Bai et al. *Boundary Value Problems* (2015) 2015:231
DOI 10.1186/s13661-015-0500-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Positive solutions of discrete Neumann boundary value problems with sign-changing nonlinearities

Dingyong Bai[1*], Johnny Henderson[2] and Yunxia Zeng[1]

*Correspondence: baidy@gzhu.edu.cn
[1]School of Mathematics and Information Science, Guangzhou University, Guangzhou, 510006, P.R. China
Full list of author information is available at the end of the article

Abstract

Our concern is the existence of positive solutions of the discrete Neumann boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) = f(t, u(t)), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases}$$

where $f : [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \to \mathbb{R}$ is a sign-changing function. By using the Guo-Krasnosel'skiĭ fixed point theorem, the existence and multiplicity of positive solutions are established. The nonlinear term $f(t, z)$ may be unbounded below or nonpositive for all $(t, z) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+$.

MSC: 39A12; 39A10; 34B09

Keywords: difference equation; Neumann boundary value problem; positive solution; fixed point

1 Introduction

For $a, b \in \mathbb{Z}$ with $a < b$, let $[a, b]_{\mathbb{Z}} = \{a, a+1, a+2, \ldots, b-1, b\}$. We consider the following discrete Neumann boundary value problem:

$$\begin{cases} -\Delta^2 u(t-1) = f(t, u(t)), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases} \tag{1.1}$$

where $T > 1$ is a given positive integer, $\Delta u(t) = u(t+1) - u(t)$. Our purpose is to establish existence results for positive solutions of (1.1) when the nonlinearity term $f : [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \to \mathbb{R}$ is a sign-changing function.

In recent years, positive solutions of boundary value problems for difference equations have been widely studied. See [1–14] and the references therein. However, little work has been done that has referred to the existence of positive solutions for discrete boundary value problems with sign-changing nonlinearities (see [15]).

Usually, in order to obtain positive solutions of semipositone problems for ordinary differential equations or difference equations by using fixed point methods, the nonlinearity terms need to be bounded below and ultimately positive. For example, Anuradha *et al.*

[16] studied the following problem:

$$\begin{cases} (p(t)u')' + \lambda g(t,u) = 0, & t \in (a,b), \\ \gamma_1 u(a) - \gamma_2 p(a)u'(a) = 0, & \gamma_3 u(b) + \gamma_4 p(b)u'(b) = 0, \end{cases}$$

where $g : [a,b] \times \mathbb{R}^+ \to \mathbb{R}$ is continuous, bounded below (*i.e.*, $g(t,z) + M > 0$ for some $M > 0$), and $\lim_{z\to\infty} \frac{g(t,z)}{z} = \infty$ uniformly for $t \in [\alpha,\beta] \subset (a,b)$. Motivated by the method in [16], the first author and Xu [15] discussed its discrete analog and the nonlinearity term also required boundedness below, as well as a superlinear condition at ∞. We also refer to [17–26] for some references.

In this paper, our interest is with the existence and multiplicity of positive solutions of (1.1), where $f(t,z)$ may be nonpositive or unbounded below for all $(t,z) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+$. For this purpose, we make our basic assumptions as follows.

(C1) $f : [1,T]_{\mathbb{Z}} \times \mathbb{R}^+ \to \mathbb{R}$ is continuous;
(C2) there exists a function $h : [1,T]_{\mathbb{Z}} \to \mathbb{R}^+$ with $h(t) \not\equiv 0$ on $[1,T]_{\mathbb{Z}}$, and a constant number $L > 0$, such that

$$f(t,z) + Lz + h(t) \geq 0, \quad (t,z) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+. \tag{1.2}$$

Under (C1) and (C2), we give some sufficient conditions such that (1.1) has at least one positive solution and two positive solutions, respectively. Under these conditions, $f(t,z)$ is allowed to meet one of the following cases:

