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Multiplicity of solutions for impulsive differential equation on the half-line via variational methods

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Abstract
In this paper, the existence of solutions for a second-order impulsive differential equation with two parameters on the half-line is investigated. Applying variational methods, we give some new criteria to guarantee that the impulsive problem has at least one classical solution, three classical solutions and infinitely many classical solutions, respectively. Some recent results are extended and significantly improved. Two examples are presented to demonstrate the application of our main results.

MSC: 34B08; 34B15; 34B37; 58E30

Keywords: impulsive differential equation; variational methods; critical points; half-line

1 Introduction

In this paper, we consider the following boundary value problem with impulses:

$$\begin{aligned} &-u''(t) + cu(t) = \lambda g\big(t, u(t)\big), \quad \text{a.e. } t \in [0, +\infty), \\ &\Delta u'(t_j) = I_j\big(u(t_j)\big), \quad j = 1, 2, \ldots, p, \\ &u'\big(0^+\big) = h\big(u(0)\big), \quad u'(+\infty) = 0, \end{aligned} \tag{1.1}$$

where c and λ are two positive parameters, $0 = t_0 < t_1 < \cdots < t_p < +\infty$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$, $u'(0^+) = \lim_{t\to 0^+} u'(t)$, and $u'(+\infty) = \lim_{t\to+\infty} u'(t)$, $h, I_j \in C(\mathbb{R}, \mathbb{R})$, and $g \in C([0, +\infty) \times \mathbb{R}, \mathbb{R})$.

Boundary value problems (BVPs) on the half-line occur in many applications; see [1–3]. Due to its significance, many researchers have studied BVPs for differential equations on the half-line, we refer the reader to [4–11].

On the other hand, impulsive differential equations have been widely applied in biology, control theory, industrial robotics, medicine, population dynamics and so on; see [12–17]. Due to its significance, a lot of work has been done in the theory of impulsive differential equations, we refer the reader to [18–24]. Some classical approaches and tools have been used to investigate BVPs for impulsive differential equations. These classical approaches and tools include the method of upper and lower solutions [23, 25], fixed point theorems [26] and topological degree theory [27, 28].

Recently, some researchers have used variational methods to investigate the existence and multiplicity of solutions for impulsive BVPs on the finite intervals [29–37]. However, as far as we know, with the exception of [38, 39], the study of solutions of impulsive BVPs on the infinite intervals via variational methods has received considerably less attention. More precisely, in [38, 39], the authors studied the following BVP:

$$\begin{aligned} &-u''(t)+u(t)=\lambda g\bigl(t,u(t)\bigr), \quad \text{a.e. } t\in[0,+\infty),\\ &\Delta u'(t_j)=I_j\bigl(u(t_j)\bigr), \quad j=1,2,\ldots,p,\\ &u'\bigl(0^+\bigr)=h\bigl(u(0)\bigr), \quad u'(+\infty)=0, \end{aligned} \tag{1.2}$$

where λ is a positive parameter, $h, I_j\in C(\mathbb{R},\mathbb{R})$ and $g\in C([0,+\infty)\times\mathbb{R},\mathbb{R})$. They obtained the existence and multiplicity of solutions for (1.2) via variational methods.

Obviously, problem (1.1) is a generalization of problem (1.2). In fact, problem (1.2) follows from problem (1.1) by letting $c=1$.

Motivated by the above facts, in this paper, we will improve and generalize some results in [38, 39].

In this paper, we need the following conditions.

(A_1) $h(u)$, $I_j(u)$ are nondecreasing, and $h(u)u\geq 0$, $I_j(u)u\geq 0$ for any $u\in\mathbb{R}$.

(A_2) $h(u)u\geq 0$, $I_j(u)u\geq 0$ for any $u\in\mathbb{R}$ ($j=1,2,\ldots,p$) and there exist constants $L, L_j\geq 0$ such that

$$\bigl|h(u)-h(v)\bigr|\leq L|u-v|, \qquad \bigl|I_j(u)-I_j(v)\bigr|\leq L_j|u-v| \quad \text{for any } u,v\in\mathbb{R},$$

where L, L_j satisfy $L+\sum_{j=1}^{p}L_j<\frac{1}{\beta^2}$, β will be given in (2.2).

