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On nonlocal fractional q-integral boundary value problems of fractional q-difference and fractional q-integrodifference equations involving different numbers of order and q

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Abstract

In this paper, we study some new class of nonlocal three-point fractional q-integral boundary value problems of a nonlinear fractional q-difference equation and a nonlinear fractional q-integrodifference equation. Our problems contain different numbers of order and q in derivatives and integrals. The existence and uniqueness results are based on Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. In addition, some examples are presented to illustrate the importance of these results.

MSC: 34A08; 34B18; 39A13

Keywords: existence; fractional q-derivative; fractional q-integral; fractional q-difference equation; fractional q-integrodifference equation

1 Introduction

Jackson [1] initiated quantum calculus or q-difference calculus that can describe many phenomena in various fields of science and engineering. Basic definitions and properties of q-difference calculus can be found in the book [2]. For the fractional q-difference calculus originating with Al-Salam [3] and Agarwal [4], we refer to the book of Annaby and Mansour [5].

A class of integral boundary value problems appeared in different areas of applied mathematics and physics. For instance, blood flow problems, population dynamics, heat conduction, underground water flow, thermo-elasticity, plasma physics, chemical engineering and so on can be reduced to nonlocal integral boundary problems. For comments on the importance of integral boundary problems, we refer the reader to the papers by Webb and Infante [6, 7], Gallardo [8], Karakostas and Tsamatos [9], Lomtatidze and Malaguti [10], and the references therein. For information as regards the theory of integral equations and their applications to integral boundary value problems, we refer to the books of Agarwal and O'Regan [11] and Corduneanu [12].

For some recent works on q-integral boundary value problems, we refer to [13–29] and the references cited therein. For example, Ahmad *et al.* [21] considered the following boundary value problem of nonlinear fractional q-difference equation with nonlocal and sub-strip type boundary conditions:

$$\begin{cases} {}^{C}D_q^{\nu}x(t) = f(t,x(t)), \quad t \in [0,1], \\ x(0) = x_0 + g(x), \qquad x(\xi) = b\int_\eta^1 x(s)\,d_qs, \end{cases} \tag{1.1}$$

where $1 < \nu \leq 2$, $0 < q < 1$, $0 < \xi < \eta < 1$, ${}^{C}D_q^{\nu}$ is the Caputo fractional q-derivative of order ν, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, and $g \in C([0,1], \mathbb{R})$ are given functions. The existence results for the problem (1.1) are shown by applying Banach's contraction mapping principle and a fixed point theorem due to O'Regan.

Almeida and Martins [26] proposed the following fractional q-difference equation with three-point integral boundary conditions:

$$\begin{cases} {}^{C}D_q^{\alpha}[x](t) = g(t,x(t)), \quad t \in [0,1], \\ x(0) = \gamma_0, \qquad D_q[x](0) = \gamma_2, \qquad x(1) = \gamma_2\int_0^\eta x(s)\,d_qs, \end{cases} \tag{1.2}$$

where $2 < \alpha < 3$, $0 < \eta \leq 1$, ${}^{C}D_q^{\alpha}$ is the Caputo fractional q-derivative of order α, $g \in C([0,1] \times \mathbb{R}, \mathbb{R})$. They presented the existence and uniqueness results for the problem (1.2) by employing Banach's contraction mapping principle, Krasnoselskii fixed point theorem and Leray-Schauder alternative.

Presently, there is a development of boundary value problems for fractional q-difference equations showing an operation of the investigative function. The study may also have another function, related to our interest. These creations are incorporating nonlocal conditions that are both extensive and more complex.

The results mentioned above are the motivation for this research. In this article, we discuss the existence and uniqueness results of solutions to a nonlinear fractional q-difference equation with nonlocal three-point fractional q-integral boundary conditions of the form

$$\begin{cases} D_q^{\alpha}x(t) = f(t,x(t),D_w^{\nu}x(t)), \quad t \in [0,T], \\ x(\eta) = \rho(x), \\ I_p^{\beta}g(T)x(T) = \frac{1}{\Gamma_p(\beta)}\int_0^T g(s)(T-ps)^{(\beta-1)}x(s)\,d_ps = 0, \end{cases} \tag{1.3}$$

and a nonlinear fractional q-integrodifference equation with nonlocal three-point fractional q-integral boundary conditions of the form

$$\begin{cases} D_q^{\alpha}x(t) = f(t,x(t),\Psi_w^{\gamma}x(t)), \quad t \in [0,T], \\ x(\eta) = \rho(x), \\ I_p^{\beta}g(T)x(T) = \frac{1}{\Gamma_p(\beta)}\int_0^T g(s)(T-ps)^{(\beta-1)}x(s)\,d_ps = 0, \end{cases} \tag{1.4}$$

where $p, q, w \in (0,1)$, $\alpha \in (1,2]$, $\nu \in (0,1]$, $\beta, \gamma > 0$, and $\eta \in (0,T)$ are given constants, D_q^{α} and D_w^{ν} are the Riemann-Liouville fractional q-derivative of order α and w-derivative of order ν, respectively, $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g \in C([0,T], \mathbb{R}^{+})$ are given functions,

$\rho \in C([0,T],\mathbb{R}) \to \mathbb{R}$ is a given functional, and for $\varphi \in C([0,T]\times[0,T],[0,\infty))$,

$$\Psi_w^\gamma x(t) := \bigl(I_w^\gamma \varphi x\bigr)(t) = \frac{1}{\Gamma_w(\gamma)}\int_0^t (t-ws)^{(\gamma-1)}\varphi(t,s)x(s)\,d_ws.$$

The plan of this paper is as follows. In Section 2, we recall some definitions and basic lemmas. In Sections 3 and 4, we prove the existence and uniqueness results for the boundary value problems (1.3) and (1.4) by employing Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. Some illustrative examples are presented in the last section.

2 Preliminaries

In the following, there are notations, definitions, and lemmas which are used in the main results. Let $q \in (0,1)$ and define

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analog of the power function $(a-b)^{(n)}$ with $n \in \mathbb{N}_0 := [0,1,2,\ldots]$ is

$$(a-b)^{(0)} := 1, \qquad (a-b)^{(n)} := \prod_{k=0}^{n-1}\bigl(a-bq^k\bigr), \quad a,b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} := a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b=0$ then $a^{(\alpha)} = a^\alpha$. We also use the notation $0^{(\alpha)} = 0$ for $\alpha > 0$. The q-gamma function is defined by

$$\Gamma_q(x) := \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R}\setminus\{0,-1,-2,\ldots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

Remark [13] We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$.

Definition 2.1 [4] For $\alpha \ge 0$ and f defined on $[0,T]$, the fractional q-integral of the Riemann-Liouville type is defined by

$$\begin{aligned}
\bigl(I_q^\alpha f\bigr)(x) &:= \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)}f(t)\,d_qt \\
&= \frac{x(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\bigl(x-x^{n+1}\bigr)^{(\alpha-1)}f\bigl(xq^n\bigr) \\
&= \frac{x^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\bigl(1-q^{n+1}\bigr)^{(\alpha-1)}f\bigl(xq^n\bigr),
\end{aligned}$$

and $(I_q^0 f)(x) = f(x)$.

Definition 2.2 [15] For $\alpha \geq 0$ and f defined on $[0,T]$, the fractional q-derivative of the Riemann-Liouville type of order α is defined by

$$\left(D_q^{\alpha} f\right)(x) := \left(D_q^m I_q^{m-\alpha} f\right)(x), \quad \alpha > 0,$$

and $(D_q^0 f)(x) = f(x)$, where m is the smallest integer that is greater than or equal to α.

Definition 2.3 [2] For any $x, s > 0$,

$$\begin{aligned} B_q(x,s) &:= \int_0^1 t^{(x-1)}(1-qt)^{(s-1)}\, d_q t \\ &= (1-q)\sum_{n=0}^{\infty} q^n \left(1-q^{n+1}\right)^{(\alpha-1)} \left(q^n\right)^{(x-1)} \\ &= \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)}, \end{aligned}$$

is called the q-beta function.

