Wan *Boundary Value Problems* (2015) 2015:239
DOI 10.1186/s13661-015-0508-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Some remarks on Phragmén-Lindelöf theorems for weak solutions of the stationary Schrödinger operator

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Abstract

In this paper, we not only give the asymptotic behavior at the origin for the maximum modulus of weak solutions of the stationary Schrödinger equation in a cone but also obtain the property of the negative parts of them, which generalize the Phragmén-Lindelöf type theorems for subfunctions.

Keywords: stationary Schrödinger equation; weak solution; cone

1 Introduction and main results

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space, where $n \geq 2$. Let E be an open set in $\mathbf{R}^n$, the boundary and the closure of it are denoted by ∂E and $\overline{E}$, respectively. A point P is denoted by (X, x_n), where $X = (x_1, x_2, \ldots, x_{n-1})$. For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at P and radius r in $\mathbf{R}^n$.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to the cartesian coordinates $(X, x_n) = (x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

Let $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$ denote the unit sphere and the upper half unit sphere, respectively. For $\Omega \subset \mathbf{S}^{n-1}$, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$, and the set $\{\Theta; (1, \Theta) \in \Omega\}$ are simply denoted by Θ and Ω respectively. The set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$, where $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$. Especially, the set $\mathbf{R}_+ \times \Omega$ by $C_n(\Omega)$, where $\mathbf{R}_+$ is the set of positive real number and $\Omega \subset \mathbf{S}^{n-1}$.

Let $C_n(\Omega; I)$ and $S_n(\Omega; I)$ denote the sets $I \times \Omega$ and $I \times \partial\Omega$, respectively, where I is an interval on $\mathbf{R}$ and $\mathbf{R}$ is the set of real numbers. Especially, the set $S_n(\Omega)$ denotes $S_n(\Omega; (0, +\infty))$, which is $\partial C_n(\Omega) - \{O\}$.

Let Δ^* be the spherical part of the Laplace operator Δ (see [1]),

$$\Delta = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Delta^*}{r^2},$$

and Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary. We consider the Dirichlet problem (see [2], p.41)

$$\bigl(\Delta^* + \lambda\bigr)\varphi(\Theta) = 0 \quad \text{on } \Omega,$$

$$\varphi(\Theta) = 0 \quad \text{on } \partial\Omega.$$

The least positive eigenvalue of the above boundary value problem is denoted by λ and the normalized positive eigenfunction corresponding to λ by $\varphi(\Theta)$, $\int_\Omega \varphi^2(\Theta)\,d\Omega = 1$, where $d\Omega$ denotes the $(n-1)$-dimensional volume element.

We put a rather strong assumption on Ω: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (*e.g.* see [3], pp.88-89, for the definition of the $C^{2,\alpha}$-domain).

Let $\mathscr{A}_a$ denote the class of nonnegative radial potentials $a(P)$, *i.e.* $0 \leq a(P) = a(r)$, $P = (r,\Theta) \in C_n(\Omega)$, such that $a \in L^b_{\mathrm{loc}}(C_n(\Omega))$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

Let I be the identical operator. If $a \in \mathscr{A}_a$, then the stationary Schrödinger operator

$$SSE_a = -\Delta + a(P)I$$

can be extended in the usual way from the space $C_0^\infty(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [4], Chapter 13). We will denote it SSE_a as well. This last one has a Green-Sch function $G^a_\Omega(P,Q)$ which is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G^a_\Omega(P,Q)/\partial n_Q \geq 0$, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$.

In this paper, we are concerned with the weak solutions of the inequality

$$SSE_a u(P) \leq 0, \tag{1.1}$$

where $P = (r,\Theta) \in C_n(\Omega)$.

We will also consider the class $\mathscr{B}_a$, consisting of the potentials $a \in \mathscr{A}_a$ such that there exists the finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0,\infty)$, moreover, $r^{-1}|r^2 a(r) - k| \in L(1,\infty)$.