(1) $f(t,z)$ may be unbounded below and even be nonpositive for all $(t,z) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+$ (see the first parts of Theorems 3.1 and 3.3, in which the existence of at least one positive solution is presented);
(2) $f(t,z)$ is ultimately nonpositive, *i.e.*, $f(t,z) \leq 0$ for all $t \in [1,T]_{\mathbb{Z}}$ and $z > 0$ sufficiently large (see the second part of Theorem 3.1, in which the existence of at least two positive solutions is presented);
(3) $f(t,z)$ is ultimately nonnegative, *i.e.*, $f(t,z) \geq 0$ for all $t \in [1,T]_{\mathbb{Z}}$ and $z > 0$ sufficiently large, which implies that $f(t,z)$ is bounded below (see the second part of Theorem 3.3, in which the existence of at least two positive solutions is presented);
(4) $\lim_{z\to\infty} \frac{f(t,z)}{z} = 0$ uniformly for $t \in [1,T]_{\mathbb{Z}}$, which implies that $f(t,z)$ may be either bounded or unbounded below, and that $f(t,z)$ may be ultimately nonpositive or nonnegative or oscillating (see Corollary 3.2, in which the existence of at least two positive solutions is presented);
(5) $\lim_{z\to\infty} \frac{f(t,z)}{z} = \infty$ uniformly for $t \in [1,T]_{\mathbb{Z}}$, which is a special case of (3) and implies that $f(t,z)$ is bounded below (see Corollary 3.4, in which the existence of at least two positive solutions is presented).

The idea of this paper comes from the method in [27] by Henderson and Kosmatov, in which the Neumann boundary value problem of ordinary differential equation at resonance

$$\begin{cases} -u''(t) = f(t,u(t)), & 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.3}$$

was first studied. The nonlinear term $f(t,z)$ satisfied a similar inequality as (1.2). The problem (1.3) was transformed into a non-resonant positone problem and an existence result for at least one positive solution was obtained by means of the Guo-Krasnosel'skiǐ fixed point theorem. In our results, we present not only the existence, but also the multiplicity of positive solutions for the discrete Neumann boundary value problem (1.1).

The remaining part of this paper is organized as follows. In Section 2, we provide some preliminary results for later use. Then, in Section 3, we show and prove the existence and multiplicity of positive solutions for boundary value problem (1.1).

2 Preliminaries

For convenience, let

$$A = \frac{1}{2}\bigl(L + 2 + \sqrt{L^2 + 4L}\bigr), \qquad \rho = \bigl(A^T - A^{-T}\bigr)\bigl(A^2 - 1\bigr).$$

Consider the linear Neumann boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) + Lu(t) = h(t), & t \in [1,T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0. \end{cases} \tag{2.1}$$

The following lemma is easy to check.

Lemma 2.1 *Problem* (2.1) *has unique solution*

$$u_0(t) = \sum_{s=1}^{T} G(t,s)h(s), \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} (A^s + A^{-s+1})(A^{t-T} + A^{T-t+1}), & 1 \le s \le t \le T+1, \\ (A^t + A^{-t+1})(A^{s-T} + A^{T-s+1}), & 0 \le t \le s \le T. \end{cases}$$

In addition, $G(t,s) > 0$ *for all* $(t,s) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$.

Let

$$q(t) = \frac{1}{A^T + A^{-T+1}} \min\bigl\{A^{t-T} + A^{T-t+1}, A^t + A^{-t+1}\bigr\}, \quad t \in [0,T+1]_{\mathbb{Z}}.$$

It is easy to see that $0 < q(t) < 1$, $t \in [1,T]_{\mathbb{Z}}$.

Lemma 2.2 $q(t)G(t,s) \le G(t,s) \le G(s,s)$, $(t,s) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$.

Proof If $s \le t$, we have, for $(t,s) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$,

$$\frac{A^{t-T} + A^{T-t+1}}{A^T + A^{-T+1}} \le \frac{G(t,s)}{G(s,s)} = \frac{A^{t-T} + A^{T-t+1}}{A^{s-T} + A^{T-s+1}} \le 1.$$

If $t \le s$, we have, for $(t,s) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$,

$$\frac{A^t + A^{-t+1}}{A^T + A^{-T+1}} \le \frac{G(t,s)}{G(s,s)} = \frac{A^t + A^{-t+1}}{A^s + A^{-s+1}} \le 1.$$

Therefore, $q(t)G(t,s) \leq G(t,s) \leq G(s,s)$, $(t,s) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$. The proof is complete. □

Lemma 2.3 *$q(t) \geq \mu u_0(t)$, $t \in [1,T]_{\mathbb{Z}}$, where $\mu = \frac{(A^2-1)(A^T-A^{-T})}{(A^T+A^{-T+1})^2 \sum_{s=1}^{T} h(s)}$.*