Lemma 2.1 [4] *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,T]$. Then the next formulas hold*:

(i) $\left(I_q^{\beta} I_q^{\alpha} f\right)(x) = \left(I_q^{\alpha+\beta} f\right)(x)$,

(ii) $\left(D_q^{\alpha} I_q^{\alpha} f\right)(x) = f(x)$.

Lemma 2.2 [15] *Let $\alpha > 0$ and N be a positive integer. Then the following equality holds*:

$$\left(I_q^{\alpha} D_q^N f\right)(x) = \left(D_q^N I_q^{\alpha} f\right)(x) - \sum_{k=0}^{N-1} \frac{x^{\alpha-N+k}}{\Gamma_q(\alpha+k-N+1)} \left(D_q^k f\right)(0).$$

Lemma 2.3 [16] *Let $\alpha, \beta \geq 0$ and $0 < p, q < 1$. Then the following formulas hold*:

(i) $\displaystyle\int_0^{\eta} (\eta - qt)^{(\alpha-1)} t^{(\beta)}\, d_q t = \eta^{\alpha+\beta} B_q(\alpha, \beta+1)$,

(ii) $\displaystyle\int_0^{\eta}\int_0^{s} (\eta - ps)^{(\alpha-1)} (s-qt)^{(\beta-1)}\, d_q t\, d_p s = \frac{\eta^{\alpha+\beta}}{[\beta]_q} B_p(\alpha, \beta+1)$.

To define the solution of the boundary value problems (1.3) and (1.4), we need the following lemma, which deals with a linear variant of the boundary value problems (1.3) and (1.4) and gives a representation of the solution.

Lemma 2.4 *Let $p, q \in (0,1)$, $\alpha \in (1,2]$, $\beta > 0$, $\eta \in (0,T)$, functions $y \in C([0,T],\mathbb{R})$ and $g \in C([0,T],\mathbb{R}^+)$, and a functional $\rho : C([0,T],\mathbb{R}) \to \mathbb{R}$. Then the boundary value problem*

$$D_q^{\alpha} x(t) = y(t), \quad t \in [0,T], \tag{2.1}$$

$$x(\eta) = \rho(x), \qquad I_p^{\beta} g(T)x(T) = 0, \tag{2.2}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} y(s)\, d_qs \\ & - \frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha) \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_ps} \\ & \times \int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)} y(r)\, d_qr\, d_ps \\ & + \frac{t^{\alpha-2} \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(t-s)\, d_ps}{\eta^{\alpha-2}\Gamma_q(\alpha) \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_ps} \\ & \times \left[\int_0^\eta (\eta-qs)^{(\alpha-1)} y(s)\, d_qs - \Gamma_q(\alpha)\rho(x) \right]. \end{aligned} \tag{2.3}$$

Proof Consider $m = 2$. By Definition 2.2 and Lemma 2.1, we obtain

$$\left(I_q^\alpha D_q^2 I_q^{2-\alpha} x\right)(t) = \left(I_q^\alpha y\right)(t) \tag{2.4}$$

and

$$x(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} y(s)\, d_qs. \tag{2.5}$$

The first condition of (2.2) implies

$$C_1 \eta^{\alpha-1} + C_2 \eta^{\alpha-2} = \rho(x) - \frac{1}{\Gamma_q(\alpha)} \int_0^\eta (\eta-qs)^{(\alpha-1)} y(s)\, d_qs. \tag{2.6}$$

Taking the fractional p-integral of order $\beta > 0$ for (2.6) and the second condition of (2.2), we get

$$\begin{aligned} & \frac{C_1}{\Gamma_p(\beta)} \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-1}\, d_ps + \frac{C_2}{\Gamma_p(\beta)} \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}\, d_ps \\ & \quad = -\frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)} \int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)} y(r)\, d_qr\, d_ps. \end{aligned} \tag{2.7}$$

Solving the system of linear equations (2.6) and (2.7), for the unknown constants C_1 and C_2, we have

$$\begin{aligned} C_1 = {} & -\frac{1}{\eta^{\alpha-2}\Gamma_q(\alpha) \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_ps} \\ & \times \left[\eta^{\alpha-2} \int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)} y(r)\, d_qr\, d_ps \right. \\ & \left. - \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}\, d_ps \left(\int_0^\eta (\eta-qs)^{(\alpha-1)} y(s)\, d_qs - \Gamma_q(\alpha)\rho(x) \right) \right], \end{aligned}$$

$$C_2 = \frac{1}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \times \biggl[\eta^{\alpha-1}\int_0^T\int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)}y(r)\,d_qr\,d_ps - \int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-1}\,d_ps\biggl(\int_0^\eta(\eta-qs)^{(\alpha-1)}y(s)\,d_qs-\Gamma_q(\alpha)\rho(x)\biggr)\biggr].$$

Substituting the constants C_1 and C_2 into (2.5), we obtain (2.3).

This completes the proof. □

3 Existence results of the problem (1.3)

Let $\mathcal{C}_\mathcal{A} = C([0,T],\mathbb{R})$ be a Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$, endowed with the norm defined by

$$\|x\|_{\mathcal{C}_\mathcal{A}} = \max\bigl\{\|x\|, \bigl\|D_\omega^\nu x\bigr\|\bigr\},$$

where $\|x\| = \sup_{t\in[0,T]}|x(t)|$ and $\|D_\omega^\nu x\| = \sup_{t\in[0,T]}|D_\omega^\nu x(t)|$. Define the operator $\mathcal{A}:\mathcal{C}_\mathcal{A}\to\mathcal{C}_\mathcal{A}$ by

$$\begin{aligned}(\mathcal{A}x)(t) :={}& \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}f\bigl(s,x(s),D_w^\nu x(s)\bigr)\,d_qs \\ &- \frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\ &\times\int_0^T\int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)}f\bigl(r,x(r),D_w^\nu x(r)\bigr)\,d_qr\,d_ps \\ &+ \frac{t^{\alpha-2}\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(t-s)\,d_ps}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\ &\times\biggl[\int_0^\eta(\eta-qs)^{(\alpha-1)}f\bigl(s,x(s),D_w^\nu x(s)\bigr)\,d_qs-\Gamma_q(\alpha)\rho(x)\biggr].\end{aligned} \tag{3.1}$$

Observe that the problem (1.3) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

Now, we are in the position to establish the main results. Our first result is based on Banach's fixed point theorem.

Theorem 3.1 *Assume that a functional $\rho\in C([0,T],\mathbb{R})\to\mathbb{R}$, $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:[0,T]\to\mathbb{R}$ are continuous functions satisfying the following conditions*:

(H_1) *There exist positive numbers L_1, L_2 such that, for each $t\in[0,T]$ and $x,y\in\mathcal{C}_\mathcal{A}$,*

$$\bigl|f\bigl(t,x,D_w^\nu x\bigr)-f\bigl(t,y,D_w^\nu y\bigr)\bigr|\le L_1|x-y|+L_2\bigl|D_w^\nu x-D_w^\nu y\bigr|.$$

(H_2) *There exists a positive number τ such that, for each $x,y\in\mathcal{C}_\mathcal{A}$,*

$$\bigl|\rho(x)-\rho(y)\bigr|\le\tau\|x-y\|_{\mathcal{C}_\mathcal{A}}.$$

(H_3) *For each* $t \in [0,T]$, $0 < n < g(t) < N$.
(H_4) $\Theta := \lambda(\Omega + \Lambda) + \frac{\tau\Gamma_q(\alpha+1)}{\eta^2}\Lambda < 1$,

where

$$\begin{aligned}
\lambda &= \max\{L_1 + L_2\},\\
\Omega &= \frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{N|T-\eta|T^\alpha\Gamma_p(\alpha+1)}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|[\alpha+\beta]_p\Gamma_q(\alpha+1)\Gamma_p(\alpha-1)},\\
\Lambda &= \frac{NT^{\alpha-1}\eta^2|[\alpha-1]_p - [\alpha+\beta-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|\Gamma_q(\alpha+1)}.
\end{aligned} \tag{3.2}$$

Then the boundary value problem (1.3) *has a unique solution*.

