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Successively iterative method for a class of high-order fractional differential equations with multi-point boundary value conditions on half-line

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Abstract

In this paper, we study the existence of the positive solutions for a class of high-order differential equations with multi-point boundary value conditions involving the Caputo fractional derivative on infinite interval. Moreover, we develop two computable explicit monotone iterative sequences for approximating the two minimal and maximal positive solutions. An example is given to show the applicability of our main results.

MSC: 34A08; 34B18

Keywords: Caputo fractional derivative; multi-point boundary value problem; monotone iteration method; infinite interval; existence of solutions

1 Introduction

In this paper, we investigate the multi-point boundary value problem for the fractional differential equation

$$ {}^{c}D_{0^+}^{\alpha}u(t) = a(t)f\bigl(t,u(t)\bigr), \quad 0 < t < +\infty, \tag{1.1}$$

$$ u(0) = 0, \qquad u^{(q)}(0) = 0, \qquad {}^{c}D_{0^+}^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2}\beta_i u(\xi_i), \tag{1.2}$$

where ${}^{c}D_{0^+}^{\alpha}$ and ${}^{c}D_{0^+}^{\alpha-1}$ are the Caputo fractional derivatives, $n-1 < \alpha \leq n$ $(n > 2)$, $q = 2,3,\dots,n-1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, and $\beta_i > 0$, $i = 1,2,\dots,m-2$, $m \geq 3$, satisfy $0 < \sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1} < \Gamma(\alpha)$.

Recently, the theory on existence of positive solutions of fractional differential equations is a rapidly growing area of research. For more details on the basic theory of fractional calculus and fractional differential equations, one can see the monographs of [1–6] and the references therein. However, the theory of the boundary value problem for nonlinear fractional differential equations is still limited to the finite interval [7–11], and many aspects of this theory need to be explored. In the near future, there has been a significant development on boundary value problems for fractional differential equations on infinite intervals

[12–22]. To the best of our knowledge, results as regards the boundary problem referring to the differential equations involving the Caputo fractional derivative on infinite interval are relatively scarce.

Liang *et al.* [19] discussed the following nonlinear fractional differential equations with multi-point boundary value problem on an unbounded domain:

$$D_{0^+}^{\alpha}u(t) + a(t)f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < +\infty, \tag{1.3}$$

$$u(0) = u'(0) = 0, \qquad D^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \tag{1.4}$$

where $D_{0^+}^{\alpha}$ and $D^{\alpha-1}$ are the Riemann-Liouville fractional derivatives, $2 < \alpha \leq 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, and $\beta_i > 0$, $i = 1, 2, \ldots, m-2$, satisfy $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$. By using a fixed point theorem on a cone, they obtained the existence of multiple positive solutions.

Assia *et al.* [20] considered the following fractional boundary value problem on the half-line:

$${}^cD_{0^+}^{q}u(t) = f\bigl(t, u(t), u'(t)\bigr), \quad t > 0, \tag{1.5}$$

$$u(0) = u''(0) = 0, \qquad \lim_{t\to\infty} {}^cD^{q-1}u(t) = \alpha u(1), \tag{1.6}$$

where ${}^cD_{0^+}^{q}$ and ${}^cD^{q-1}$ are the Caputo fractional derivatives, $2 < q < 3$. By using the nonlinear alternative of Leray-Schauder and the Guo-Krasnosel'skii fixed point theorem on a cone, they obtained the existence of positive solutions.

Liang *et al.* [21] investigated the following fractional boundary value problem on an infinite interval:

$$D_{0^+}^{\gamma}\bigl(\phi_p\bigl(D_{0^+}^{\alpha}u(t)\bigr)\bigr) + a(t)f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < +\infty, \tag{1.7}$$

$$u(0) = u'(0) = 0, \qquad D^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \qquad D_{0^+}^{\alpha}u(t)|_{t=0} = 0, \tag{1.8}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivatives, $2 < \alpha \leq 3$, $0 < \gamma \leq 1$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$. They established solvability of the above fractional boundary value problems by means of the properties of the Green function and some fixed-point theorems.

Wang *et al.* [22] also investigated the problem (1.3)-(1.4), the difference is in using monotone iterative method to develop two computable explicit monotone iterative sequences for approximating the minimal and maximal positive solutions.

Our main results of this paper are in extending the results in [19] from the low order to the high order case. In addition, our research topic involves the Caputo fractional derivative, which is different from [19]. We employ the monotone iterative method [22–25], which is indeed an important and useful contribution to the ones used in relevant papers.

The plan of this paper is as follows. In Section 2, we shall give some definitions and lemmas to prove our main results. In Section 3, we employ the monotone iterative method to establish the two computable explicit monotone iterative sequences for approximating the minimal and maximal positive solutions of boundary value problems (1.1) and (1.2). In Section 4, an example is presented to illustrate the main results.

In order to facilitate our study, we make the following assumptions:

(H_1) $f \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$, $f(t,0) \not\equiv 0$ on any subinterval of $[0,+\infty)$, and $f(t,(1+t^{\alpha-1})u)$ is bounded when u is bounded on $[0,+\infty)$;

(H_2) $a(\cdot) : [0,+\infty) \to [0,+\infty)$ does not identically vanish on any subinterval of $[0,+\infty)$ and $0 < \int_0^{+\infty} a(t)\,dt < \infty$.

2 Preliminaries

For convenience of the reader, we present here some necessary definitions and lemmas from the fractional calculus theory.

Definition 2.1 ([3]) The fractional integral of order α ($\alpha > 0$) of a function $f : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\,ds,$$

where $\Gamma(\cdot)$ is the Gamma function, provided that the right side is point-wise defined on $(0,+\infty)$.