Proof On the one hand, for $t \in [1,T]_{\mathbb{Z}}$,

$$\begin{aligned} q(t) &= \frac{1}{A^T + A^{-T+1}} \min\left\{A^{t-T} + A^{T-t+1}, A^t + A^{-t+1}\right\} \\ &\geq \min\left\{\frac{A^{t-T} + A^{T-t+1}}{A^T + A^{-T+1}}, \frac{A^t + A^{-t+1}}{A^T + A^{-T+1}}\right\} \cdot \max\left\{\frac{A^{t-T} + A^{T-t+1}}{A^T + A^{-T+1}}, \frac{A^t + A^{-t+1}}{A^T + A^{-T+1}}\right\} \\ &= \frac{(A^{t-T} + A^{T-t+1})(A^t + A^{-t+1})}{(A^T + A^{-T+1})^2} \\ &= \frac{(A^2-1)^2(A^T - A^{-T})}{(A^T + A^{-T+1})^2} G(t,t). \end{aligned}$$

On the other hand,

$$u_0(t) = \sum_{s=1}^{T} G(t,s)h(s) \leq G(t,t) \sum_{s=1}^{T} h(s), \quad t \in [1,T]_{\mathbb{Z}}.$$

Thus, $q(t) \geq \mu u_0(t)$, $t \in [1,T]_{\mathbb{Z}}$. The proof is complete. □

Define

$$\tilde{f}(t,z) = \begin{cases} f(t,z) + Lz + h(t), & (t,z) \in [1,T]_{\mathbb{Z}} \times (0,\infty), \\ f(t,0) + h(t), & (t,z) \in [1,T]_{\mathbb{Z}} \times (-\infty,0), \end{cases}$$

and consider

$$\begin{cases} -\Delta^2 v(t-1) + Lv(t) = \tilde{f}(t, v(t) - u_0(t)), & t \in [1,T]_{\mathbb{Z}}, \\ \Delta v(0) = \Delta v(T) = 0. \end{cases} \tag{2.3}$$

It is easy to check that the following lemma holds.

Lemma 2.4 *u is a positive solution of the boundary value problem* (1.1) *if and only if $v = u + u_0$ is a solution of the boundary value problem* (2.3) *with $v(t) > u_0(t)$ in* $[1,T]_{\mathbb{Z}}$.

The proofs of our main results are based on the Guo-Krasnosel'skiǐ fixed point theorem [28].

Lemma 2.5 *Let X be a Banach space and $K \subset X$ be a cone. Assume Ω_1, Ω_2 are open sets of X with $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be a completely continuous operator such that*

(i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In this section, we show the existence and multiplicity of positive solutions for (1.1). Let $q_0 = \min_{t\in[1,T]_{\mathbb{Z}}} q(t)$ and define

$$\phi(r) = \max\{\tilde{f}(t,z) : t \in [1,T]_{\mathbb{Z}}, z \in [0,r]\}, \quad \text{for } r > 0,$$

and

$$\psi(r) = \min\left\{\tilde{f}(t,z) : t \in [1,T]_{\mathbb{Z}}, z \in \left[\left(r - \frac{1}{\mu}\right)q_0, r\right]\right\}, \quad \text{for } r > \frac{1}{\mu}.$$

Theorem 3.1 *Assume that* (C1) *and* (C2) *hold. Suppose that there exist* $r, R > 0$ *such that* $\frac{1}{\mu} < r < R$ *and*

(C3) $\phi(r) \le \frac{r}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$ *and* $\psi(R) \ge \frac{R}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$.

Then problem (1.1) *has at least one positive solution. In addition, if*

(C4) $f(t,z) \le 0$ *for all* $t \in [1,T]_{\mathbb{Z}}$ *and* $z > 0$ *sufficiently large, and*
(C5) $L < \frac{1}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$,

then problem (1.1) *has at least two positive solutions.*

Proof Assume (C3) holds. We first prove by Lemma 2.5 that problem (2.3) has at least one positive solution v. In the Banach space $\mathbb{E} = \{v : [0,T+1]_{\mathbb{Z}} \to \mathbb{R}\}$ endowed with the norm $\|v\| = \max_{[0,T+1]_{\mathbb{Z}}} |v(t)|$, define

$$Fv(t) = \sum_{s=1}^{T} G(t,s)\tilde{f}\big(s, v(s) - u_0(s)\big). \tag{3.1}$$

Then by (C1), $F : \mathbb{E} \to \mathbb{E}$ is completely continuous. Define the cone

$$P = \{v \in \mathbb{E} : v(t) \ge q(t)\|v\|, t \in [1,T]_{\mathbb{Z}}\}. \tag{3.2}$$

By Lemma 2.2, $F(P) \subset P$. Thus, a fixed point of F in P is a positive solution of problem (2.3).