Proof We transform the boundary value problem (1.3) into a fixed point problem $x = \mathcal{A}x$, where $\mathcal{A}: \mathcal{C}_\mathcal{A} \to \mathcal{C}_\mathcal{A}$ is defined by (3.1). Assuming that $\sup_{t\in[0,T]}|f(t,0,0)| = M$ and $\sup_{x\in\mathcal{C}_\mathcal{A}}|\rho(x)| = K$, we choose a constant R satisfied with

$$R \geq \frac{M(\Omega+\Lambda) + \frac{\Gamma_q(\alpha+1)}{\eta^2}K\Lambda}{1-\Theta}. \tag{3.3}$$

Now, we will show that $\mathcal{A}B_R \subset B_R$, where $B_R = \{x \in \mathcal{C}_\mathcal{A} : \|x\|_{\mathcal{C}_\mathcal{A}} \leq R\}$. For all $x, y \in \mathcal{C}_\mathcal{A}$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&|\mathcal{A}x|\\
&\quad\leq \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\left(\left|f\left(s,x(s),D_w^\nu x(s)\right) - f(s,0,0)\right| + \left|f(s,0,0)\right|\right) d_qs\\
&\qquad+ \frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps}\int_0^T\int_0^s g(r)(T-ps)^{(\beta-1)}\\
&\qquad\times(s-qr)^{(\alpha-1)}\left(\left|f\left(r,x(r),D_w^\nu x(r)\right) - f(r,0,0)\right| + \left|f(r,0,0)\right|\right) d_qr\,d_ps\\
&\qquad+ \frac{t^{\alpha-2}\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(t-s)\,d_ps}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps}\Bigg[\Gamma_q(\alpha)\left(\left|\rho(x)-\rho(0)\right| + \left|\rho(0)\right|\right)\\
&\qquad+ \int_0^\eta(\eta-qs)^{(\alpha-1)}\left(\left|f\left(s,x(s),D_w^\nu x(s)\right) - f(s,0,0)\right| + \left|f(s,0,0)\right|\right) d_qs\Bigg]\\
&\quad\leq \left(L_1\|x\| + L_2\left\|D_\omega^\nu x\right\| + M\right)\left[\frac{t^\alpha}{\Gamma_q(\alpha+1)}\right] + \left(L_1\|x\| + L_2\left\|D_\omega^\nu x\right\| + M\right)\\
&\qquad\times\left[\frac{Nt^{\alpha-2}T^2|t-\eta|\Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right]\\
&\qquad+ \left(L_1\|x\| + L_2\left\|D_\omega^\nu x\right\| + M\right)\left[\frac{N\eta^2t^{\alpha-2}|t[\alpha+\beta-1]_p - T[\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right]\\
&\qquad+ \left(\tau\|x\|_{\mathcal{C}_\mathcal{A}} + K\right)\left[\frac{Nt^{\alpha-2}|t[\alpha+\beta-1]_p - T[\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right]\\
&\quad\leq \left(\lambda\|x\|_{\mathcal{C}_\mathcal{A}} + M\right)\left\{\frac{NT^\alpha|T-\eta|\Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{T^{\alpha}}{\Gamma_q(\alpha+1)}+\frac{N\eta^2T^{\alpha-1}|[\alpha+\beta-1]_p-[\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p-\eta[\alpha+\beta-1]_p|}\biggr\} \\
&+\bigl(\tau\|x\|_{\mathcal{C}_{\mathcal{A}}}+K\bigr)\biggl\{\frac{NT^{\alpha-1}|[\alpha+\beta-1]_p-[\alpha-1]_p|}{n|T[\alpha-1]_p-\eta[\alpha+\beta-1]_p|}\biggr\} \\
={}& R\Theta+M(\Omega+\Lambda)+\frac{\Gamma_q(\alpha+1)}{\eta^2}K\Lambda
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigl|D_w^{\nu}\mathcal{A}x\bigr| \\
&\quad=\bigl|D_wI_w^{1-\nu}\mathcal{A}x\bigr| \\
&\quad=\biggl|D_w\biggl\{-\frac{\int_0^T\int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{\alpha-1}f(r,x(r),D_w^{\nu}x(r))\,d_qr\,d_ps}{\Gamma_w(1-\nu)\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\
&\qquad\times\int_0^t(t-ws)^{(-\nu)}s^{\alpha-2}(s-\eta)\,d_ws \\
&\qquad\times\frac{\int_0^{\eta}(\eta-qs)^{\alpha-1}f(s,x(s),D_w^{\nu}x(s))\,d_ws-\Gamma_q(\alpha)\rho(x)}{\eta^{\alpha-2}\Gamma_w(1-\nu)\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\
&\qquad\times\biggl[\int_0^t(t-ws)^{(-\nu)}s^{\alpha-1}\,d_w\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}\,d_ps \\
&\qquad-\int_0^t(t-ws)^{(-\nu)}s^{\alpha-2}\,d_ws\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-1}\,d_ps\biggr] \\
&\qquad+\frac{1}{\Gamma_q(\alpha)\Gamma_w(1-\nu)}\int_0^t\int_0^s(t-ws)^{(-\nu)}(s-qr)^{\alpha-1}f\bigl(r,x(r),D_w^{\nu}x(r)\bigr)\,d_qr\,d_ws\biggr\}\biggr| \\
&\quad=\frac{[\int_0^{wt}(wt-ws)^{(-\nu)}s^{\alpha-2}(s-\eta)\,d_ws-\int_0^t(t-ws)^{(-\nu)}s^{\alpha-2}(s-\eta)\,d_ws]}{(1-w)t\Gamma_w(1-\nu)\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\
&\qquad\times\biggl[\int_0^T\int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{\alpha-1}\bigl(\bigl|f\bigl(r,x,D_w^{\nu}x\bigr)-f(r,0,0)\bigr|+\bigl|f(r,0,0)\bigr|\bigr)\,d_qr\,d_ps\biggr] \\
&\qquad+\frac{\int_0^{\eta}(\eta-qs)^{\alpha-1}(|f(s,x,D_w^{\nu}x)-f(s,0,0)|+|f(s,0,0)|)\,d_ws+\Gamma_q(\alpha)(|\rho(x)-\rho(0)|+|\rho(0)|)}{(1-w)t\eta^{\alpha-2}\Gamma_w(1-\nu)\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\
&\qquad\times\biggl[\biggl|\int_0^{wt}(wt-ws)^{(-\nu)}s^{\alpha-1}\,d_w-\int_0^t(t-ws)^{(-\nu)}s^{\alpha-1}\,d_w\biggr| \\
&\qquad\times\int_0^T g(s)(T-ps)^{\beta-1}s^{\alpha-2}\,d_ps+\biggl|\int_0^{wt}(wt-ws)^{(-\nu)}s^{\alpha-2}\,d_w \\
&\qquad-\int_0^t(t-ws)^{(-\nu)}s^{\alpha-2}\,d_w\biggr|\int_0^T g(s)(T-ps)^{\beta-1}s^{\alpha-1}\,d_ps\biggr] \\
&\qquad+\frac{1}{(1-w)t\Gamma_q(\alpha)\Gamma_w(1-\nu)}\biggl|\int_0^{wt}\int_0^s(wt-ws)^{(-\nu)}(s-qr)^{\alpha-1} \\