We denote by $SbH_a(\Omega)$ the class of all weak solutions of the inequality (1.1) for any $P = (r,\Theta) \in C_n(\Omega)$, which are continuous when $a \in \mathscr{B}_a$ (see [5]). We denote by $SpH_a(\Omega)$ the class of $u(P)$ satisfying $-u(P) \in SbH_a(\Omega)$. If $u(P) \in SbH_a(\Omega)$ and $u(P) \in SpH_a(\Omega)$, then $u(P)$ is the solution of $SSE_a u(P) = 0$ for any $P = (r,\Theta) \in C_n(\Omega)$. In our terminology we follow Nirenberg [6]. Other authors have under similar circumstances used various terms such as subfunctions, subsolutions, submetaharmonic function, subelliptic functions, panharmonic functions, *etc.*; see, for example, Duffin, Littman, Qiao *et al.*, Topolyansky, Vekua (see [7–11]).

Solutions of the ordinary differential equation

$$-\Pi''(r) - \frac{n-1}{r}\Pi'(r) + \left(\frac{\lambda}{r^2} + a(r)\right)\Pi(r) = 0, \quad 0 < r < \infty, \tag{1.2}$$

play an essential role in this paper. It is well known (see, for example, [12]) that if the potential $a \in \mathscr{A}_a$, then equation (1.2) has a fundamental system of positive solutions $\{V, W\}$ such that V is non-decreasing with

$$0 \leq V(0+) \leq V(r) \nearrow \infty \quad \text{as } r \to +\infty,$$

and W is monotonically decreasing with

$$+\infty = W(0+) > W(r) \searrow 0 \quad \text{as } r \to +\infty.$$

Denote

$$\iota_k^{\pm} = \frac{2-n \pm \sqrt{(n-2)^2 + 4(k+\lambda)}}{2},$$

then the solutions to equation (1.2) have the following asymptotic (see [3]):

$$V(r) \approx r^{\iota_k^+}, \qquad W(r) \approx r^{\iota_k^-}, \quad \text{as } r \to \infty.$$

Let $u(P)$ ($P = (r,\Theta) \in C_n(\Omega)$) be a function. We introduce the following notations: $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$, $M_u(r) = \sup_{\Theta\in\Omega} u(P)$, $l = \max_{\Theta\in\Omega} \varphi(\Theta)$,

$$S_u(r) = \sup_{\Theta\in\Omega} \frac{u(P)}{\varphi(\Theta)}, \qquad L_u = \limsup_{r\to 0} \frac{S_u(r)}{W(r)}, \qquad J_u = \sup_{P\in C_n(\Omega)} \frac{u(P)}{W(r)\varphi(\Theta)}.$$

For any two positive numbers δ an r, we put

$$E_0^u(r;\delta) = \bigl\{\Theta \in \Omega : u(P) \le -\delta W(r)\bigr\}$$

and

$$\xi_u(\delta) = \limsup_{r\to 0} \int_{E_0^u(r;\delta)} \varphi(\Theta)\, d\Omega.$$

The integral

$$\int_\Omega u(r,\Theta)\varphi(\Theta)\, d\Omega,$$

is denoted by $N_u(r)$, when it exists. The finite or infinite limits

$$\lim_{r\to\infty} \frac{N_u(r)}{V(r)} \quad \text{and} \quad \lim_{r\to 0} \frac{N_u(r)}{W(r)}$$

are denoted by μ_u and η_u, respectively, when they exist.

We shall say that $u(P)$ ($P = (r,\Theta) \in C_n(\Omega)$) satisfies the Phragmén-Lindelöf boundary condition on $S_n(\Omega)$, if

$$\limsup_{P\to Q, Q\in S_n(\Omega)} u(P) \le 0$$

for every $Q \in S_n(\Omega)$.

Throughout this paper, unless otherwise specified, we will always assume that $u(P) \in SbH_a(\Omega)$ and satisfy the Phragmén-Lindelöf boundary condition on $S_n(\Omega)$. Recently, about the Phragmén-Lindelöf theorems for subfunctions in a cone, Qiao and Deng (see [9], Theorem 3) proved the following result.