Definition 2.2 ([3]) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,+\infty) \to \mathbb{R}$ is given by

$${}^{c}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $\Gamma(\cdot)$ is the Gamma function, provided that the right side is point-wise defined on $(0,+\infty)$, and $n = \lceil\alpha\rceil$, where $\lceil\alpha\rceil$ is the ceiling function of α.

Definition 2.3 ([3]) Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called cone if

(1) if $x \in K$ and $\lambda > 0$, then $\lambda x \in K$;

(2) if $x \in K$ and $-x \in K$, then $x = 0$.

Lemma 2.1 ([3]) *Let* $\alpha, \beta > 0$ *and* $n = \lceil\alpha\rceil$. *Then*

$${}^{c}D_{0^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \quad \beta > n,$$

$${}^{c}D_{0^+}^{\alpha} t^{k} = 0, \quad k = 0, 1, 2, \ldots, n-1.$$

Lemma 2.2 ([3]) *Let* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0, f \in L^p(a,b)$ $(1 \le p \le \infty)$. *Then*

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f(t) = I_{a^+}^{\alpha+\beta} f(t) = I_{a^+}^{\beta} I_{a^+}^{\alpha} f(t),$$

for all $t \in [a,b]$. *If* $\alpha + \beta > 1$, *then the relation holds at any point of* $[a,b]$.

Lemma 2.3 ([3]) *Let* $\operatorname{Re}\alpha > 0$ *and* $f \in L^p(a,b)$ $(1 \le p \le \infty)$. *Then*

$$^cD^\alpha_{a^+} I^\alpha_{a^+} f(t) = f(t),$$

for all $t \in [a,b]$.

Lemma 2.4 ([4]) *For* $\alpha > 0$, $g \in C([0,+\infty))$, *the homogeneous fractional differential equation* $^cD^\alpha_{0^+} g(t) = 0$ *has a solution*

$$g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1,\ldots,n$, *and* $n = \lceil \alpha \rceil$.

Lemma 2.5 ([4]) *Assume that* $u(t) \in C[0,\infty) \cap L^1[0,\infty)$ *with the derivative of order* n *that belongs to* $C[0,\infty) \cap L^1[0,\infty)$, *let the Caputo fractional derivative of order* $\alpha > 0$. *Then*

$$I^\alpha_{0^+}\, {}^cD^\alpha_{0^+} u(t) = u(t) + c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, *and* $n = \lceil \alpha \rceil$.

The following lemma is fundamental in the proofs of our main results.

Lemma 2.6 ([26]) *Let* $V = \{u \in C_\infty, \|u\| < l$, *where* $l > 0\}$, $V(t) = \{\frac{u(t)}{1+t^{\alpha-1}}, u \in V\}$. *Then* V *is relatively compact on* C_∞, *if* $V(t)$ *is equicontinuous on any finite subinterval of* $[0,+\infty)$ *and equiconvergent at infinity, that is, for any* $\varepsilon > 0$, *there exists* $N = N(\varepsilon) > 0$ *such that*

$$\left| \frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}} \right| < \varepsilon,$$

where $C_\infty = \{u \in C([0,+\infty),\mathbb{R}) : \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty\}$ *and for all* $u \in V$, $t_1, t_2 \ge N$.

3 Main results

Lemma 3.1 *Let* $h(t) \in L^1[0,\infty)$ *be a nonnegative continuous function. Then the boundary value problem of fractional differential equation*

$$^cD^\alpha_{0^+} u(t) = h(t), \quad 0 < t < +\infty, \tag{3.1}$$

$$u(0) = 0, \qquad u^{(q)}(0) = 0, \qquad {}^cD^{\alpha-1}_{0^+} u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \tag{3.2}$$

has a unique solution

$$u(t) = \int_0^\infty G(t,s) h(s)\, ds,$$

where

$$G(t,s)=\begin{cases}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i}-\frac{t\sum_{i=1}^{m-2}\beta_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i},\\ \quad 0\le s\le\min(t,\xi_1)<\infty,\\ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i},\\ \quad 0\le\xi_{m-2}\le s\le t<\infty,\\ \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i}-\frac{t\sum_{i=\kappa}^{m-2}\beta_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=\kappa}^{m-2}\beta_i\xi_i},\\ \quad 0\le\xi_{\kappa-1}<s\le\xi_\kappa\le t<\infty,\kappa=2,3,\dots,m-2,\\ \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i}-\frac{t\sum_{i=1}^{m-2}\beta_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i},\\ \quad 0\le t\le s\le\xi_1<\infty,\\ \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i}-\frac{t\sum_{i=\kappa}^{m-2}\beta_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=\kappa}^{m-2}\beta_i\xi_i},\\ \quad 0\le t\le\xi_{\kappa-1}<s\le\xi_\kappa<\infty,\kappa=2,3,\dots,m-2,\\ \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i},\\ \quad 0\le\max(t,\xi_{m-2})\le s<\infty.\end{cases}$$

Proof In view of Lemma 2.4, it is clear that equation (3.1) is equivalent to the integral form

$$u(t)=I_{0^+}^{\alpha}h(t)+c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1},$$

for some $c_i\in\mathbb{R}$, $i=1,2,\dots,n$, $n=\lceil\alpha\rceil$.