(A_3) There exist $d,q>0$ such that

$$\frac{d^2}{\beta^2}<\frac{(1+c)q^2}{2}+2\sum_{j=1}^{p}\int_0^{qe^{-t_j}}I_j(s)\,ds+2\int_0^{q}h(s)\,ds$$

and

$$\alpha_1:=\frac{2\beta^2\int_0^{+\infty}\max_{|\xi|\leq d}G(t,\xi)\,dt}{d^2}<\alpha_2:=\frac{\int_0^{+\infty}G(t,qe^{-t})\,dt}{\frac{1+c}{4}q^2+\sum_{j=1}^{p}\int_0^{qe^{-t_j}}I_j(s)\,ds+\int_0^{q}h(s)\,ds},$$

where $G(t,u)=\int_0^u g(t,s)\,ds$, β will be given in (2.2).

Let $|\cdot|_k$ denotes the usual norm on $L^k[0,+\infty)$. Now, we state our main results.

Theorem 1.1 *Assume that* (A_1) (*or* (A_2)), (A_3) *hold and the following conditions are satisfied.*

(A_4) *There exist a positive constant* $\alpha\in(1,2)$ *and* $a_1,a_2,a_3\in L^1[0,+\infty)$ *such that*

$$\bigl|g(t,u)\bigr|\leq a_1(t)|u|+a_2(t)|u|^{\alpha-1}+a_3(t)$$

for a.e. $t\in[0,+\infty)$ *and all* $u\in\mathbb{R}$.

(A_5) *There exists a constant* m *satisfying*

$$\frac{(1+c)m^2}{2} + 2\sum_{j=1}^{p}\int_0^{me^{-t_j}} I_j(s)\,ds + 2\int_0^{m} h(s)\,ds \le \frac{d^2}{\beta^2}$$

such that

$$|a_1|_1 \le \frac{\int_0^{+\infty} G(t, me^{-t})\,dt}{d^2}.$$

Then, for each $\lambda \in]\frac{1}{\alpha_2}, \frac{1}{\alpha_1}[$, *problem* (1.1) *has at least three classical solutions.*

Remark 1.1 In (H_2) of [38], $l > 1$ is needed; see (2.5) of [38].

Remark 1.2 Obviously, Theorem 1.1 generalizes Theorem 3.1 in [38]. Furthermore, the function

$$g(t,u) = \begin{cases} \sqrt{\beta}e^{-t}, & u \le \beta, \\ e^{-t}(\frac{u}{100} + 600u^{\frac{1}{2}} - 599\sqrt{\beta} - \frac{\beta}{100}), & u > \beta, \end{cases} \tag{1.3}$$

does not satisfy (H_2) in [38], while it satisfies (A_4), and there are indeed many functions h and I_j not satisfying (H_1) in [38], while they satisfy (A_2), for example, $h(u) = -\theta_1 u$ and $I_j(u) = \theta_2 u(1 + \sin u)$, where $0 < \theta_1 < \frac{1}{2\beta^2}$ and $0 < \theta_2 < \frac{1}{4p\beta^2}$.

Theorem 1.2 *Assume that the following conditions are satisfied.*

(A_6) *There exist positive constants* c_3, $1 < \sigma < 2$, *and* $c_1, c_2, c_4, c_5, c_6 \in L^1[0,+\infty)$ *such that*

$$\left|G(t,u)\right| \le c_1(t)|u|^2 + c_2(t)\left(|u|^\sigma + c_3\right), \qquad \left|g(t,u)\right| \le c_4(t)|u| + c_5(t)|u|^{\sigma-1} + c_6(t)$$

for a.e. $t \in [0,+\infty)$ *and all* $u \in \mathbb{R}$.

(A_7) $h(u)u \ge 0$, $I_j(u)u \ge 0$ *for any* $u \in \mathbb{R}$ $(j = 1, 2, \ldots, p)$.

Then, for each $\lambda \in]0, \frac{1}{2\beta^2|c_1|_1}[$, *problem* (1.1) *has at least one classical solution.*

Remark 1.3 Let $c = 1$, it is clear that Theorem 1.2 improves Theorem 3.1 in [39]. In fact, there are many functions not satisfying the condition (S2) in [39], while they satisfy (A_6), for example, the function $g(t,u) = e^{-t}(u + u^{\frac{1}{2}})$.