Theorem A *If*

$$\mu_{u^+} = \eta_{u^+} = 0,$$

then

$$u(P) \le 0$$

for any $P = (r, \Theta) \in C_n(\Omega)$.

A stronger version of a Phragmén-Lindelöf type theorem is also due to Qiao and Deng (see [9], Theorem 3).

Theorem B *If*

$$\liminf_{r\to\infty} \frac{M_u(r)}{V(r)} < +\infty \tag{1.3}$$

and

$$\liminf_{r\to 0} \frac{M_u(r)}{W(r)} < +\infty, \tag{1.4}$$

then

$$u(P) \le \bigl(\mu_u V(r) + \eta_u W(r)\bigr)\varphi(\Theta) \tag{1.5}$$

for any $P = (r, \Theta) \in C_n(\Omega)$.

However, they do not tell us in [9] whether or not the limit

$$B_u = \lim_{r\to 0} \frac{M_u(r)}{W(r)}$$

exists. In this paper, we first of all answer this question positively and prove the following result.

Theorem 1 *If* (1.3) *is satisfied, then the limit* B_u $(0 \le B_u \le +\infty)$ *exists and*

$$B_u = (L_u)^+ l, \tag{1.6}$$

where

$$(L_u)^+ = \eta_{u^+}. \tag{1.7}$$

Remark It is obvious that $\eta_u \le L_u$. On the other hand, we have $\eta_u \ge L_u$ from (1.5). Thus, if (1.3) and (1.4) are satisfied, then we have $\eta_u = L_u$.

As an application of Theorem 1 we immediately have the following result by using Lemma 3 in Section 2.

Corollary *If*

$$\liminf_{r\to\infty} \frac{M_u(r)}{V(r)} \le 0, \tag{1.8}$$

then

$$B_u = (J_u)^+ l. \tag{1.9}$$

In [9], the authors gave the properties of the positive part of weak solutions satisfying the Phragmén-Lindelöf boundary condition on $S_n(\Omega)$. Finally, we shall show one of the properties of its negative part.

From the remark, we have

$$\eta_{u^+} = L_{u^+} = (L_u)^+ = (\eta_u)^+.$$

Since

$$N_u(r) = N_{u^+}(r) - N_{u^-}(r),$$

Theorem 2 follows immediately.

Theorem 2 *Under the conditions of Theorem* B, *if* $\eta_u \geq 0$, *then*

$$\lim_{r\to 0} \frac{N_{u^-}(r)}{W(r)} = 0.$$

2 Some lemmas

Lemma 1 (see [9], Lemma 8)

(1) *Both of the limits* μ_u *and* η_u $(-\infty < \mu_u, \eta_u \leq +\infty)$ *exist.*

(2) *If* $\eta_u \leq 0$, *then* $V^{-1}(r)N_u(r)$ *is non-decreasing on* $(0, +\infty)$.

(3) *If* $\mu_u \leq 0$, *then* $W^{-1}(r)N_u(r)$ *is non-increasing on* $(0, +\infty)$.

Lemma 2 *If* (1.3) *is satisfied and there exists a positive number R such that* $u(P) \leq 0$ *for any* $P = (r, \Theta) \in C_n(\Omega; (0, R))$, *then for any positive number* δ, *we have*

$$u(P) \leq \bigl(\mu_u V(r) - \delta\xi_u(\delta) W(r)\bigr)\varphi(\Theta) \tag{2.1}$$

for any $P = (r, \Theta) \in C_n(\Omega)$.