&\qquad\times\bigl(\bigl|f\bigl(r,x,D_w^{\nu}x\bigr)-f(r,0,0)\bigr|+\bigl|f(r,0,0)\bigr|\bigr)\,d_qr\,d_ws \\
&\qquad-\int_0^t\int_0^s(t-ws)^{(-\nu)}(s-qr)^{\alpha-1}\bigl(\bigl|f\bigl(r,x,D_w^{\nu}x\bigr)-f(r,0,0)\bigr|+\bigl|f(r,0,0)\bigr|\bigr)\,d_qr\,d_ws\biggr| \\
&\quad\le\bigl(L_1\|x\|+L_2\|D_{\omega}^{\nu}x\|+M\bigr) \\
&\qquad\times\biggl[\frac{NTt^{\alpha-\nu-2}\Gamma_p(\alpha+1)}{n\Gamma_q(\alpha+1)[\alpha+\beta]_p\Gamma_p(\alpha-1)|[\alpha-1]_p-T[\alpha+\beta-1]_p|}\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&\times \left| \frac{t\Gamma_w(\alpha)(1-w^{\alpha-\nu})}{(1-w)\Gamma_w(\alpha-\nu+1)} - \frac{\Gamma_w(\alpha-1)(1-w^{\alpha-\nu-1})}{(1-w)\Gamma_w(\alpha-\nu)} \right| \\
&+ \Bigg[\big(L_1\|x\| + L_2\|D_\omega^\nu x\| + M\big) \\
&\times \left(\frac{N\eta^2 t^{\alpha-\nu-2}}{nT\Gamma_q(\alpha+1)\Gamma_p(\alpha-1)\big|[\alpha-1]_p - T[\alpha+\beta-1]_p\big|} \right) \\
&+ \big(\tau\|x\|_{\mathcal{C}_\mathcal{A}} + K\big) \left(\frac{Nt^{\alpha-\nu-2}}{nT\Gamma_p(\alpha-1)|[\alpha-1]_p - T[\alpha+\beta-1]_p|} \right) \Bigg] \\
&\times \left| \frac{t\Gamma_w(\alpha)(1-w^{\alpha-\nu})[\alpha+\beta-1]_p}{(1-w)\Gamma_w(\alpha-\nu+1)} - \frac{\Gamma_w(\alpha-1)(1-w^{\alpha-\nu-1})T[\alpha-1]_p}{(1-w)\Gamma_w(\alpha-\nu)} \right| \\
&+ \big(L_1\|x\| + L_2\|D_\omega^\nu x\| + M\big) \left[\frac{t^{\alpha-\nu}\Gamma_w(\alpha+1)}{\Gamma_q(\alpha+1)\Gamma_w(\alpha-\nu+2)} \right] \left| \frac{1-w^{\alpha-\nu+1}}{1-w} \right| \\
\leq{}& \big(\lambda\|x\|_{\mathcal{C}_\mathcal{A}} + M\big) \Bigg\{ \left[\frac{NT^{\alpha-\nu-1}\Gamma_p(\alpha+1)}{n\Gamma_q(\alpha+1)[\alpha+\beta]_p\Gamma_p(\alpha-1)|[\alpha-1]_p - T[\alpha+\beta-1]_p|} \right] \\
&\times \left| \frac{T\Gamma_w(\alpha)(1-w^{\alpha-\nu})}{(1-w)\Gamma_w(\alpha-\nu+1)} - \frac{\Gamma_w(\alpha-1)(1-w^{\alpha-\nu-1})}{(1-w)\Gamma_w(\alpha-\nu)} \right| \\
&+ \left[\frac{N\eta^2 T^{\alpha-\nu-1}}{n\Gamma_q(\alpha+1)\Gamma_p(\alpha-1)|[\alpha-1]_p - T[\alpha+\beta-1]_p|} \right] \\
&\times \left| \frac{\Gamma_w(\alpha)(1-w^{\alpha-\nu})[\alpha+\beta-1]_p}{(1-w)\Gamma_w(\alpha-\nu+1)} - \frac{\Gamma_w(\alpha-1)(1-w^{\alpha-\nu-1})[\alpha-1]_p}{(1-w)\Gamma_w(\alpha-\nu)} \right| \\
&+ \left[\frac{T^{\alpha-\nu}\Gamma_w(\alpha+1)}{\Gamma_q(\alpha+1)\Gamma_w(\alpha-\nu+2)} \right] \left| \frac{1-w^{\alpha-\nu+1}}{1-w} \right| \Bigg\} \\
&+ \big(\tau\|x\|_{\mathcal{C}_\mathcal{A}} + K\big) \left[\frac{NT^{\alpha-\nu-1}}{n\Gamma_p(\alpha-1)|[\alpha-1]_p - T[\alpha+\beta-1]_p|} \right] \\
&\times \left| \frac{\Gamma_w(\alpha)(1-w^{\alpha-\nu})[\alpha+\beta-1]_p}{(1-w)\Gamma_w(\alpha-\nu+1)} - \frac{\Gamma_w(\alpha-1)(1-w^{\alpha-\nu-1})[\alpha-1]_p}{(1-w)\Gamma_w(\alpha-\nu)} \right| \\
={}& (\lambda R + M) \Bigg\{ \left(\frac{T^\alpha}{\Gamma_q(\alpha+1)} \right) \left[\frac{1}{T^\nu} \cdot \frac{\Gamma_w(\alpha+1)}{\Gamma_w(\alpha-\nu+2)} \cdot \left| \frac{1-w^{\alpha-\nu+1}}{1-w} \right| \right] \\
&+ \left(\frac{NT^\alpha|T-\eta|\Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right) \\
&\times \Bigg[\frac{1}{T^{\nu+1}} \cdot \frac{|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}{|T[\alpha+\beta-1]_p - [\alpha-1]_p|} \\
&\times \frac{1}{|T-\eta|} \left| T\frac{\Gamma_w(\alpha)}{\Gamma_w(\alpha-\nu+1)} \cdot \frac{(1-w^{\alpha-\nu})}{(1-w)} - \frac{\Gamma_w(\alpha-1)}{\Gamma_w(\alpha-\nu)} \cdot \frac{(1-w^{\alpha-\nu-1})}{(1-w)} \right| \Bigg] \\
&+ \left(\frac{N\eta^2 T^{\alpha-1}|[\alpha-1]_p - [\alpha+\beta-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right) \\
&\times \Bigg[\frac{1}{T^\nu} \cdot \frac{|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}{|T[\alpha+\beta-1]_p - [\alpha-1]_p|} \cdot \frac{1}{|[\alpha+\beta-1]_p - [\alpha-1]_p|} \\
&\times \left| \frac{\Gamma_w(\alpha)}{\Gamma_w(\alpha-\nu+1)} \cdot \frac{(1-w^{\alpha-\nu})}{(1-w)}[\alpha+\beta-1]_p - \frac{\Gamma_w(\alpha-1)}{\Gamma_w(\alpha-\nu)} \cdot \frac{(1-w^{\alpha-\nu-1})}{(1-w)}[\alpha-1]_p \right| \Bigg] \Bigg\} \\
&+ (\tau R + K) \Bigg\{ \left(\frac{NT^{\alpha-1}|[\alpha-1]_p - [\alpha+\beta-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right) \\
&\times \Bigg[\frac{1}{T^\nu} \cdot \frac{|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}{|T[\alpha+\beta-1]_p - [\alpha-1]_p|} \cdot \frac{1}{|[\alpha+\beta-1]_p - [\alpha-1]_p|}
\end{aligned}
$$

$$\begin{aligned}
&\times \left| \frac{\Gamma_w(\alpha)}{\Gamma_w(\alpha-\nu+1)} \cdot \frac{(1-w^{\alpha-\nu})}{(1-w)}[\alpha+\beta-1]_p - \frac{\Gamma_w(\alpha-1)}{\Gamma_w(\alpha-\nu)} \cdot \frac{(1-w^{\alpha-\nu-1})}{(1-w)}[\alpha-1]_p \right| \Bigg] \Bigg\} \\
&< R\Theta + M\Omega + \left(M + \frac{\Gamma_q(\alpha+1)}{\eta^2} \right)\Lambda.
\end{aligned}$$

Therefore, we obtain $\|\mathcal{A}x\|_{\mathcal{C}_\mathcal{A}} \leq R$ and hence $\mathcal{A}B_R \subset B_R$.