Let

$$\Omega_1 = \{v \in \mathbb{E} : \|v\| < r\}, \qquad \Omega_2 = \{v \in \mathbb{E} : \|v\| < R\}. \tag{3.3}$$

For $v \in P \cap \partial\Omega_1$, we have $v(s) - u_0(s) \ge q(s)\|v\| - u_0(s) \ge (\mu r - 1)u_0(s) > 0$, $s \in [1,T]_{\mathbb{Z}}$. It follows that $\tilde{f}(s, v(s) - u_0(s)) \le \phi(r)$, $s \in [1,T]_{\mathbb{Z}}$. Notice that $G(0,s) = G(1,s)$ and $G(T,s) = G(T+1,s)$, $s \in [1,T]_{\mathbb{Z}}$. Then, by (C3), we have

$$\begin{aligned}\|Fv\| &= \max_{t\in[0,T+1]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)\tilde{f}\big(s, v(s) - u_0(s)\big) \\ &\le \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)\phi(r) \\ &\le r.\end{aligned}$$

That is, $\|Fv\| \le \|v\|$ for $v \in P \cap \partial\Omega_1$.

For $v \in P \cap \partial\Omega_2$, we have by Lemma 2.3, for $s \in [1, T]$,

$$R \geq v(s) - u_0(s) \geq q(s)\|v\| - \frac{q(s)}{\mu} \geq \left(R - \frac{1}{\mu}\right)q_0. \tag{3.4}$$

This implies $\tilde{f}(s, v(s) - u_0(s)) \geq \psi(R)$ for $s \in [1, T]$, $v \in P \cap \partial\Omega_2$. Then, by (C3), we have, for $v \in P \cap \partial\Omega_2$,

$$\begin{aligned}\|Fv\| &= \max_{t\in[0,T+1]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)\tilde{f}\big(s, v(s) - u_0(s)\big) \\ &\geq \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)\psi(R) \\ &\geq R.\end{aligned}$$

That is, $\|Fv\| \geq \|v\|$ for $v \in P \cap \partial\Omega_2$.

Therefore, by Lemma 2.5, F has a fixed point $v_1 \in P$ satisfying $r \leq \|v_1\| \leq R$, which is a positive solution of problem (2.3). By Lemma 2.3, $u_1(t) = v_1(t) - u_0(t) \geq q(t)\|v_1\| - u_0(t) \geq (\mu r - 1)u_0(t) > 0$, $t \in [1, T]_{\mathbb{Z}}$. Therefore, by Lemma 2.4, u_1 is a positive solution of problem (1.1).

Now, let (C4) and (C5) also hold. We prove that problem (1.1) has a distinct second positive solution $u_2(t)$. By (C4), there exists $D > 0$ such that, for $z > D$,

$$\tilde{f}(t,z) = f(t,z) + Lz + h(t) \leq Lz + h(t), \quad t \in [1, T]_{\mathbb{Z}}.$$

By (C5), we can choose $R_\infty > \max\{R, \frac{D}{q_0} + \frac{1}{\mu}\}$ such that

$$L + \frac{\bar{h}}{R_\infty} \leq \frac{1}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}, \tag{3.5}$$

where $\bar{h} = \max_{t\in[1,T]_{\mathbb{Z}}} h(t)$. Let $\Omega_3 = \{v \in \mathbb{E} : \|v\| < R_\infty\}$. For $v \in P \cap \partial\Omega_3$, similar to (3.4), we have

$$v(t) - u_0(t) \geq \left(R_\infty - \frac{1}{\mu}\right)q_0 > D, \quad t \in [1, T]_{\mathbb{Z}}, \tag{3.6}$$

which implies that

$$\tilde{f}\big(t, v(t) - u_0(t)\big) \leq L\big(v(t) - u_0(t)\big) + h(t) \leq LR_\infty + \bar{h}, \quad t \in [1, T]_{\mathbb{Z}}.$$