Proof Let δ be any given positive number and $\{r_k\}$ be a sequence such that

$$\lim_{k\to\infty} r_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \int_{E_0^u(r_k;\delta)} \varphi(\Theta)\, d\Omega = \xi_u(\delta).$$

Then we have

$$N_u(r_k) \leq \int_{E_0^u(r_k;\delta)} u(r_k, \Theta)\varphi(\Theta)\, d\Omega \leq -\delta W(r_k) \int_{E_0^u(r_k;\delta)} \varphi(\Theta)\, d\Omega$$

for any $0 < r_k < R$ and hence

$$\eta_u \leq -\delta\xi_u(\delta).$$

Thus we obtain (2.1) from Theorem B. □

Lemma 3 *Under the conditions of the corollary, $L_u > -\infty$ and $J_u = L_u$.*

Proof It is evident that

$$J_u \geq L_u. \tag{2.2}$$

Hence, we shall prove that $J_u = L_u$ under the assumption that $L_u < +\infty$. Since (1.3) and (1.4) are satisfied and (1.8) gives $\mu_u \leq 0$, we have

$$u(P) \leq \eta_u W(r)\varphi(\Theta)$$

for any $P = (r, \Theta) \in C_n(\Omega)$ from Theorem B, which gives

$$\eta_u \geq J_u. \tag{2.3}$$

Since Lemma 1 and the remark give $\eta_u > -\infty$ and $\eta_u = L_u$, respectively, we have the conclusion from (2.2) and (2.3).

Given a continuous function ψ defined on the truncated cone $\partial C_n(\Omega; (R_1, R_2))$, where R_1 and R_2 are two positive real numbers satisfying $R_1 < R_2$, then the solution of the Dirichlet-Sch problem on $C_n(\Omega; (R_1, R_2))$ with ψ is denoted by $H_\psi(P; C_n(\Omega; (R_1, R_2)))$. □

Lemma 4 *If*

$$\mu_{u^+} < +\infty \quad \textit{and} \quad \eta_{u^+} < +\infty, \tag{2.4}$$

are satisfied, then

$$B_u \leq \eta_{u^+}.$$

Proof Take any $P = (r, \Theta) \in C_n(\Omega)$ and any pair of numbers R_1, R_2 satisfying $0 < 2R_1 < r < \frac{1}{2}R_2 < \infty$. If $\psi(P)$ is a boundary function on $\partial C_n(\Omega; (R_1, R_2))$ satisfying

$$\psi(P) = \begin{cases} u(R_i, \Phi) & \text{on } \{R_i\} \times \Omega \ (i = 1, 2), \\ 0 & \text{on } S_n(\Omega; (R_1, R_2)), \end{cases}$$

then we have

$$\begin{aligned} u(P) &\lesssim H_\psi\bigl(P; C_n\bigl(\Omega; (R_1, R_2)\bigr)\bigr) \\ &= \int_\Omega u^+(R_1, \Phi) \frac{G^a_{C_n(\Omega;(R_1,R_2))}(P, (R_1, \Phi))}{\partial y} R_1^{n-1}\, d\Omega \\ &\quad - \int_\Omega u^+(R_2, \Phi) \frac{G^a_{C_n(\Omega;(R_1,R_2))}(P, (R_2, \Phi))}{\partial y} R_2^{n-1}\, d\Omega, \end{aligned}$$

where $G^a_{C_n(\Omega;(R_1,R_2))}(\cdot, \cdot)$ is the Green-Sch function on $C_n(\Omega; (R_1, R_2))$ with the pole at P.

Here we use the following inequalities (see [1], p.124):

$$\frac{\partial(G^a_{C_n(\Omega;(R_1,R_2))}(P, (R_1, \Phi)))}{\partial R} \lesssim c_1 \frac{W(r)}{W(R_1)} \frac{\varphi(\Theta)\varphi(\Phi)}{R_1^{n-1}}$$

and

$$\frac{\partial (G^a_{C_n(\Omega;(R_1,R_2))}(P,(R_2,\Phi)))}{\partial R} \gtrsim -c_2 \frac{V(r)}{V(R_2)} \frac{\varphi(\Theta)\varphi(\Phi)}{R_2^{n-1}},$$

where c_1 and c_2 are two positive constants.

