \forall x \in \Omega, \tag{A.2}
$$

where ψ is the solution of problem (1.4) *and $\overline{v} \in W_0^{1,p}(\Omega)$ is the unique solution for problem*

$$-\Delta_p \overline{v} = b(x), \qquad \overline{v}(x) > 0, \quad x \in \Omega, \qquad \overline{v}|_{\partial\Omega} = 0. \tag{A.3}$$

Remark 4 By Lemma 2(3), we can see that f satisfies the *Keller-Osserman* condition under our hypotheses on f in Theorem 1.

Remark 5 By the similar argument in [46], we show that (f_1) and the *Keller-Osserman* condition imply (f_2). Indeed, if we can prove that there exist two positive numbers ρ and M such that

$$\frac{f^{p'-1}(s)}{s} \geq \rho^{p'} \quad \text{for } s \geq M, \tag{A.4}$$

then it will be done since

$$F(s) = \int_0^s f(t)\,dt \leq sf(s) \leq \frac{f^{p'}(s)}{\rho^{p'}} \quad \text{for } s \geq M,$$

which, in turn, yields $[F(s)]^{-\frac{1}{p}} \geq \frac{\rho^{\frac{1}{p-1}}}{f^{\frac{1}{p-1}}(s)}$, so that the *Keller-Osserman* condition implies (A.4). Then, we will prove (A.4) by contradiction. Assume that there exists an increasing sequence s_j of real numbers such that $\lim_{j\to\infty} s_j = \infty$ and $\frac{f(s_j)}{s_j} < \frac{1}{j}$ for all j. Since f is increasing, we have $f(s) \leq f(s_j)$ for all $s \in [0, s_j]$, which, in turn, produces $F(s) \leq sf(s) \leq sf(s_j) \to \infty$ for $s \in [0, s_j]$. Hence,

$$\begin{aligned}
\int_{s_1}^{s_j} \left[F(s)\right]^{-\frac{1}{p}} ds &\geq \int_{s_1}^{s_j} \left[F(s)\right]^{-\frac{1}{p}} ds \geq \left[\frac{j}{s_j}\right]^{\frac{1}{p'}} \int_{s_1}^{s_j} s^{-\frac{1}{p}}\, ds \\
&= \frac{p}{p-1} j^{\frac{1}{p'}} \left(1 - \left(\frac{s_1}{s_j}\right)\right) \to \infty
\end{aligned}$$

as $j \to \infty$, which contradicts the *Keller-Osserman* condition (2.7). Thus, (A.4) must be true, and then (f_2) holds.

Proof Let

$$v = \phi(u) = \int_u^\infty f^{-\frac{1}{p-1}}(v)\,dv, \quad u > 0. \tag{A.5}$$

We see that problem (1.1) is equivalent to the following problem:

$$-\Delta_p v + g(v)|\nabla v|^p = b(x), \quad v > 0, x \in \Omega, \qquad \overline{v}|_{\partial\Omega} = 0, \tag{A.6}$$

where $g(v) = f'(\psi(v))$, and ϕ is also the inverse function of ψ.

Now let $v \in W_{\mathrm{loc}}^{1,p}(\Omega) \cap L_{\mathrm{loc}}^\infty(\Omega)$ be any solution of problem (A.6). We claim that

$$v(x) \leq \overline{v}(x), \quad \forall x \in \Omega. \tag{A.7}$$

Indeed, assume on the contrary that $\{x \in \Omega : v(x) > \overline{v}(x)\} \neq \emptyset$. Then, on its arbitrary connected component D, we have $-\Delta_p(v - \overline{v})(x) \leq 0$, $x \in D$, since $g(v) \geq 0$. It follows by $(v-\overline{v})|_{\partial\Omega} = 0$ and the maximum principle that $v(x) \leq \overline{v}(x)$ for all $x \in D$. This is a contradiction. Thus, (A.7) holds, *i.e.*, any weak solution u of problem (1.1) satisfies (A.1). Moreover, by the definition of ψ and the condition $\int_0^1 \frac{ds}{f(s)} = \infty$ we see that $\psi(\overline{v}(x)) > 0$, $\forall x \in \Omega$, and (A.2) holds. Next, we consider the perturbed problem

$$\Delta_p u = b(x)f(u), \quad x \in \Omega, \qquad u|_{\partial\Omega} = m \in N. \tag{A.8}$$

By (b_1) and (f_1) we see that $\overline{u}_m = m$ is a supersolution of problem (A.8). To construct a subsolution $\underline{u}_1$ of problem (A.8), we let $\overline{v}_1 \in C^{2+\alpha}(\overline{\Omega})$ be the unique solution of the problem

$$-\Delta_p\overline{v}_1 = b(x), \qquad \overline{v}(x) > 0, \quad x \in \Omega, \qquad \overline{v}_1|_{\partial\Omega} = \int_1^\infty f^{-\frac{1}{p-1}}(s)\,ds, \tag{A.9}$$

and $\overline{u}_1 = \psi(\overline{v}_1)$. Then we see that $\underline{u}_1|_{\partial\Omega} = 1 \leq m$ and

$$-\Delta_p\overline{v}_1 = \frac{\Delta_p\underline{u}_1}{f(\underline{u}_1)} - \frac{f'(\underline{u}_1)}{f^2(\underline{u}_1)}|\nabla\underline{u}_1|^p = b(x), \quad x \in \Omega,$$