Thus, by (3.5), we have, for $v \in P \cap \partial\Omega_3$,

$$\begin{aligned}\|Fv\| &= \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)\tilde{f}\big(s, v(s) - u_0(s)\big) \\ &\leq (LR_\infty + \bar{h}) \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s) \\ &\leq R_\infty.\end{aligned}$$

That is, $\|Fv\| \leq \|v\|$ for $v \in P \cap \partial\Omega_3$.

Therefore, by Lemma 2.5, F has a fixed point $v_2 \in P$ such that $R \leq \|v_2\| \leq R_\infty$. By Lemma 2.4, $u_2(t) = v_2(t) - u_0(t)$ is a second positive solution of problem (1.1). The proof is complete. □

Corollary 3.2 *Let* (C1) *and* (C2) *hold. Assume that there exist* $\frac{1}{\mu} < r < R$ *such that* (C3) *hold. Then, if* $\lim_{z\to\infty} \frac{f(t,z)}{z} = 0$ *uniformly for* $t \in [1,T]_{\mathbb{Z}}$ *and* $L < \frac{1}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$, *problem* (1.1) *has at least two positive solutions.*

Proof Let F, P, and Ω_1, Ω_2 be defined as (3.1), (3.2), and (3.3), respectively. From the proof of Theorem 3.1, we know by (C3) that F has a fixed point v_1 such that $r \leq \|v_1\| \leq R$ and $u_1(t) = v_1(t) - u_0(t)$ is a positive solution of (1.1). Now, we prove that F has a second fixed point $v_2 \in P$.

Take $\epsilon > 0$ sufficiently small such that $L + \epsilon \leq \frac{1}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$. Note that $\lim_{z\to\infty} \frac{f(t,z)}{z} = 0$ implies

$$\lim_{z\to\infty} \frac{\tilde{f}(t,z)}{z} = L$$

uniformly for $t \in [1,T]_{\mathbb{Z}}$. Then there exists $D > 0$ such that, for $z > D$, $\tilde{f}(t,z) \leq (L+\epsilon)z$ holds for all $t \in [1,T]_{\mathbb{Z}}$. Choose $R_\infty = \max\{R, \frac{D}{q_0} + \frac{1}{\mu}\} + 1$ and $\Omega_3 = \{v \in \mathbb{E} : \|v\| < R_\infty\}$. For $v \in P \cap \partial\Omega_3$, we see that (3.6) holds and hence $\tilde{f}(t, v(t) - u_0(t)) \leq (L+\epsilon)(v(t) - u_0(t)) \leq (L+\epsilon)R_\infty$ for $t \in [1,T]_{\mathbb{Z}}$. Thus, for $v \in P \cap \partial\Omega_3$,

$$\begin{aligned} \|Fv\| &= \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)\tilde{f}\big(s, v(s) - u_0(s)\big) \\ &\leq (L+\epsilon)R_\infty \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s) \\ &\leq R_\infty. \end{aligned}$$

That is, $\|Fv\| \leq \|v\|$ for $v \in P \cap \partial\Omega_3$. Therefore, by Lemma 2.5, F has a fixed point $v_2 \in P$ such that $R \leq \|v_2\| \leq R_\infty$, and hence $u_2(t) = v_2(t) - u_0(t)$ is a second positive solution of problem (1.1). The proof is complete. □

Theorem 3.3 *Assume that* (C1) *and* (C2) *hold. Suppose that there exist* $r, R > 0$ *such that* $\frac{1}{\mu} < r < R$ *and*

(C3)* $\phi(R) \leq \frac{R}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$ *and* $\psi(r) \geq \frac{r}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$.