Then we have

$$u(P) \leq c_3 W^{-1}(R_1) N_{u^+}(R_1) W(r)\varphi(\Theta) + c_4 V^{-1}(R_2) N_{u^+}(R_2) V(r)\varphi(\Theta), \tag{2.5}$$

where c_3 and c_4 are two positive constants.

As $R_1 \to 0$ and $R_2 \to \infty$ in (2.5), we obtain

$$M_u(r) \leq \bigl(c_3 \eta_{u^+} W(r) + c_4 \mu_{u^+} V(r)\bigr) \max_{\Theta\in\Omega} \varphi(\Theta)$$

from Lemma 1, which gives the conclusion of Lemma 4 from (2.4). □

3 Proof of Theorem 1

Put

$$\tau = \liminf_{r\to 0} \frac{M_u(r)}{W(r)}.$$

Since

$$N_u(r) \leq M_u(r) \int_\Omega \varphi(\Theta)\, d\Omega,$$

and Lemma 1 gives

$$\eta_u > -\infty, \tag{3.1}$$

we immediately see that $\tau > -\infty$.

Now we distinguish two cases.

Case 1 $\tau = +\infty$.

In this case B_u exists and is equal to $+\infty$. It is obvious that for any positive number r

$$\frac{M_{u^+}(r)}{W(r)} \leq l \frac{S_{u^+}(r)}{W(r)}, \tag{3.2}$$

which gives $L_u = +\infty$.

These results show that (1.7) holds in this case.

Case 2 $\tau < +\infty$.

From Theorem B, we see that $L_u < +\infty$. On the other hand, we have $L_u > -\infty$ from (3.1).

Subcase 2.1 $0 \leq L_u < +\infty$.

There exists a positive number R_ϵ such that

$$u(P) \leq (L_u + \epsilon) W(r)\varphi(\Theta)$$

for any $\epsilon > 0$, where $P = (r,\Theta) \in C_n(\Omega;(0,R_\epsilon))$.

This gives

$$\limsup_{r\to 0} \frac{M_u(r)}{W(r)} \le L_u l. \tag{3.3}$$

Now, assume that $\tau < L_u l$. There exist a positive number δ_1 and a set $E_u \subset \Omega$ such that

$$\int_{E_u} \varphi(\Theta)\, d\Omega > 0$$

and

$$L_u \varphi(\Theta) - \tau \ge 2\delta_1 \tag{3.4}$$

for $\Theta \in E_u$.

We define $\nu_1(P)$ ($P = (r, \Theta) \in C_n(\Omega)$) by

$$\nu_1(P) = u(P) - (L_u + \epsilon) W(r) \varphi(\Theta) \tag{3.5}$$

and apply Lemma 2 to $\nu_1(P)$. It gives

$$u(P) \le \bigl[\bigl\{L_u + \epsilon - \delta_1 \xi_{\nu_1}(\delta_1)\bigr\} W(r) + \mu_{\nu_1} V(r)\bigr] \varphi(\Theta)$$

for any $P = (r, \Theta) \in C_n(\Omega)$.

So we have

$$L_u \le L_u - \delta_1 \xi_{\nu_1}(\delta_1).$$

If we can show that

$$\xi_{\nu_1}(\delta_1) > 0, \tag{3.6}$$

then we have a contradiction.

To prove (3.6), take a sequence $\{r_k\}$, with $\lim_{k\to\infty} r_k = 0$, such that

$$\frac{M_u(r_k)}{W(r_k)} \le \tau + \delta_1 \quad (k = 1, 2, 3, \ldots).$$

From (3.4) and (3.5) we have

$$\frac{\nu_1(r_k, \Theta)}{W(r_k)} \le \frac{u(r_k, \Theta)}{W(r_k)} - L_u \varphi(\Theta) \le -\delta_1$$

for any $\Theta \in E_u$, which gives

$$E_u \subset E_0^{\nu_1}(r_k; \delta_1) \quad (k = 1, 2, 3, \ldots).$$

Hence

$$\xi_{\nu_1}(\delta_1) \ge \int_{E_u} \varphi(\Theta)\, d\Omega > 0.$$

Thus from (3.3) we can simultaneously prove the existence of B_u and (1.6).