By the boundary value conditions $u(0)=0$, $u^{(q)}(0)=0$, we imply that

$$c_1=0,\qquad c_3=c_4=c_5=\cdots=c_n=0$$

and

$$u(t)=I_{0^+}^{\alpha}h(t)+c_2t.$$

Applying Lemma 2.1 and the boundary condition ${}^cD_{0^+}^{\alpha-1}u(+\infty)=\sum_{i=1}^{m-2}\beta_iu(\xi_i)$, we obtain

$$\begin{aligned}{}^cD_{0^+}^{\alpha-1}u(+\infty)&=\lim_{t\to+\infty}\bigl(I_{0^+}h(t)+{}^cD_{0^+}^{\alpha-1}(c_2t)\bigr)\\&=\lim_{t\to+\infty}I_{0^+}h(t)=\int_0^\infty h(s)\,ds,\\ u(\xi_i)&=I_{0^+}^{\alpha}h(\xi_i)+c_2\xi_i,\end{aligned}$$

consequently

$$c_2=\frac{\int_0^\infty h(s)\,ds}{\sum_{i=1}^{m-2}\beta_i\xi_i}-\frac{I_{0^+}^{\alpha}\sum_{i=1}^{m-2}\beta_ih(\xi_i)}{\sum_{i=1}^{m-2}\beta_i\xi_i}.$$

Substituting c_2 by its value, it yields

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds + \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i}\int_0^\infty h(s)\,ds \\ &\quad - \frac{t\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}h(s)\,ds}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} \\ &= \int_0^\infty G(t,s)h(s)\,ds. \end{aligned}$$

The proof is completed. □

Define $C_\infty = \{u \in C([0,+\infty),\mathbb{R}) : \sup_{t\in[0,+\infty)}\frac{|u(t)|}{1+t^{\alpha-1}} < +\infty\}$ endowed with the norm

$$\|u\|_{C_\infty} = \sup_{t\in[0,+\infty)}\frac{|u(t)|}{1+t^{\alpha-1}}.$$

Lemma 3.2 ([27]) *C_∞ is a Banach space.*

Define a cone $K \subset C_\infty$ by

$$K = \{u \in C_\infty : u(t) \geq 0, t \in [0,+\infty)\}.$$

Define an operator $T : K \to C_\infty$ as follows:

$$Tu(t) = \int_0^\infty G(t,s)a(s)f\big(s,u(s)\big)\,ds.$$

Set $h(t) = a(t)f(t,u(t))$ in Lemma 3.1. We deduce that u is a solution of the boundary value problem (1.1)-(1.2) if and only if it is a fixed point of the operator T.

Lemma 3.3 *The function $G(t,s)$ in Lemma* 3.1 *satisfies the following properties*:

(i) *$G(t,s)$ is continuous on $[0,+\infty)\times[0,+\infty)$*;
(ii) *$G(t,s) > 0$, for any $t,s \in (0,+\infty)$*;
(iii) *$0 < \frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{2}{\sum_{i=1}^{m-2}\beta_i}$, for any $t,s \in (0,+\infty)$.*

Proof It is easy to see that (i) holds. So we prove that the rest are true. Let

$$g_1(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i} - \frac{t\sum_{i=1}^{m-2}\beta_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i}, \quad 0 \leq s \leq \min(t,\xi_1) < \infty,$$

$$g_2(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i}, \quad 0 \leq \xi_{m-2} \leq s \leq t < \infty,$$

$$g_3(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i} - \frac{t\sum_{i=\kappa}^{m-2}\beta_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=\kappa}^{m-2}\beta_i\xi_i},$$

$$0 \leq \xi_{\kappa-1} < s \leq \xi_\kappa \leq t < \infty, \kappa = 2,3,\ldots,m-2,$$

$$g_4(t,s) = \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i} - \frac{t\sum_{i=1}^{m-2}\beta_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i}, \quad 0 \leq t \leq s \leq \xi_1 < \infty,$$

$$g_5(t,s) = \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i} - \frac{t\sum_{i=\kappa}^{m-2}\beta_i(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=\kappa}^{m-2}\beta_i\xi_i},$$
$$0 \le t \le \xi_{\kappa-1} < s \le \xi_\kappa < \infty, \kappa = 2,3,\ldots,m-2,$$
$$g_6(t,s) = \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i}, \quad 0 \le \max(t,\xi_{m-2}) \le s < \infty.$$

Let $g_k(t,s)$ ($k = 1,2,3,4,5,6$) be defined by the above formulas. We will show that

$$g_4(t,s) \ge 0, \quad 0 \le t \le s \le \xi_1 < \infty.$$

Since

$$\begin{aligned} g_4(t,s) &= \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i} - \frac{t\sum_{i=1}^{m-2}\beta_i(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} \\ &\ge \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i} - \frac{t\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} \\ &= t\left(\frac{\Gamma(\alpha) - \sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i}\right) \\ &\ge 0, \end{aligned}$$

we deduce

$$g_4(t,s) \ge 0, \quad 0 \le t \le s \le \xi_1 < \infty.$$

By using an analogous argument, we can conclude that

$$\begin{aligned} &g_1(t,s) \ge 0, \quad 0 \le s \le \min(t,\xi_1) < \infty, \\ &g_2(t,s) \ge 0, \quad 0 \le \xi_{m-2} \le s \le t < \infty, \\ &g_3(t,s) \ge 0, \quad 0 \le \xi_{\kappa-1} < s \le \xi_\kappa \le t < \infty, \\ &g_5(t,s) \ge 0, \quad 0 \le t \le \xi_{\kappa-1} < s \le \xi_\kappa < \infty, \kappa = 2,3,\ldots,m-2, \end{aligned}$$

and

$$g_6(t,s) \ge 0, \quad 0 \le \max(t,\xi_{m-2}) \le s < \infty.$$

Therefore, we get $G(t,s) > 0$, for any $t,s \in (0,+\infty)$.