Theorem 1.3 *Assume that the following conditions are satisfied.*

(A_8) *There exist constants* $c', c'_j > 0$ *and* $\delta, \delta_j \in (0,1)$ *such that*

$$\left|I_j(u)\right| \le c'_j|u|^{\delta_j}, \qquad \left|h(u)\right| \le c'|u|^{\delta} \quad \textit{for any } u \in \mathbb{R}.$$

(A_9) *There exist* $k_1, k_2 \in L^1[0,+\infty)$ *and* $\gamma_1 \in (0,1)$ *such that*

$$g(t,u) \le k_1(t)|u|^{\gamma_1} + k_2(t), \quad \textit{for a.e. } t \in [0,+\infty) \textit{ and all } u \in \mathbb{R}.$$

(A_{10}) *There exist an open set* $J \subset [0, +\infty)$ *and constants* $T, \eta > 0$ *and* $\gamma_2 \in (1, 2)$ *with* $\gamma_2 < \min\{\min_{1\leq j\leq p}\{\delta_j\}, \delta\} + 1$ *such that*

$$G(t,u) \geq \eta|u|^{\gamma_2}, \quad \forall (t,u) \in J \times \mathbb{R}, |u| \leq T.$$

Furthermore, suppose that $g(t,u)$, $I_j(u)$, *and* $h(u)$ *are odd about* u. *Then problem* (1.1) *has infinitely many classical solutions for* $\lambda > 0$.

Remark 1.4 By (S3) and (3.3) in [39], one has $d \in L^{\frac{2}{2-\alpha}}([0,+\infty),[0,+\infty))$ (in (S3)). Let $c = 1$, it is clear that Theorem 1.1 generalizes Theorem 3.2 in [39]. Furthermore, there are many functions g, h, and I_{ij} satisfying our Theorem 1.3 and not satisfying Theorem 3.2 in [39]. For example, let $I_j(u) = -u^{\frac{3}{5}}$, $h(u) = -u^{\frac{3}{5}}$, and $g(t,u) = (\frac{1}{(1+t^2)^2} - \frac{1}{(1+t)^2})u^{\frac{1}{3}}$.

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we give the proof of Theorems 1.1-1.3. Finally, two examples are presented to illustrate the main results.

2 Preliminaries

In order to prove Theorem 1.1, we will need to the following critical points theorem.

Theorem 2.1 ([40, 41]) *Let* X *be a reflexive real Banach space, let* $\Phi : X \to \mathbb{R}$ *be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on* X^*, *and let* $\Psi : X \to \mathbb{R}$ *be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist* $r \in \mathbb{R}$ *and* $x_0, x_1 \in X$, *with* $\Phi(x_0) < r < \Phi(x_1)$ *and* $\Psi(x_0) = 0$ *such that*

(i) $\sup_{\Phi(x)\leq r} \Psi(x) < (r - \Phi(x_0))\frac{\Psi(x_1)}{\Phi(x_1)-\Phi(x_0)}$,

(ii) *for each* $\lambda \in \Lambda_r :=]\frac{\Phi(x_1)-\Phi(x_0)}{\Psi(x_1)}, \frac{r-\Phi(x_0)}{\sup_{\Phi(x)\leq r}\Psi(x)}[$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then for each $\lambda \in \Lambda_r$, *the functional* $\Phi - \lambda\Psi$ *has at three distinct critical points in* X.

In order to prove Theorem 1.3, we will need to the following definitions and theorems. Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $e \in \mathbb{R}$. Let

$$\Sigma := \{J \subset X - \{0\} : J \text{ is closed in } X \text{ and symmetric with respect to } 0\},$$

$$K_e := \{u \in X : \varphi(u) = e, \varphi'(u) = 0\}, \quad \varphi^e := \{u \in X : \varphi(u) \leq e\}.$$

Definition 2.1 ([42]) For $A \in \Sigma$, we say the genus of A is n (denoted by $\gamma(A) = n$) if there is an odd $f \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property.

Definition 2.2 Suppose that X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$. If any sequence $\{u_n\} \subset X$ for which $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in X, we say that φ satisfies the Palais-Smale condition.