Subcase 2.2 $-\infty \leq L_u < 0$.

Take any small number $\epsilon > 0$ satisfying $L_u + \epsilon < 0$. There exists a positive number R_ϵ such that

$$u(P) \leq (L_u + \epsilon)W(r)\varphi(\Theta)$$

for any $P = (r, \Theta) \in C_n(\Omega; (0, R_\epsilon))$.

This gives

$$\limsup_{r\to 0} \frac{M_u(r)}{W(r)} \leq 0. \tag{3.7}$$

Now suppose that $\tau < 0$. There are a sequence $\{r_k\}$ tending to 0 and a positive number δ_2 such that

$$\frac{M_u(r_k)}{W(r_k)} \leq -2\delta_2 \quad (k = 1, 2, 3, \ldots).$$

Define $v_2(P)$ ($P = (r, \Theta) \in C_n(\Omega)$) by

$$v_2(P) = u(P) - (L_u + \epsilon)W(r)\varphi(\Theta)$$

and apply Lemma 2 to $v_2(P)$. Then we obtain

$$u(P) \leq \bigl[\bigl\{L_u + \epsilon - \delta_2\xi_{v_2}(\delta_2)\bigr\}W(r) + \mu_{v_2}V(r)\bigr]\varphi(\Theta)$$

for any $P = (r, \Theta) \in C_n(\Omega)$, which gives

$$L_u \leq L_u - \delta_2\xi_{v_2}(\delta_2).$$

If we can show that

$$\xi_{v_2}(\delta_2) > 0, \tag{3.8}$$

then we have a contradiction.

To prove (3.8), write

$$F_u = \bigl\{\Theta \in \Omega; -L_u\varphi(\Theta) \leq \delta_2\bigr\}.$$

It is evident that

$$\int_{F_u} \varphi(\Theta)\, d\Omega > 0.$$

For every $\Theta \in F_u$, we have

$$\frac{v_1(r_k, \Theta)}{W(r_k)} \leq \frac{u(r_k, \Theta)}{W(r_k)} - L_u\varphi(\Theta) \leq -\delta_2,$$

which shows that

$$F_u \subset E_0^{\nu_2}(r_k;\delta_2) \quad (k = 1, 2, 3, \ldots).$$

Hence we have

$$\xi_{\nu_2}(\delta_2) \geq \int_{F_u} \varphi(\Theta)\, d\Omega > 0.$$

Thus we can prove that $\tau \geq 0$. With (3.7), this also gives the existence of B_u and

$$B_u = 0 = (L_u)^+ l.$$

Lastly, we shall show that (1.7) holds.

If $\eta_{u^+} = +\infty$, then it is evident that $B_{u^+} = +\infty$. This together with (3.2) gives $L_{u^+} = +\infty$. Since

$$L_{u^+} = (L_u)^+, \tag{3.9}$$

we know that (1.7) holds.

Next suppose that $\eta_{u^+} < +\infty$. We have $B_u < +\infty$ by Lemma 4 and hence $L_{u^+} = \eta_{u^+}$ by the remark. With (3.9), this gives (1.7).

Competing interests
The author declares that there is no conflict of interests regarding the publication of this article.

Acknowledgements
The author would like to thank the referees for their comments. This work was supported by the National Social Science Foundation of China (No. 13BGL047), the Humanities and Social Science Fund of Ministry of Education (No. 2013-ZD-011) and the 2015 Universities Philosophy Social Sciences Innovation Team of Henan Province: the Institutional Arrangements for Mixed Ownership Reform of Stateowned Enterprises in Henan Province (No. 2015-CXTD-09).

Received: 29 October 2015 Accepted: 8 December 2015 Published online: 22 December 2015

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