Next, we will prove (iii) is true. We will show that $\frac{g_2(t,s)}{1+t^{\alpha-1}} \le \frac{2}{\sum_{i=1}^{m-2}\beta_i}$, $0 \le \xi_{m-2} \le s \le t < \infty$.

$$\begin{aligned} \frac{g_2(t,s)}{1+t^{\alpha-1}} &= \frac{(t-s)^{\alpha-1}}{(1+t^{\alpha-1})\Gamma(\alpha)} + \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i(1+t^{\alpha-1})} \\ &= \frac{(t-s)^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\xi_i + t\Gamma(\alpha)}{\sum_{i=1}^{m-2}\beta_i\xi_i(1+t^{\alpha-1})\Gamma(\alpha)} \\ &\le \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\xi_i + t\Gamma(\alpha)}{\sum_{i=1}^{m-2}\beta_i\xi_i(1+t^{\alpha-1})\Gamma(\alpha)} \le \frac{t^{\alpha-1}\sum_{i=1}^{m-2}\beta_i\xi_i + t\Gamma(\alpha)}{\sum_{i=1}^{m-2}\beta_i\xi_i\Gamma(\alpha)} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i} \leq \frac{t^{\alpha-1}}{\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}} + \frac{t}{\sum_{i=1}^{m-2}\beta_i\xi_i} \\
&\leq \frac{1}{\sum_{i=1}^{m-2}\beta_i}(1+1) = \frac{2}{\sum_{i=1}^{m-2}\beta_i},
\end{aligned}$$

so we have $0 \leq \frac{g_2(t,s)}{1+t^{\alpha-1}} \leq \frac{2}{\sum_{i=1}^{m-2}\beta_i}$, $0 \leq \xi_{m-2} \leq s \leq t < \infty$.

By using an analogous argument, we can conclude that

$$\begin{aligned}
&0 \leq \frac{g_1(t,s)}{1+t^{\alpha-1}} \leq \frac{2}{\sum_{i=1}^{m-2}\beta_i}, \quad 0 \leq s \leq \min(t,\xi_1) < \infty, \\
&0 \leq \frac{g_3(t,s)}{1+t^{\alpha-1}} \leq \frac{2}{\sum_{i=1}^{m-2}\beta_i}, \quad 0 \leq \xi_{\kappa-1} < s \leq \xi_\kappa \leq t < \infty, \\
&0 \leq \frac{g_4(t,s)}{1+t^{\alpha-1}} \leq \frac{2}{\sum_{i=1}^{m-2}\beta_i}, \quad 0 \leq t \leq s \leq \xi_1 < \infty, \\
&0 \leq \frac{g_5(t,s)}{1+t^{\alpha-1}} \leq \frac{2}{\sum_{i=1}^{m-2}\beta_i}, \quad 0 \leq t \leq \xi_{\kappa-1} < s \leq \xi_\kappa < \infty,
\end{aligned}$$

$\kappa = 2,3,\ldots,m-2$, and

$$0 \leq \frac{g_6(t,s)}{1+t^{\alpha-1}} \leq \frac{2}{\sum_{i=1}^{m-2}\beta_i}, \quad 0 \leq \max(t,\xi_{m-2}) \leq s < \infty.$$

Therefore, we get $0 < \frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{2}{\sum_{i=1}^{m-2}\beta_i}$, for any $s,t \in (0,+\infty)$. The proof is completed. □

Lemma 3.4 *If* (H_1) *and* (H_2) *hold, the operator* $T: K \to K$ *is completely continuous.*

Proof We divide the proof into the following five steps.

Step 1: We show that $T: K \to K$.

In view of the continuous and nonnegative of $G(t,s), f \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$, and $a(t) \in L^1[0,\infty)$ is nonnegative, it is easy to see that $Tu(t) \geq 0$ for $t \in [0,+\infty)$.

By condition (H_1) and Lemma 3.3, for any fixed $u \in K$, we have

$$\frac{u(t)}{1+t^{\alpha-1}} \leq \|u\|_{C_\infty}, \quad t \in [0,+\infty)$$

and there exists Υ_u such that

$$\begin{aligned}
\sup_{t\in[0,+\infty)} \frac{|(Tu)(t)|}{1+t^{\alpha-1}} &= \sup_{t\in[0,+\infty)} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f\bigl(s,u(s)\bigr)\,ds \\
&\leq \sup_{t\in[0,+\infty)} \frac{2}{\sum_{i=1}^{m-2}\beta_i} \int_0^\infty a(s) f\left(s, \bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right) ds \\
&\leq \frac{2\Upsilon_u}{\sum_{i=1}^{m-2}\beta_i} \int_0^\infty a(s)\,ds < \infty,
\end{aligned}$$

so $T(K) \subset K$.

Step 2: We show that $T : K \to K$ is continuous.

Let $u_n \to u$ as $n \to +\infty$ in K. Then $\frac{u_n(s)}{1+s^{\alpha-1}} \rightrightarrows \frac{u(s)}{1+s^{\alpha-1}}$ as $n \to +\infty$ on $[0,\infty)$. Hence

$$\begin{aligned}
&\left|\frac{Tu_n(t)}{1+t^{\alpha-1}} - \frac{Tu(t)}{1+t^{\alpha-1}}\right| \\
&\quad = \left|\int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f\bigl(s,u_n(s)\bigr)\,ds - \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f\bigl(s,u(s)\bigr)\,ds\right| \\
&\quad \le \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} a(s) \bigl|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u(s)\bigr)\bigr|\,ds \\
&\quad \le \frac{2}{\sum_{i=1}^{m-2}\beta_i} \int_0^\infty a(s) \left| f\left(s, \bigl(1+s^{\alpha-1}\bigr)\frac{u_n(s)}{1+s^{\alpha-1}}\right) - f\left(s, \bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right)\right|\,ds.
\end{aligned}$$

With the help of Lebesgue's dominated convergence theorem and the continuity of f, we have

$$\|Tu_n - Tu\|_{C_\infty} = \sup_{t\in[0,+\infty)} \left|\frac{(Tu_n)(t)}{1+t^{\alpha-1}} - \frac{(Tu)(t)}{1+t^{\alpha-1}}\right| \to 0, \quad \text{as } n \to \infty,$$

that is, T is continuous.

Now take $\Omega \subset K$ be bounded, *i.e.*, there exists a positive constant l such that $\|u\|_{C_\infty} \le l$ for all $u \in \Omega$.