Next, we will show that $\mathcal{A}$ is a contraction. Denote

$$\mathcal{S}\big[t,x,y,D_w^\nu x, D_w^\nu y\big] = \big| f\big(t,x(t),D_w^\nu x(t)\big) - f\big(t,y(t),D_w^\nu y(t)\big)\big|.$$

For all $x,y \in \mathcal{C}_\mathcal{A}$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
&|\mathcal{A}x - \mathcal{A}y| \\
&\quad\leq \Bigg| \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \mathcal{S}\big[s,x,y,D_w^\nu x, D_w^\nu y\big]\, d_qs \\
&\qquad - \frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\, d_ps} \int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)} \\
&\qquad \times (s-qr)^{(\alpha-1)} \mathcal{S}\big[r,x,y,D_w^\nu x, D_w^\nu y\big]\, d_qr\, d_ps \\
&\qquad + \frac{t^{\alpha-2}\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(t-s)\, d_ps}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\, d_ps} \\
&\qquad \times \left(\int_0^\eta (\eta-qs)^{(\alpha-1)} \mathcal{S}\big[s,x,y,D_w^\nu x, D_w^\nu y\big]\, d_qs - \Gamma_q(\alpha)\big|\rho(x)-\rho(y)\big| \right) \Bigg| \\
&\quad\leq \lambda\|x-y\|_{\mathcal{C}_\mathcal{A}} \left\{ \frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{NT^\alpha|T-\eta|\Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right. \\
&\qquad \left. + \frac{N\eta^2T^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&\qquad + \tau\|x-y\|_{\mathcal{C}_\mathcal{A}} \left\{ \frac{NT^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&\quad= \|x-y\|_{\mathcal{C}_\mathcal{A}}\Theta
\end{aligned}$$

and

$$\begin{aligned}
&\big|D_w^\nu\mathcal{A}x - D_w^\nu\mathcal{A}y\big| \\
&\quad= \big|D_w\big(I_w^{1-\nu}\mathcal{A}x - I_w^{1-\nu}\mathcal{A}y\big)\big| \\
&\quad= \Bigg| D_w \Bigg\{ - \frac{\int_0^T\int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{\alpha-1}\mathcal{S}[r,x,y,D_w^\nu x, D_w^\nu y]\, d_qr\, d_ps}{\Gamma_w(1-\nu)\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\, d_ps} \\
&\qquad \times \int_0^t (t-ws)^{(-\nu)}s^{\alpha-2}(s-\eta)\, d_ws \\
&\qquad + \frac{\int_0^\eta (\eta-qs)^{\alpha-1}\mathcal{S}[s,x,y,D_w^\nu x, D_w^\nu y]\, d_ws - \Gamma_q(\alpha)|\rho(x)-\rho(y)|}{\eta^{\alpha-2}\Gamma_w(1-\nu)\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\, d_ps} \\
&\qquad \times \Bigg[\int_0^t (t-ws)^{(-\nu)}s^{\alpha-1}\, d_w \int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}\, d_ps
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^t (t-ws)^{(-\nu)} s^{\alpha-2}\, d_w s \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-1}\, d_p s \Big] \\
&+ \frac{1}{\Gamma_q(\alpha)\Gamma_w(1-\nu)} \int_0^t \int_0^s (t-ws)^{(-\nu)} (s-qr)^{\alpha-1} \mathcal{S}\big[r, x, y, D_w^\nu x, D_w^\nu y\big]\, d_q r\, d_w s \Big\} \Big| \\
\leq{}& \lambda \|x-y\|_{\mathcal{C}_\mathcal{A}} \left\{ \frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{N T^\alpha |T-\eta| \Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p |T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right. \\
&\left. + \frac{N\eta^2 T^{\alpha-1} |[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&+ \tau \|x-y\|_{\mathcal{C}_\mathcal{A}} \left\{ \frac{N T^{\alpha-1} |[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
={}& \|x-y\|_{\mathcal{C}_\mathcal{A}} \Theta.
\end{aligned}
$$

Thus, $\|\mathcal{A}x - \mathcal{A}y\|_{\mathcal{C}_\mathcal{A}} \leq \Theta \|x-y\|_{\mathcal{C}_\mathcal{A}}$. From ($H_4$), we see that $\mathcal{A}$ is a contraction.

Hence, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof. □

Our second result is based on the following Krasnoselskii fixed point theorem.

Theorem 3.2 (Krasnoselskii fixed point theorem) [30] *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be operators such that*

(i) *$Ax + By \in K$ whenever $x, y \in K$,*

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then there exists $z \in K$ such that $z = Az + Bz$.

Theorem 3.3 (Arzela-Ascoli theorem) [30] *Let $D \subseteq \mathbb{R}^n$ be a bounded domain, $K \subseteq C(\overline{D}, \mathbb{R})$ be bounded and the following property of equicontinuity holds. For every $\epsilon > 0$, there exists $\delta > 0$, so that*

$$
\|x-y\| < \delta \quad \Rightarrow \quad \big|u(x) - u(y)\big| < \epsilon, \quad \forall x, y \in \overline{D}, \forall u \in K.
$$

Then $\overline{K}$ is compact.

Theorem 3.4 *Assume that* (H_2)-(H_3) *hold. In addition, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following condition*:

(H_5) *For all $(t, x, D_w^\nu x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, with $\mu \in C([0,T], \mathbb{R}^+)$,*

$$
\big|f\big(t, x, D_w^\nu x\big)\big| \leq \mu(t).
$$

If

$$
\Phi := \|\mu\|(\Omega + \Lambda) + \frac{\alpha\tau}{\eta^2}\Lambda < 1, \tag{3.4}
$$

then the boundary value problem (1.3) *has at least one solution on* $[0,T]$.

Proof Set $\sup_{t\in[0,T]}|\mu(t)| = \|\mu\|$, and choose a constant

$$R \geq \Phi. \tag{3.5}$$

In view of Lemma 2.4, we define the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on the ball $B_R = \{x \in \mathcal{C}_{\mathcal{A}} : \|x\|_{\mathcal{C}_{\mathcal{A}}} \leq R\}$ by

$$\begin{aligned}
(\mathcal{A}_1 x)(t) = {}& -\frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\
&\times \int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)} f\bigl(r, x(r), D_w^\nu x(r)\bigr)\,d_qr\,d_ps \\
&+ \frac{t^{\alpha-2}\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(t-s)\,d_ps}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\
&\times \left(\int_0^\eta (\eta-qs)^{(\alpha-1)} f\bigl(s, x(s), D_w^\nu x(s)\bigr)\,d_qs - \Gamma_q(\alpha)\rho(x)\right), \qquad (3.6)\\
(\mathcal{A}_2 x)(t) = {}& \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f\bigl(s, x(s), D_w^\nu x(s)\bigr)\,d_qs. \qquad (3.7)
\end{aligned}$$

For all $x, y \in B_R$, by computing directly, we have

$$\begin{aligned}
&|\mathcal{A}_1 x + \mathcal{A}_2 y| \\
&\quad \leq \|\mu\| \left\{ \frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{NT^\alpha|T-\eta|\Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right. \\
&\qquad \left. + \frac{N\eta^2 T^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} + \tau \left\{ \frac{NT^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&\quad = \|\mu\|(\Omega + \Lambda) + \frac{\alpha\tau}{\eta^2}\Lambda \\
&\quad = \Phi \leq R.
\end{aligned}$$

Similarly to the proof above and Theorem 3.1, we obtain $\|D_w^\nu \mathcal{A}_1 x + D_w^\nu \mathcal{A}_2 y\| < R$, and hence $\|\mathcal{A}_1 x + \mathcal{A}_2 y\|_{\mathcal{C}_{\mathcal{A}}} < R$. Therefore,

$$\mathcal{A}_1 x + \mathcal{A}_2 y \in B_R.$$

The condition (3.3) implies that $\mathcal{A}_2$ is a contraction mapping.