Then problem (1.1) *has at least one positive solution. In addition, if*

(C4)* $f(t,z) \geq 0$ *for all* $t \in [1,T]_{\mathbb{Z}}$ *and* $z > 0$ *sufficiently large, and*
(C5)* $L > \frac{2}{q_0 \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$,

then problem (1.1) *has at least two positive solutions.*

Proof Let F, P, and Ω_1, Ω_2 be defined as (3.1), (3.2), and (3.3), respectively. Consider the fixed point of operator F in the cone P. Similar to the arguments in the proof of Theorem 3.1, if (C3)* holds, then we have $\|Fv\| \geq \|v\|$ for $v \in P \cap \partial\Omega_1$ and $\|Fv\| \leq \|v\|$ for

$v \in P \cap \partial\Omega_2$. Thus, by Lemma 2.5, F has a fixed point $v_1 \in P$ satisfying $r \leq \|v_1\| \leq R$, which is a positive solution of problem (2.3) and satisfies $v_1(t) - u_0(t) > 0$, $t \in [1,T]_{\mathbb{Z}}$. Therefore, by Lemma 2.4, $u_1 = v_1(t) - u_0(t)$ is a positive solution of problem (1.1).

Now, let (C4)* and (C5)* hold. By (C4)*, there exists $D > 0$ such that, for $z > D$,

$$\tilde{f}(t,z) = f(t,z) + Lz + h(t) \geq Lz + h(t), \quad t \in [1,T]_{\mathbb{Z}}.$$

By (C5)*, one can choose $R_\infty > R + \max\{\frac{2D}{q_0}, \frac{2}{\mu}\}$ such that

$$\frac{1}{2}q_0L + \frac{h_0}{R_\infty} \geq \frac{1}{\max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}, \tag{3.7}$$

where $h_0 = \min_{t\in[1,T]_{\mathbb{Z}}} h(t)$. Let $\Omega_3 = \{v \in \mathbb{E} : \|v\| < R_\infty\}$. Then, for $v \in P \cap \partial\Omega_3$, we have

$$v(t) - u_0(t) \geq \left(R_\infty - \frac{1}{\mu}\right)q_0 \geq \frac{1}{2}q_0R_\infty > D, \quad t \in [1,T]_{\mathbb{Z}},$$

which implies that

$$\tilde{f}\bigl(t, v(t) - u_0(t)\bigr) \geq L\bigl(v(t) - u_0(t)\bigr) + h(t) \geq \frac{1}{2}Lq_0R_\infty + h_0, \quad t \in [1,T]_{\mathbb{Z}}.$$

Thus, by (3.7), we have, for $v \in P \cap \partial\Omega_3$,

$$\begin{aligned} \|Fv\| &= \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)\tilde{f}\bigl(s, v(s) - u_0(s)\bigr) \\ &\geq \left(\frac{1}{2}Lq_0R_\infty + h_0\right) \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s) \\ &\geq R_\infty. \end{aligned}$$

That is, $\|Fv\| \geq R_\infty = \|v\|$ for $v \in P \cap \partial\Omega_3$. Therefore, F has another fixed point $v_2 \in P$ such that $r \leq \|v_1\| \leq R \leq \|v_2\| \leq R_\infty$. By Lemma 2.4, problem (1.1) has two positive solutions $u_1(t) = v_1(t) - u_0(t)$ and $u_2(t) = v_2(t) - u_0(t)$. The proof is complete. □

The following result can be obtained directly from Theorem 3.3.

Corollary 3.4 *Let* (C1) *and* (C2) *hold. Assume that there exist* $\frac{1}{\mu} < r < R$ *such that* (C3)* *hold. Then, if* $\lim_{z\to\infty} \frac{f(t,z)}{z} = \infty$ *uniformly for* $t \in [1,T]_{\mathbb{Z}}$ *and* $L > \frac{2}{q_0 \max_{t\in[1,T]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)}$, *problem* (1.1) *has at least two positive solutions*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

Author details
[1]School of Mathematics and Information Science, Guangzhou University, Guangzhou, 510006, P.R. China. [2]Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA.

Acknowledgements
Supported partially by PCSIRT of China (No. IRT1226) and NSF of China (No. 11171078).