Step 3: $T(\Omega)$ is uniformly bounded.

By condition (H_1), let

$$\Upsilon_l = \sup\bigl\{f\bigl(t, \bigl(1+t^{\alpha-1}\bigr)u\bigr), (t,u) \in [0,+\infty) \times [0,l]\bigr\}.$$

For any $u \in \Omega$, by Lemma 3.3, we have

$$\begin{aligned}
\|Tu\|_{C_\infty} &= \sup_{t\in[0,+\infty)} \frac{|(Tu)(t)|}{1+t^{\alpha-1}} \\
&= \sup_{t\in[0,+\infty)} \int_0^\infty \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f\bigl(s,u(s)\bigr)\,ds \\
&\le \sup_{t\in[0,+\infty)} \frac{2}{\sum_{i=1}^{m-2}\beta_i} \int_0^\infty a(s) f\left(s, \bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right)\,ds \\
&\le \frac{2\Upsilon_l}{\sum_{i=1}^{m-2}\beta_i} \int_0^\infty a(s)\,ds < \infty,
\end{aligned}$$

therefore $T(\Omega)$ is uniformly bounded.

Step 4: We show that $T(\Omega)$ is locally equicontinuous on any finite subinterval of $[0,+\infty)$.

For any $\theta > 0$, $t_1, t_2 \in [0,\theta]$ and $u \in \Omega$, without loss of generality, we assume that $t_2 > t_1$. Then

$$\begin{aligned}
&\left|\frac{(Tu)(t_2)}{1+t_2^{\alpha-1}} - \frac{(Tu)(t_1)}{1+t_1^{\alpha-1}}\right| \\
&\quad = \left|\int_0^\infty \frac{G(t_2,s)}{1+t_2^{\alpha-1}} a(s) f\bigl(s,u(s)\bigr)\,ds - \int_0^\infty \frac{G(t_1,s)}{1+t_1^{\alpha-1}} a(s) f\bigl(s,u(s)\bigr)\,ds\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_0^\infty \left(\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}} \right) a(s) f\bigl(s,u(s)\bigr)\,ds \right| \\
&= \int_0^\infty \left| \frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}} + \frac{G(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}} \right| a(s) f\bigl(s,u(s)\bigr)\,ds \\
&\le \int_0^\infty \left(\left| \frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}} \right| + \left| \frac{G(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}} \right| \right) a(s) f\bigl(s,u(s)\bigr)\,ds.
\end{aligned}
$$

Furthermore, we deduce that

$$
\begin{aligned}
&\int_0^\infty \left| \frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}} \right| a(s) f\bigl(s,u(s)\bigr)\,ds \\
&\quad = \frac{1}{1+t_2^{\alpha-1}} \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} a(s) f\bigl(s,u(s)\bigr)\,ds + \frac{t_2}{\sum_{i=1}^{m-2}\beta_i\xi_i} \int_0^\infty a(s) f\bigl(s,u(s)\bigr)\,ds \right. \\
&\qquad - \frac{t_2 \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} a(s) f(s,u(s))\,ds}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} a(s) f\bigl(s,u(s)\bigr)\,ds \\
&\qquad \left. - \frac{t_1}{\sum_{i=1}^{m-2}\beta_i\xi_i} \int_0^\infty a(s) f\bigl(s,u(s)\bigr)\,ds + \frac{t_1 \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} a(s) f(s,u(s))\,ds}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} \right| \\
&\quad \le \frac{1}{1+t_2^{\alpha-1}} \left(\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} a(s) f\bigl(s,u(s)\bigr)\,ds \right.\right. \\
&\qquad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} a(s) f\bigl(s,u(s)\bigr)\,ds \right| \\
&\qquad + \left| \frac{t_2}{\sum_{i=1}^{m-2}\beta_i\xi_i} \int_0^\infty a(s) f\bigl(s,u(s)\bigr)\,ds - \frac{t_1}{\sum_{i=1}^{m-2}\beta_i\xi_i} \int_0^\infty a(s) f\bigl(s,u(s)\bigr)\,ds \right| \\
&\qquad + \left| \frac{t_1 \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} a(s) f(s,u(s))\,ds}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} \right. \\
&\qquad \left.\left. - \frac{t_2 \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} a(s) f(s,u(s))\,ds}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} \right| \right) \\
&\quad \le \frac{1}{1+t_2^{\alpha-1}} \left(\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \bigl((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr) a(s) f\left(s, \bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right) ds \right| \right. \\
&\qquad + \left| \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} a(s) f\left(s, \bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right) ds \right| \\
&\qquad + \left| \frac{(t_2-t_1)}{\sum_{i=1}^{m-2}\beta_i\xi_i} \int_0^\infty a(s) f\left(s, \bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right) ds \right| \\
&\qquad \left. + \left| \frac{(t_1-t_2)\sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} a(s) f(s,(1+s^{\alpha-1})\frac{u(s)}{1+s^{\alpha-1}})\,ds}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} \right| \right) \\
&\quad \le \frac{\Upsilon_l}{1+t_2^{\alpha-1}} \left(\frac{t_2-t_1}{\Gamma(\alpha)} \int_0^{t_1} a(s)\,ds + \left| \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} a(s)\,ds \right| \right. \\
&\qquad \left. + \frac{(t_2-t_1)}{\sum_{i=1}^{m-2}\beta_i\xi_i} \int_0^\infty a(s)\,ds + \frac{(t_2-t_1)\sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-1} a(s)\,ds}{\Gamma(\alpha)\sum_{i=1}^{m-2}\beta_i\xi_i} \right).
\end{aligned} \tag{3.3}
$$

Since $0 < \int_0^{+\infty} a(s)\,ds < \infty$, by the integration of Cauchy's test for convergence, we can get

$$\int_0^{\infty} \left| \frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_2^{\alpha-1}} \right| a(s) f\bigl(s,u(s)\bigr)\,ds \to 0,$$

uniformly as $t_1 \to t_2$.