which yields

$$\Delta_p\underline{u}_1 \geq b(x)f(\underline{u}_1), \quad x \in \Omega,$$

i.e., $\underline{u}_1$ is a subsolution of problem (A.8). Moreover, $\underline{u}_1 \leq 1 \leq m, x \in \Omega$, due to the maximum principle. Thus, problem (A.8) has one solution $u_m \in W_0^{1,p}$ in the order interval $[u_1, m]$, and the maximum principle again yields that the map $m \to u_m$ is increasing. On the other hand, the classical Keller-Osserman condition guarantees that the problem

$$\Delta_p u = b_0 f(u), \quad x \in \Omega_0, \qquad u|_{\partial\Omega_0} = \infty, \tag{A.10}$$

has one solution $u_{\Omega_0} \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for each $\Omega_0 \subset\subset \Omega$, where $b_0 = \min_{x\in\overline{\Omega}_0} b(x)$. By the maximum principle we have $u_m \leq u_{\Omega_0}(x)$, $x \in \Omega_0$, and $u(x) := \lim_{m\to\infty} u_m(x)$ exists for $x \in \Omega_0$. Thus, u is the desired solution for problem (1.1) by the standard bootstrap argument, the arbitrariness of Ω_0, and (A.1). □

A.2 The uniqueness of the solution for problem (1.1)

In the second part, we prove the uniqueness of the solution for problem (1.1). The method is similar to the idea in [40, 58].

Theorem 4 *Under the hypotheses in Theorem* 1, *problem* (1.1) *admits a unique solution*.

Proof Since $\frac{f(s)}{s^q}$ is increasing in $[S_0, \infty)$ for some $q > p - 1$ and S_0 large enough, by Lemma 2(2) we have

$$\frac{f(s)}{s} \text{ is also increasing in } [S_0, \infty). \tag{A.11}$$

Let u_0 be the minimal solution for problem (1.3), and u be another solution for problem (1.1). We prove that $u = u_0$ in Ω. In fact, by the maximum principle we have

$$u_0 \le u \quad \text{in } \Omega. \tag{A.12}$$

Moreover, by the asymptotic behavior (1.3) we deduce that

$$\lim_{d(x)\to 0} \frac{u_0(x)}{u(x)} = 1. \tag{A.13}$$

For any $\epsilon > 0$, setting $w = (1+\epsilon)u_0$, we have

$$\lim_{d(x)\to 0} \bigl(w(x) - u(x)\bigr) = \lim_{d(x)\to 0} \left(\frac{(1+\epsilon)u_0(s)}{u(s)} - 1 \right) = +\infty. \tag{A.14}$$

Now, for small $\epsilon > 0$, we define the open set

$$D_\epsilon = \bigl\{ x \in \Omega : w(x) < u(x) \bigr\}. \tag{A.15}$$

We may assume that D_ϵ is nonempty for ϵ small enough; otherwise, there is nothing to prove. Indeed, notice that D_ϵ increases as $\epsilon \to 0$. Moreover, we may also assume that $D_\epsilon \to \Omega$ as $\epsilon \to 0$; if there exists $x_0 \in \Omega$ and a sequence $\epsilon_n \to 0$ such that $x_0 \in D_{\epsilon_n}$ for all n, then we have $(1+\epsilon_n)u_0(x_0) \ge u(x_0)$. Then the strong maximum principle yields $u \equiv u_0$ in Ω. Finally, we have $D_\epsilon \subset \Omega$ by (A.13).

Next, we choose $\eta > 0$ such that $u_0 \ge S_0$ in Ω_η and define $D_{\epsilon,\eta} = D_\epsilon \cap D_\eta$. Notice that $D_{\epsilon,\eta}$ is a nonempty open set for small ϵ. Moreover, by (A.11) we have

$$\Delta_p w = (1+\epsilon) b(x) f(u_0) \le b(x) f(w), \quad x \in D_{\epsilon,\eta}. \tag{A.16}$$

It follows by (f_1) that

$$\Delta_p (u - w) \ge b(x)\bigl(f(u) - f(w)\bigr) \ge 0, \quad x \in D_{\epsilon,\eta}. \tag{A.17}$$

Thus, by the maximum principle we obtain

$$u(x) - w(x) \le \max_{\partial D_{\epsilon,\eta}} (u - w), \quad x \in D_{\epsilon,\eta}. \tag{A.18}$$

Since $\partial D_{\epsilon,\eta} = (\partial D_\epsilon \cap D_\eta) \cup (D_\epsilon \cap \partial D_\eta)$, $D_\epsilon \cap \partial\Omega = \emptyset$, and $(u - w)|_{\partial D_\epsilon} = 0$, we see that the maximum of $u - w$ is achieved on $D_\epsilon \cap \partial D_\eta = (D_\epsilon \cap x : d(x) = \eta)$. Hence,

$$u(x) - w(x) \le \max_{(D_\epsilon \cap x : d(x) = \eta)} (u - w), \quad x \in D_{\epsilon,\eta}. \tag{A.19}$$

Letting $\epsilon \to 0$ in (A.19), we obtain

$$u - u_0 \le \max_{d(x)=\eta} (u - u_0) := \theta \quad \text{in } \Omega_\eta. \tag{A.20}$$

On the other hand, by (A.12) and (f_1) we have

$$\Delta_p(u-u_0) = b(x)\big(f(u)-f(u_0)\big) \geq 0, \quad x \in \Omega^\eta = x \in \Omega: d(x) > \eta. \tag{A.21}$$

The maximum principle implies that $u-u_0 \leq \theta$ in Ω^η, and hence $u-u_0 \leq \theta$ in the whole Ω. Then the strong maximum principle gives $u-u_0 \equiv \theta$. We obtain that $f(u) = f(u+\theta)$ in Ω, which can only hold if $\theta = 0$. Thus, $u = u_0$, which shows the uniqueness. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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