Next, we will show that $\mathcal{A}_1$ is compact and continuous. Continuity of f coupled with the assumption (H_4) implies that the operator $\mathcal{A}_1$ is continuous and uniformly bounded on B_R. For $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\bigl|\mathcal{A}_1 x(t_2) - \mathcal{A}_1 x(t_1)\bigr| \\
&\quad \leq \bigl(\bigl|t_2^{\alpha-2} - t_1^{\alpha-2}\bigr| + \eta\bigl|t_2^{\alpha-1} - t_1^{\alpha-1}\bigr|\bigr) \\
&\qquad \times \frac{\int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)} f(r, x(r), D_w^\nu x(r))\,d_qr\,d_ps}{\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps}
\end{aligned}$$

$$
\begin{aligned}
&+ \left|t_2^{\alpha-2} - t_1^{\alpha-2}\right| \left(\int_0^\eta (\eta - qs)^{(\alpha-1)} f\big(s, x(s), D_w^\nu x(s)\big)\, d_q s + \Gamma_q(\alpha)\rho(x) \right) \\
&\times \frac{\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(t-s)\, d_p s}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_p s} \\
\leq{}& \|\mu\| \left\{ \frac{(|t_2^{\alpha-2} - t_1^{\alpha-2}| + \eta|t_2^{\alpha-1} - t_1^{\alpha-1}|) N T^2 \Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p |T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right. \\
&\left. + \frac{N\eta^2 T |t_2^{\alpha-2} - t_1^{\alpha-2}| |[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&+ \tau \left\{ \frac{NT|t_2^{\alpha-2} - t_1^{\alpha-2}| |[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\}.
\end{aligned}
$$

Similarly to the proof above and Theorem 3.1, we obtain

$$
\begin{aligned}
&\left|D_w^\mu \mathcal{A}_1 x(t_2) - D_w^\mu \mathcal{A}_1 x(t_1)\right| \\
&\quad < \left|\mathcal{A}_1 x(t_2) - \mathcal{A}_1 x(t_1)\right| \\
&\quad \leq \|\mu\| \left\{ \frac{(|t_2^{\alpha-2} - t_1^{\alpha-2}| + \eta|t_2^{\alpha-1} - t_1^{\alpha-1}|) N T^2 \Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p |T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right. \\
&\qquad \left. + \frac{N\eta^2 T |t_2^{\alpha-2} - t_1^{\alpha-2}| |[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&\qquad + \tau \left\{ \frac{NT|t_2^{\alpha-2} - t_1^{\alpha-2}| |[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\}.
\end{aligned}
$$

Actually, as $|t_2 - t_1| \to 0$, the right-hand side of the above inequality tends to be zero. So $\mathcal{A}_1$ is relatively compact on B_R. Hence, by the Arzela-Ascoli theorem, $\mathcal{A}_1$ is compact on B_R.

Therefore, all the assumptions of Theorem (3.2) are satisfied and the conclusion of Theorem 3.2 implies that the boundary value problem (1.3) has at least one solution on $[0, T]$. This completes the proof. □

4 Existence results of the problem (1.4)

Let $\mathcal{C} = C([0,T], \mathbb{R})$ be a Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$ endowed with the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Define the operator $\mathcal{B} : \mathcal{C} \to \mathcal{C}$ as follows:

$$
\begin{aligned}
(\mathcal{B}x)(t) ={}& \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f\big(s, x(s), \Psi_w^\gamma x(s)\big)\, d_q s \\
&- \frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_p s} \\
&\times \int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)} f\big(r, x(r), \Psi_w^\gamma x(r)\big)\, d_q r\, d_p s \\
&+ \frac{t^{\alpha-2}\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(t-s)\, d_p s}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_p s} \\
&\times \left[\int_0^\eta (\eta-qs)^{(\alpha-1)} f\big(s, x(s), \Psi_w^\gamma x(s)\big)\, d_q s - \Gamma_q(\alpha)\rho(x) \right].
\end{aligned} \tag{4.1}
$$

Observe that the problem (1.4) has solutions if and only if the operator $\mathcal{B}$ has fixed points.

Our first result is based on Banach's fixed point theorem.

Theorem 4.1 *Assume that a functional $\rho : C([0,T],\mathbb{R}) \to \mathbb{R}$ is continuous, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and maps bounded subsets of $[0,T] \times \mathbb{R} \times \mathbb{R}$ into relatively compact subsets of $\mathbb{R}$, $g : [0,T] \to \mathbb{R}^+$, and $\varphi : [0,T] \times [0,T] \to [0,\infty)$ are continuous functions. Let $\varphi_0 = \sup_{(t,s)\in[0,T]\times[0,T]}\{\varphi(t,s)\}$ and* (H_3) *hold. In addition, ρ and f satisfy the following conditions*:

(H_6) *There exists a positive number τ such that, for all $x, y \in \mathcal{C}$,*

$$\left|\rho(x) - \rho(y)\right| \le \tau \|x - y\|.$$

(H_7) *There exist positive numbers l_1, l_2 such that, for each $t \in [0,T]$ and $x, y \in \mathcal{C}$,*

$$\left|f\left(t, x, \Psi_w^\gamma x\right) - f\left(t, y, \Psi_w^\gamma y\right)\right| \le l_1|x - y| + l_2\left|\Psi_w^\gamma x - \Psi_w^\gamma y\right|.$$

(H_8) $\Upsilon := (l_1 + l_2\frac{\varphi_0 T^\gamma}{\Gamma_w(\gamma+1)})\Omega + \frac{\tau\Gamma_q(\alpha+1)}{\eta^2}\Lambda < 1$, *where Ω, Λ are defined as* (3.2).

Then the boundary value problem (1.4) *has a unique solution.*

Proof We transform the boundary value problem (1.4) into a fixed point problem $x = \mathcal{B}x$, where $\mathcal{B} : \mathcal{C} \to \mathcal{C}$ is defined by (4.1). Assuming that $\sup_{t\in[0,T]} |f(t,0,0)| = M$ and $\sup_{x\in\mathcal{C}} \rho(x) = K$, we choose a constant ρ satisfied with

$$\rho \ge \frac{M(\Omega + \Lambda) + \frac{\Gamma_q(\alpha+1)}{\eta^2}K\Lambda}{1 - \Upsilon}. \tag{4.2}$$

Now, we will show that $\mathcal{B}B_\rho \subset B_\rho$, where $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$. For all $x \in B_\rho$, we have

$$\begin{aligned}
&|\mathcal{B}x| \\
&\quad\le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t - qs)^{(\alpha-1)}\left(\left|f\left(s, x(s), \Psi_w^\gamma x(s)\right) - f(s,0,0)\right| + \left|f(s,0,0)\right|\right) d_q s \\
&\qquad + \frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha)\int_0^T g(s)(T - ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_p s}\int_0^T\int_0^s g(r)(T - ps)^{(\beta-1)} \\
&\qquad \times (s - qr)^{(\alpha-1)}\left(\left|f\left(r, x(r), \Psi_w^\gamma x(r)\right) - f(r,0,0)\right| + \left|f(r,0,0)\right|\right) d_q r\, d_p s \\
&\qquad + \frac{t^{\alpha-2}\int_0^T g(s)(T - ps)^{(\beta-1)}s^{\alpha-2}(t-s)\,d_p s}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T - ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_p s}\biggl(\Gamma_q(\alpha)\left(\left|\rho(x) - \rho(0)\right| + \left|\rho(0)\right|\right) \\
&\qquad + \int_0^\eta (\eta - qs)^{(\alpha-1)}\left(\left|f\left(s, x(s), \Psi_w^\gamma x(s)\right) - f(s,0,0)\right| + \left|f(s,0,0)\right|\right) d_q s\biggr) \\
&\quad\le \left[\left(l_1 + l_2\frac{\varphi_0 T^\gamma}{\Gamma_w(\gamma+1)}\right)\|x\| + M\right] \\
&\qquad \times \left\{\frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{NT^\alpha|T-\eta|\Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right. \\
&\qquad \left. + \frac{N\eta^2 T^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \big(\tau \|x\| + K\big)\left\{\frac{NT^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right\} \\
&= \rho\Upsilon + M(\Omega+\Lambda) + \frac{\Gamma_q(\alpha+1)}{\eta^2}K\Lambda \\
&\leq \rho.
\end{aligned}
$$

Therefore, $\mathcal{B}B_\rho \subset B_\rho$.