Similar to (3.3), we can deduce that

$$\int_0^{\infty} \left| \frac{G(t_1,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}} \right| a(s) f\bigl(s,u(s)\bigr)\,ds \to 0,$$

uniformly as $t_1 \to t_2$.

Thus, we conclude that

$$\left| \frac{(Tu)(t_2)}{1+t_2^{\alpha-1}} - \frac{(Tu)(t_1)}{1+t_1^{\alpha-1}} \right| \to 0$$

uniformly as $t_1 \to t_2$, and hence $T(\Omega)$ is locally equicontinuous on any finite subinterval of $[0,+\infty)$.

Step 5: We show that $T : K \to K$ is equiconvergent at ∞.

For convenience, we denote $\Delta = \frac{\sum_{i=1}^{m-2} \beta_i (\xi_i - s)^{\alpha-1}}{\Gamma(\alpha) \sum_{i=1}^{m-2} \beta_i \xi_i}$ and $\Lambda = \sum_{i=1}^{m-2} \beta_i \xi_i$.

Since $\lim_{t\to\infty} \frac{t}{1+t^{\alpha-1}} = 0$, there exists $N_1 > 0$ such that

$$\left| \frac{t_2}{1+t_2^{\alpha-1}} - \frac{t_1}{1+t_1^{\alpha-1}} \right| \le \left| 1 - \frac{t_2}{1+t_2^{\alpha-1}} \right| + \left| 1 - \frac{t_1}{1+t_1^{\alpha-1}} \right| < \varepsilon_1 \tag{3.4}$$

for any $t_2 > t_1 > N_1$, $\varepsilon_1 > 0$.

Similarly, there exist $M > 0$, $N_2 > 0$ such that $\lim_{t\to\infty} \frac{(t-M)^{\alpha-1}}{1+t^{\alpha-1}} = 1$ and

$$\begin{aligned}
\left| \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} \right| &\le \left| 1 - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} \right| + \left| 1 - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} \right| \\
&\le \left| 1 - \frac{(t_2-M)^{\alpha-1}}{1+t_2^{\alpha-1}} \right| + \left| 1 - \frac{(t_1-M)^{\alpha-1}}{1+t_1^{\alpha-1}} \right| < \varepsilon_2
\end{aligned} \tag{3.5}$$

for any $t_2 > t_1 > N_2$, $\varepsilon_2 > 0$ and $0 \le s \le M$.

Let $N > \max\{N_1, N_2\}$. For any $u \in \Omega$, by (3.4) and (3.5), we have

$$\begin{aligned}
&\left| \frac{(Tu)(t_2)}{1+t_2^{\alpha-1}} - \frac{(Tu)(t_1)}{1+t_1^{\alpha-1}} \right| \\
&\quad = \left| \int_0^{\infty} \frac{G(t_2,s)}{1+t_2^{\alpha-1}} a(s) f\bigl(s,u(s)\bigr)\,ds - \int_0^{\infty} \frac{G(t_1,s)}{1+t_1^{\alpha-1}} a(s) f\bigl(s,u(s)\bigr)\,ds \right| \\
&\quad = \left| \int_0^{\infty} \left(\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}} \right) a(s) f\bigl(s,u(s)\bigr)\,ds \right| \\
&\quad \le \int_0^{t_1} \left(\left| \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_1^{\alpha-1})} \right| + \left| \frac{t_2}{\Lambda(1+t_2^{\alpha-1})} - \frac{t_1}{\Lambda(1+t_1^{\alpha-1})} \right| \right. \\
&\qquad \left. + \left| \frac{t_1 \Delta}{(1+t_1^{\alpha-1})} - \frac{t_2 \Delta}{(1+t_2^{\alpha-1})} \right| \right) a(s) f\bigl(s,u(s)\bigr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\left(\left|\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)(1+t_2^{\alpha-1})}\right|+\left|\frac{t_2}{\Lambda(1+t_2^{\alpha-1})}-\frac{t_1}{\Lambda(1+t_1^{\alpha-1})}\right|\right.\\
&+\left.\left|\frac{t_1\Delta}{(1+t_1^{\alpha-1})}-\frac{t_2\Delta}{(1+t_2^{\alpha-1})}\right|\right)a(s)f\bigl(s,u(s)\bigr)\,ds\\
&+\int_{t_2}^{+\infty}\left(\left|\frac{t_2}{\Lambda(1+t_2^{\alpha-1})}-\frac{t_1}{\Lambda(1+t_1^{\alpha-1})}\right|+\left|\frac{t_1\Delta}{(1+t_1^{\alpha-1})}-\frac{t_2\Delta}{(1+t_2^{\alpha-1})}\right|\right)a(s)f\bigl(s,u(s)\bigr)\,ds\\
\le{}&\int_0^{t_1}\left(\frac{\varepsilon_2}{\Gamma(\alpha)}+\frac{\varepsilon_1}{\Lambda}+\varepsilon_1\Delta\right)a(s)f\left(s,\bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right)ds\\
&+\int_{t_1}^{t_2}\left(1+\frac{\varepsilon_1}{\Lambda}+\varepsilon_1\Delta\right)a(s)f\left(s,\bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right)ds\\
&+\int_{t_2}^{+\infty}\left(\frac{\varepsilon_1}{\Lambda}+\varepsilon_1\Delta\right)a(s)f\left(s,\bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right)ds\\
\le{}&\Upsilon_l\left(\left(\frac{\varepsilon_2}{\Gamma(\alpha)}+\frac{\varepsilon_1}{\Lambda}+\varepsilon_1\Delta\right)\int_0^{t_1}a(s)\,ds+\left(1+\frac{\varepsilon_1}{\Lambda}+\varepsilon_1\Delta\right)\int_{t_1}^{t_2}a(s)\,ds\right.\\
&+\left.\left(\frac{\varepsilon_1}{\Lambda}+\varepsilon_1\Delta\right)\int_{t_2}^{+\infty}a(s)\,ds\right)\\
\to{}&0
\end{aligned}
$$

uniformly as $t_1 \to t_2$.