Received: 7 October 2015 Accepted: 29 November 2015 Published online: 09 December 2015

References

1. Agarwal, RP, Perera, K, O'Regan, D: Multiple positive solutions of singular discrete p-Laplacian problems via variational methods. Adv. Differ. Equ. **2**, 93-99 (2005)
2. Anderson, DR: Discrete third-order three-point right-focal boundary value problems. Comput. Math. Appl. **45**, 861-871 (2003)
3. Anderson, D, Avery, RI: Multiple positive solutions to a third-order discrete focal boundary value problem. Comput. Math. Appl. **42**, 333-340 (2001)
4. Aykut, N: Existence of positive solutions for boundary value problems of second order functional difference equations. Comput. Math. Appl. **48**, 517-527 (2004)
5. Bai, D: A global result for discrete ϕ-Laplacian eigenvalue problems. Adv. Differ. Equ. **2013**, 264 (2013)
6. Bai, D, Xu, Y: Nontrivial solutions of boundary value problems of second order difference equations. J. Math. Anal. Appl. **326**(1), 297-302 (2007)
7. Bai, D, Xu, X: Existence and multiplicity of difference ϕ-Laplacian boundary value problems. Adv. Differ. Equ. **2013**, 267 (2013)
8. Ji, D, Ge, W: Existence of multiple positive solutions for Sturm-Liouville-like four-point boundary value problem with p-Laplacian. Nonlinear Anal. **68**, 2638-2646 (2008)
9. Jiang, L, Zhou, Z: Existence of nontrivial solutions for discrete nonlinear two point boundary value problems. Appl. Math. Comput. **180**, 318-329 (2006)
10. Karaca, IY: Discrete third-order three-point boundary value problem. J. Comput. Appl. Math. **205**, 458-468 (2007)
11. Liu, Y: Existence results for positive solutions of non-homogeneous BVPs for second order difference equations with one-dimensional p-Laplacian. J. Korean Math. Soc. **47**, 135-163 (2010)
12. Yang, Y, Meng, F: Eigenvalue problem for finite difference equations with p-Laplacian. J. Appl. Math. Comput. **40**, 319-340 (2012)
13. Yu, J, Guo, Z: On generalized discrete boundary value problems of Emden-Fowler equation. Sci. China Ser. A **36**, 721-732 (2006)
14. Yu, J, Guo, Z: On boundary value problems for a discrete generalized Emden-Fowler equation. J. Differ. Equ. **231**, 18-31 (2006)
15. Bai, D, Xu, Y: Positive solutions for semipositone BVPs of second-order difference equations. Indian J. Pure Appl. Math. **39**, 59-68 (2008)
16. Anuradha, A, Hai, DD, Shivaji, R: Existence results for superlinear semipositone BVP's. Proc. Am. Math. Soc. **124**(3), 757-763 (1996)
17. Anuradha, V, Maya, C, Shivaji, R: Positive solutions for a class of nonlinear boundary value problems with Neumann-Robin boundary conditions. J. Math. Anal. Appl. **236**, 94-124 (1999)
18. Bai, D, Xu, Y: Existence of positive solutions for boundary value problems of second-order delay differential equations. Appl. Math. Lett. **18**, 621-630 (2005)
19. Bai, D, Xu, Y: Positive solutions of second-order two-delay differential systems with twin-parameter. Nonlinear Anal. **63**, 601-617 (2005)
20. Bai, D, Yu, J: Semipositone problems of second order ordinary differential equations. Sciencepaper Online (10 October 2009); http://www.paper.edu.cn/
21. Castro, A, Gadam, S, Shivaji, R: Evolution of positive solution curves in semipositone problems with concave nonlinearities. J. Math. Anal. Appl. **245**, 282-293 (2000)
22. Hai, DD, Schmitt, K, Shivaji, R: Positive solutions of quasilinear boundary value problems. J. Math. Anal. Appl. **217**, 672-686 (1998)
23. Hai, DD, Shivaji, R: An existence result on positive solutions for a class of p-Laplacian systems. Nonlinear Anal. **56**, 1007-1010 (2004)
24. Ma, R: Multiple positive solutions for a semipositone fourth-order boundary value problem. Hiroshima Math. J. **33**, 217-227 (2003)
25. Maya, C, Shivaji, R: Multiple positive solutions for a class of semilinear elliptic boundary value problems. Nonlinear Anal. **38**, 497-504 (1999)
26. Sun, J, Wei, J: Existence of positive solutions of positive solution for semi- positone second-order three-point boundary-value problems. Electron. J. Differ. Equ. **2008**, 41 (2008)
27. Henderson, J, Kosmatov, N: Positive solutions of the semipositone Neumann boundary value problem. Math. Model. Anal. **20**, 578-584 (2015)
28. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, Orlando (1988)