Next, we will show that $\mathcal{B}$ is a contraction. For all $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, we have

$$
\begin{aligned}
&|\mathcal{B}x - \mathcal{B}y| \\
&\quad \leq \Bigg| \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \big| f\big(s, x(s), \Psi_w^\gamma x(s)\big) - f\big(s, y(s), \Psi_w^\gamma y(s)\big)\big| \, d_q s \\
&\qquad - \frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_p s} \int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)} \\
&\qquad \times (s-qr)^{(\alpha-1)} \big| f\big(r, x(r), \Psi_w^\gamma x(r)\big) - f\big(r, y(r), \Psi_w^\gamma y(s)\big)\big| \, d_q r \, d_p s \\
&\qquad + \frac{t^{\alpha-2}\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(t-s)\, d_p s}{\eta^{\alpha-2}\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_p s} \\
&\qquad \times \bigg(\int_0^\eta (\eta-qs)^{(\alpha-1)} \big| f\big(s, x(s), \Psi_w^\gamma x(s)\big) - f\big(s, y(s), \Psi_w^\gamma y(s)\big)\big| \, d_q s \\
&\qquad - \Gamma_q(\alpha)\big(\rho(x) - \rho(y)\big)\bigg)\Bigg| \\
&\quad \leq \left[\left(l_1 + l_2 \frac{\varphi_0 T^\gamma}{\Gamma_w(\gamma+1)}\right)\|x-y\|\right] \\
&\qquad \times \left\{ \frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{NT^\alpha|T-\eta|\Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right. \\
&\qquad \left. + \frac{N\eta^2 T^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&\qquad + \tau\left\{ \frac{NT^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&\quad = \Upsilon\|x-y\|.
\end{aligned}
$$

By (H_8), we have $\mathcal{B}$ is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof. □

Our second result is based on the following Krasnoselskii's fixed point theorem.

Theorem 4.2 *Assume that* (H_3) *and* (H_6) *hold. In addition,* $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the following condition*:

(H_9) *For all* $(t, x, \Psi_w^\gamma x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, *with* $\sigma \in C([0,T], \mathbb{R}^+)$,

$$
\big| f\big(t, x, \Psi_w^\gamma x\big)\big| \leq \sigma(t).
$$

If

$$\chi := \|\sigma\|(\Omega + \Lambda) + \frac{\alpha\tau}{\eta^2}\Lambda < 1, \tag{4.3}$$

then the boundary value problem (1.4) *has at least one solution on* $[0, T]$.

Proof Set $\sup_{t\in[0,T]} |\sigma(t)| = \|\sigma\|$ and choose a constant

$$\rho \geq \chi. \tag{4.4}$$

In view of Lemma 2.4, we define the operators $\mathcal{B}_1$ and $\mathcal{B}_2$ on the ball $B_\rho = \{x \in \mathcal{C} : \|x\| \leq \rho\}$ by

$$\begin{aligned}
(\mathcal{B}_1x)(t) = {}& -\frac{t^{\alpha-2}(t-\eta)}{\Gamma_q(\alpha)\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\
&\times \int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)}f\bigl(r, x(r), \Psi_w^\gamma x(r)\bigr)\,d_qr\,d_ps \\
&+ \frac{t^{\alpha-2}\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(t-s)\,d_ps}{\eta^{\alpha-2}\Gamma_q(\alpha)(\int_0^T g(s)(T-ps)^{(\beta-1)}s^{\alpha-2}(s-\eta)\,d_ps} \\
&\times \left(\int_0^\eta (\eta-qs)^{(\alpha-1)}f\bigl(s, x(s), \Psi_w^\gamma x(s)\bigr)\,d_qs - \Gamma_q(\alpha)\rho(x)\right),
\end{aligned} \tag{4.5}$$

$$(\mathcal{B}_2x)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}f\bigl(s, x(s), \Psi_w^\gamma x(s)\bigr)\,d_qs. \tag{4.6}$$

For $x, y \in B_\rho$, by computing directly, we have

$$\begin{aligned}
&|\mathcal{B}_1x + \mathcal{B}_2y| \\
&\quad\leq \|\sigma\|\left\{\frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{NT^\alpha|T-\eta|\Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right. \\
&\qquad\left. + \frac{N\eta^2T^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right\} \\
&\qquad + \tau\left\{\frac{NT^{\alpha-1}|[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|}\right\} \\
&\quad= \|\sigma\|(\Omega+\Lambda) + \frac{\alpha\tau}{\eta^2}\Lambda \\
&\quad= \chi \leq \rho.
\end{aligned}$$

Therefore $\mathcal{B}_1x + \mathcal{B}_2y \in B_\rho$. The condition (4.3) implies that $\mathcal{B}_2$ is a contraction mapping.

Next, we will show that $\mathcal{B}_1$ is compact and continuous. Continuity of f coupled with the assumption (H_7) implies that the operator $\mathcal{B}_1$ is continuous and uniformly bounded on B_ρ. For $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\bigl|\mathcal{B}_1x(t_2) - \mathcal{B}_1x(t_1)\bigr| \\
&\quad\leq \bigl(\bigl|t_2^{\alpha-2} - t_1^{\alpha-2}\bigr| + \eta\bigl|t_2^{\alpha-1} - t_1^{\alpha-1}\bigr|\bigr)
\end{aligned}$$

$$\begin{aligned}
&\times \frac{\int_0^T \int_0^s g(r)(T-ps)^{(\beta-1)}(s-qr)^{(\alpha-1)} f(r, x(r), \Psi_w^\gamma x(r))\, d_q r\, d_p s}{\Gamma_q(\alpha) \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_p s} \\
&+ \left|t_2^{\alpha-2} - t_1^{\alpha-2}\right| \left(\int_0^\eta (\eta - qs)^{(\alpha-1)} f\big(s, x(s), \Psi_w^\gamma x(s)\big)\, d_q s + \Gamma_q(\alpha)\rho(x) \right) \\
&\times \frac{\int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(t-s)\, d_p s}{\eta^{\alpha-2}\Gamma_q(\alpha) \int_0^T g(s)(T-ps)^{(\beta-1)} s^{\alpha-2}(s-\eta)\, d_p s} \\
\le{}& \|\sigma\| \left\{ \frac{(|t_2^{\alpha-2} - t_1^{\alpha-2}| + \eta |t_2^{\alpha-1} - t_1^{\alpha-1}|) N T^2 \Gamma_p(\alpha+1)}{n\Gamma_p(\alpha-1)\Gamma_q(\alpha+1)[\alpha+\beta]_p |T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right. \\
&\left. + \frac{N\eta^2 T |t_2^{\alpha-2} - t_1^{\alpha-2}| |[\alpha+\beta-1]_p - [\alpha-1]_p|}{n\Gamma_q(\alpha+1)|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\} \\
&+ \tau \left\{ \frac{NT |t_2^{\alpha-2} - t_1^{\alpha-2}| |[\alpha+\beta-1]_p - [\alpha-1]_p|}{n|T[\alpha-1]_p - \eta[\alpha+\beta-1]_p|} \right\}.
\end{aligned}$$

Actually, as $|t_2 - t_1| \to 0$, the right-hand side of the above inequality tends to be zero. So $\mathcal{B}_1$ is relatively compact on B_ρ. Hence, by the Arzela-Ascoli theorem, $\mathcal{B}_1$ is compact on B_ρ.