In conclusion, for any $\varepsilon > 0$, there exists a sufficiently large $N > 0$ such that for any $u \in \Omega$,

$$
\left|\frac{u(t_1)}{1+t_1^{\alpha-1}}-\frac{u(t_2)}{1+t_2^{\alpha-1}}\right|<\varepsilon,
$$

$\forall t_1, t_2 > N$.

This implies that $T : K \to K$ is equiconvergent at ∞.

By the Arzela-Ascoli theorem, we see that $T : K \to K$ is completely continuous. The proof is completed. □

Now we will list the following condition in this section:

(H_3) $f(t,\cdot)$ is nondecreasing for any $t \in [0,+\infty)$, and there exists a constant $b > 0$, such that $f(t,(1+t^{\alpha-1})u) \le \frac{b\sum_{i=1}^{m-2}\beta_i}{2\int_0^{+\infty}a(s)\,ds}$ for $(t,u) \in [0,+\infty) \times [0,b]$.

Theorem 3.1 *Assume* (H_1), (H_2), *and* (H_3) *hold. Then the multi-point boundary value problem* (1.1)-(1.2) *has the minimal and maximal positive solutions* v^*, u^* *in* $(0,b]$, *which can be obtained by the following two explicit monotone iterative sequences*:

$$
v_{n+1}=\int_0^{+\infty}G(t,s)a(s)f\bigl(s,v_n(s)\bigr)\,ds
$$

with initial value $v_0(t) = 0$,

$$
u_{n+1}=\int_0^{+\infty}G(t,s)a(s)f\bigl(s,u_n(s)\bigr)\,ds
$$

with initial value $u_0(t) = b$. Moreover,

$$\begin{aligned} &v_0 \le v_1 \le \cdots \le v_n \le \cdots \le v^* \le \cdots \\ &\quad \le u^* \le \cdots \le u_n \le \cdots \le u_1 \le u_0. \end{aligned}$$

Proof Denote $\Phi = \{u \in K, \|u\|_{C_\infty} \le b\}$. Then we have $T(\Phi) \subset \Phi$. In fact, let $u \in \Phi$. Then by (H_1), (H_3), and Lemma 3.3, we get

$$\begin{aligned} \|Tu\|_{C_\infty} &= \sup_{t\in[0,+\infty)} \left| \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} a(s) f\bigl(s, u(s)\bigr) \right| ds \\ &\le \frac{2}{\sum_{i=1}^{m-2}\beta_i} \int_0^{+\infty} a(s) f\left(s, \bigl(1+s^{\alpha-1}\bigr)\frac{u(s)}{1+s^{\alpha-1}}\right) ds \\ &\le \frac{2}{\sum_{i=1}^{m-2}\beta_i} \int_0^{+\infty} a(s)\, ds \frac{b\sum_{i=1}^{m-2}\beta_i}{2\int_0^{+\infty} a(s)\, ds} \\ &= b. \end{aligned}$$

So $T(\Phi) \subset \Phi$.

Denote that $v_0(t) = 0$, $v_1 = Tv_0$, and $v_2 = T^2 v_0 = Tv_1$, for all $t \in [0,+\infty)$. Since $v_0(t) = 0 \in \Phi$ and $T : \Phi \to \Phi$, $v_1 \in T(\Phi) \subset \Phi$ and $v_2 \in T(\Phi) \subset \Phi$. We have

$$v_1(t) = (Tv_0)(t) \ge 0 = v_0(t),$$

for all $t \in [0,+\infty)$.

By (H_3), for $u, v \in \Phi$ and $u \ge v$, we deduce

$$\begin{aligned} Tu(t) &= \int_0^{+\infty} G(t,s) a(s) f\bigl(s, u(s)\bigr)\, ds \\ &\ge \int_0^{+\infty} G(t,s) a(s) f\bigl(s, v(s)\bigr)\, ds \\ &= Tv(t). \end{aligned}$$

We know that T is a nondecreasing operator.

So we have

$$v_2(t) = (Tv_1)(t) \ge (Tv_0)(t) = v_1(t),$$

for all $t \in [0,+\infty)$.

By the induction, define $v_{n+1} = Tv_n$, $n = 0,1,2,\ldots$. The sequence $\{v_n\}_{n=1}^\infty \subset T(\Phi) \subset \Phi$ and satisfies the following relation:

$$v_{n+1}(t) \ge v_n(t),$$

for all $t \in [0,+\infty)$, $n = 0,1,2,\ldots$.

In view of T is completely continuous and $v_{n+1} = Tv_n$, $\{v_n\}_{n=1}^\infty$ is relative compact. That is to say, $\{v_n\}_{n=1}^\infty$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^\infty$ and there exists a $v^* \in \Phi$ such that $v_{n_k} \to v^*$ as $k \to \infty$.

By the above part and $v_{n+1}(t) \ge v_n(t)$, for all $t \in [0,+\infty)$, $n = 0,1,2,\ldots$, we can get $\lim_{n\to\infty} v_n = v^*$.

Since T is continuous and $v_{n+1} = Tv_n$, we have $Tv^* = v^*$. That is to say, v^* is a fixed point of the operator T.

Denote $u_0(t) = b$, $u_1 = Tu_0$, and $u_2 = T^2u_0 = Tu_1$, for all $t \in [0,+\infty)$. Since $u_0(t) \in \Phi$ and $T : \Phi \to \Phi$, $u_1 \in T(\Phi) \subset \Phi$, and $u_2 \in T(\Phi) \subset \Phi$.

By (H_3), we deduce

$$\begin{aligned} u_1(t) &= \int_0^{+\infty} G(t,s)a(s)f\bigl(s,u_0(s)\bigr)\,ds \\ &\le \frac{2}{\sum_{i=1}^{m-2}\beta_i}\int_0^{+\infty} a(s)f\bigl(s,\bigl(1+s^{\alpha-1}\bigr)u_0(s)\bigr)\,ds \\ &\le \frac{2}{\sum_{i=1}^{m-2}\beta_i}\int_0^{+\infty} a(s)\,ds\frac{b\sum_{i=1}^{m-2}\beta_i}{2\int_0^{+\infty}a(s)\,ds} \\ &= b = u_0(t), \end{aligned}$$

for all $t \in [0,+\infty)$.

Since T is a nondecreasing operator, we have

$$u_2(t) = (Tu_1)(t) \le (Tu_0)(t) = u_1(t),$$

for all $t \in [0,+\infty)$.

By the induction, define $u_{n+1} = Tu_n$, $n = 0,1,2,\ldots$. The sequence $\{u_n\}_{n=1}^{\infty} \subset T(\Phi) \subset \Phi$ and satisfies the following relation:

$$u_{n+1}(t) \le u_n(t),$$

for all $t \in [0,+\infty)$, $n = 0,1,2,\ldots$.

With an analysis exactly parallel to the proving process of $\lim_{n\to\infty} v_n = v^*$, we see that there exists a $u^* \in \Phi$ such that $\lim_{n\to\infty} u_n = u^*$.

Since T is completely continuous and $u_{n+1} = Tu_n$, we have $Tu^* = u^*$. That is to say, u^* is a fixed point of the operator T.

Now, we will show that u^* and v^* are the maximal and minimal positive solutions of the boundary value problem (1.1)-(1.2) in $(0,b]$.

Let $\phi \in [0,b]$ be any solution of the boundary value problem (1.1)-(1.2). That is, $T\phi = \phi$. Noting that T is nondecreasing and $v_0(t) = 0 \le \phi(t) \le b = u_0(t)$, we have $v_1(t) = Tv_0(t) \le \phi(t) \le Tu_0(t) = u_1(t)$, for all $t \in [0,+\infty)$.

Similarly, we can obtain

$$v_n(t) \le \phi(t) \le u_n(t),$$

for all $t \in [0,+\infty)$, $n = 0,1,2,\ldots$.

Since $\lim_{n\to\infty} u_n = u^*$ and $\lim_{n\to\infty} v_n = v^*$, by the above formulas we obtain

$$\begin{aligned} &v_0 \le v_1 \le \cdots \le v_n \le \cdots \le v^* \le \cdots \\ &\quad \le u^* \le \cdots \le u_n \le \cdots \le u_1 \le u_0. \end{aligned}$$

Since $f(t,0) \not\equiv 0$, for all $t \in [0,+\infty)$, 0 is not a solution of the boundary value problem (1.1)-(1.2). We know that u^* and v^* are the maximal and minimal positive solutions of the boundary value problem (1.1)-(1.2) in $(0,b]$, which can be obtained by the corresponding iterative sequences in

$$v_{n+1} = \int_0^{+\infty} G(t,s)a(s)f\bigl(s,v_n(s)\bigr)\,ds$$

with initial value $v_0(t) = 0$,

$$u_{n+1} = \int_0^{+\infty} G(t,s)a(s)f\bigl(s,u_n(s)\bigr)\,ds$$

with initial value $u_0(t) = b$. The proof is completed. □

4 Examples

In this section, we will present an example to illustrate our main results.

Example 4.1 Consider the following boundary value problem:

$$^{c}D_{0^+}^{\frac{5}{2}}u(t) = e^{-t}f\bigl(t,u(t)\bigr), \quad 0 < t < +\infty, \tag{4.1}$$

$$u(0) = u^{(4)}(0) = 0, \qquad ^{c}D_{0^+}^{\frac{3}{2}}u(+\infty) = \frac{3}{10}u\left(\frac{1}{4}\right) + \frac{1}{5}u(1), \tag{4.2}$$

where $a(t) = e^{-t}$ and

$$f(t,u) = \begin{cases} \frac{1}{100(1+t^4)} + \frac{1}{10}(\frac{u}{1+t^{\frac{3}{2}}})^5, & u \in [0,1], \\ \frac{1}{100(1+t^4)} + \frac{1}{10}(\frac{1}{1+t^{\frac{3}{2}}})^5, & u > 1. \end{cases}$$

Here $\alpha = \frac{5}{2}$.

It is clear that $f(t,0) \not\equiv 0$ on any subinterval of $[0,+\infty)$ and $f(t,(1+t^{\frac{3}{2}})u) \le \frac{11}{100}$, so condition ($\mathrm{H}_1$) holds.

In view of $\int_0^{+\infty} a(t)\,dt = \int_0^{+\infty} e^{-t}\,dt = 1$, so condition ($\mathrm{H}_2$) holds.

Taking $\omega(t) = \frac{1}{5}t$, $\varphi(t) = e^{-t}$, $b = 1$, by a simple computation, we have

$$f\bigl(t,\bigl(1+t^{\frac{3}{2}}\bigr)u\bigr) \le \frac{11}{100} \le \frac{\frac{3}{10} + \frac{1}{5}}{2\int_0^{+\infty} e^{-t}\,dt} = \frac{1}{4},$$

so condition (H_3) holds.

By Theorem 3.1, we see that the boundary value problem (4.1)-(4.2) has the minimal and maximal positive solutions in $(0,1]$, which can be obtained by two explicit monotone iterative sequences.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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