Therefore, all the assumptions of Theorem 4.2 are satisfied and the conclusion of Theorem 4.2 implies that the boundary value problem (1.4) has at least one solution on $[0, T]$. This completes the proof. □

5 Examples

In this section, we give some examples to illustrate our results.

Example 5.1 Consider the following fractional q-integral boundary value problem:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{4}{3}} x(t) = \frac{e^{-\cos^2(2\pi t)}}{100 + e^{\sin^2(2\pi t)}} \cdot \frac{\frac{|x(t)| + |D_{\frac{1}{4}}^{\frac{2}{3}} x(t)|}{4}}{1 + |x(t)|}, \quad t \in [0,1], \\ x(\frac{1}{4}) = 1 + \sum_{i=1}^n C_i x(t_i), \qquad I_{\frac{1}{3}}^{\frac{5}{2}} e^{\sin(2\pi t)} x(1) = 0, \end{cases} \tag{5.1}$$

where $0 < t_1, t_2, \ldots, t_n < 1$ and C_i are given positive constants with $\sum_{i=1}^n C_i < \frac{1}{400}$.

Here $\alpha = \frac{4}{3}$, $\beta = \frac{5}{2}$, $\eta = \frac{1}{4}$, $\nu = \frac{2}{3}$, $q = \frac{1}{2}$, $p = \frac{1}{3}$, $w = \frac{1}{4}$, $T = 1$, $g(t) = e^{\sin(2\pi t)}$, and $f(t,x) = \frac{e^{-\cos^2(2\pi t)}}{20 + e^{\sin^2(2\pi t)}} \cdot \frac{\frac{|x(t)| + |D_{\frac{1}{4}}^{\frac{2}{3}} x(t)|}{4}}{1 + |x(t)|}$.

Since $|f(t, x, D_{\frac{1}{2}}^{\frac{4}{3}} x) - f(t, y, D_{\frac{1}{2}}^{\frac{4}{3}} y)| \le \frac{1}{101}|x - y| + \frac{1}{101}|D_{\frac{1}{2}}^{\frac{4}{3}} x - D_{\frac{1}{2}}^{\frac{4}{3}} y|$, ($\mathrm{H}_1$) is satisfied with $L_1 = L_2 = \frac{1}{101}$, so $\lambda = \frac{2}{101}$.

Also, we get $|\rho(x) - \rho(y)| = |\sum_{i=1}^n C_i x(t_i) - \sum_{i=1}^n C_i y(t_i)| \le \sum_{i=1}^n C_i |x - y|$. So, ($\mathrm{H}_2$) holds with $\tau = \sum_{i=1}^n C_i < \frac{1}{400}$.

Since $\frac{1}{e} < g(t) < e$, (H_3) is satisfied with $N = e$, $n = \frac{1}{e}$.

We can show that

$$\Omega = \frac{1}{\Gamma_{\frac{1}{2}}(\frac{7}{3})} + \frac{e|1 - \frac{1}{4}|\Gamma_{\frac{1}{3}}(\frac{7}{3})}{\frac{1}{e}|[\frac{1}{3}]_{\frac{1}{3}} - \frac{1}{4}[\frac{17}{6}]_{\frac{1}{3}}|[\frac{23}{6}]_{\frac{1}{3}}\Gamma_{\frac{1}{2}}(\frac{7}{3})\Gamma_{\frac{1}{3}}(\frac{1}{3})} \approx 2.2358,$$

$$\Lambda = \frac{\frac{e}{16}|[\frac{1}{3}]_{\frac{1}{3}} - [\frac{17}{6}]_{\frac{1}{3}}|}{\frac{1}{e}|[\frac{1}{3}]_{\frac{1}{3}} - \frac{1}{4}[\frac{17}{6}]_{\frac{1}{3}}|\Gamma_{\frac{1}{2}}(\frac{7}{3})} \approx 5.0088.$$

Therefore, we get

$$\Theta = \lambda(\Omega + \Lambda) + \frac{\tau \Gamma_q(\alpha + 1)}{\eta^2}\Lambda \approx 0.3668 < 1.$$

Hence, by Theorem 3.1, problem (5.1) has a unique solution on $[0, 1]$.

Example 5.2 Consider the following fractional q-integral boundary value problem:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{4}{3}} x(t) = \frac{e^{-\cos^2(2\pi t)}}{20+e^{\sin^2(2\pi t)}} \cdot \frac{|x(t)|+|\psi_{\frac{1}{4}}^{\frac{8}{3}} x(t)|}{1+|x(t)|}, & t \in [0,1], \\ x(\frac{1}{4}) = 1 + \sum_{i=1}^{n} C_i x(t_i), & I_{\frac{1}{3}}^{\frac{5}{2}} e^{\sin(2\pi t)} x(T) = 0. \end{cases} \tag{5.2}$$

where $0 < t_1, t_2, \ldots, t_n < 1$ and C_i are given positive constants with $\sum_{i=1}^{n} C_i < \frac{1}{500}$.

Here $\alpha = \frac{4}{3}$, $\beta = \frac{5}{2}$, $\eta = \frac{1}{4}$, $\gamma = \frac{8}{3}$, $q = \frac{1}{2}$, $p = \frac{1}{3}$, $w = \frac{1}{4}$, $T = 1$, $g(t) = e^{\sin(2\pi t)}$, $f(t,x) = \frac{e^{-\cos^2(2\pi t)}}{20+e^{\sin^2(2\pi t)}} \cdot \frac{|x(t)|+|\Psi_{\frac{1}{4}}^{\frac{8}{3}} x(t)|}{1+|x(t)|}$, $\Psi_{\frac{1}{4}}^{\frac{8}{3}} x(t) = \frac{1}{\Gamma_{\frac{1}{4}}(\frac{8}{3})} \int_0^t (t - \frac{s}{4})^{(\frac{5}{3})} \frac{e^{-(s-t)}}{10} x(s)\, d_w s$ and $\varphi(t,s) = \frac{e^{-(s-t)}}{10}$.

Also, we get $|\rho(x) - \rho(y)| = |\sum_{i=1}^{n} C_i x(t_i) - \sum_{i=1}^{n} C_i y(t_i)| \leq \sum_{i=1}^{n} C_i |x - y|$. So, (H_2) holds with $\tau = \sum_{i=1}^{n} C_i < \frac{1}{500}$.

Since $|f(t,x,\Psi_{\frac{1}{4}}^{\frac{8}{3}} x) - f(t,y,\Psi_{\frac{1}{4}}^{\frac{8}{3}} y)| \leq \frac{1}{21}|x-y| + \frac{1}{21}|\Psi_{\frac{1}{4}}^{\frac{8}{3}} x - \Psi_{\frac{1}{4}}^{\frac{8}{3}} y|$ and $\varphi_0 = \frac{e}{10}$, (H_5) is satisfied with $l_1 = l_2 = \frac{1}{21}$. So $l_1 + \frac{el_2}{10(\frac{8}{3}+1)} = 0.0977$.

By Example 5.1, we get $\Omega \approx 2.2358$ and $\Lambda \approx 5.0088$. Therefore, we have

$$\Upsilon = \left(l_1 + \frac{el_2}{10(\frac{8}{3}+1)}\right)\Omega + \frac{\tau \Gamma_q(\alpha)}{\eta^2}\Lambda \approx 0.4059 < 1.$$

Hence, by Theorem 4.1, problem (5.2) has a unique solution on $[0, 1]$.

Competing interests

The author declares to have no competing interests.

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