Theorem 2.2 ([43]) *Let* φ *be an even* C^1 *functional on* X *and satisfy the Palais-Smale condition. For any* $n \in \mathbb{N}$, *set*

$$\Sigma_n := \{A \in \Sigma : \gamma(A) \geq n\}, \qquad d_n := \inf_{A\in\Sigma_n} \sup_{u\in A} \varphi(u).$$

(i) *If* $\Sigma_n \neq \emptyset$ *and* $d_n \in \mathbb{R}$, *then* d_n *is a critical value of* φ.
(ii) *If there exists* $k_0 \in \mathbb{N}$ *such that*

$$d_n = d_{n+1} = \cdots = d_{n+k_0} = e \in \mathbb{R},$$

and $e \neq \varphi(0)$, *then* $\gamma(K_e) \geq k_0 + 1$.

Let us recall some basic concepts. Set

$$E = \{u : [0,+\infty) \to \mathbb{R} \mid u \text{ is absolutely continuous}, u' \in L^2[0,+\infty)\}.$$

Denote the Sobolev space by

$$X := \left\{u \in E \;\Big|\; \int_0^{+\infty} \left(\left|u'(t)\right|^2 + \left|u(t)\right|^2\right) dt < \infty\right\},$$

with the norm

$$\|u\|_X = \left(\int_0^{+\infty} \left(\left|u'(t)\right|^2 + c\left|u(t)\right|^2\right) dt\right)^{\frac{1}{2}}, \tag{2.1}$$

this norm is equivalent to the usual norm. Hence, X is a reflexive Banach space.

Let $C := \{u \in C[0,+\infty) \mid \sup_{t\in[0,+\infty)} |u(t)| < +\infty\}$, with the norm $\|u\|_C = \sup_{t\in[0,+\infty)} |u(t)|$. Then C is a Banach space. In addition, X is continuously embedded in C, thus, there exists a constant $\beta > 0$ such that

$$\|u\|_C \leq \beta\|u\|_X \quad \text{for any } u \in X. \tag{2.2}$$

Suppose that $u \in C[0,+\infty)$. Moreover, assume that for every $j = 0,1,2,\ldots,p-1$, $u_j = u|_{(t_j,t_{j+1})}$ satisfy $u_j \in C^2(t_j,t_{j+1})$ and $u_p = u|_{(t_p,+\infty)} \in C^2(t_p,+\infty)$. We say u is a classical solution of problem (1.1) if it satisfies the equation in problem (1.1) a.e. on $[0,+\infty)$, the limits $u'(0^+)$, $u'(+\infty)$, $u'(t_j^+)$, $u'(t_j^-)$ ($j = 1,2,\ldots,p$) exist, and the impulsive conditions and boundary conditions in problem (1.1) hold.

For every $u \in X$, put

$$\Phi(u) = \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)\, ds + \int_0^{u(0)} h(s)\, ds \tag{2.3}$$

and

$$\Psi(u) = \int_0^{+\infty} G(t,u)\, dt. \tag{2.4}$$

It is clear that Φ is Gâteaux differentiable at any $u \in X$ and

$$\left\langle\Phi'(u), v\right\rangle = \int_0^{+\infty} \left[u'(t)v'(t) + cu(t)v(t)\right] dt + \sum_{j=1}^{p} I_j\left(u(t_j)\right)v(t_j) + h\left(u(0)\right)v(0) \tag{2.5}$$

for any $v \in X$. Obviously, $\Phi' : X \to X^*$ is continuous.

Clearly, $\Psi : X \to \mathbb{R}$ is continuously Gâteaux differentiable functional at any $u \in X$ and

$$\langle \Psi'(u), v\rangle = \int_0^{+\infty} g\bigl(t, u(t)\bigr)v(t)\,dt \tag{2.6}$$

for any $v \in X$.

Lemma 2.1 *If $u \in X$ is a critical point of $\Phi - \lambda\Psi$, then u is a classical solution of problem* (1.1).

Proof The proof is similar to that of [38], and we omit it here. □

Lemma 2.2 *Assume that* (A_2) *are satisfied, then Φ is sequentially weakly lower semicontinuous, coercive and its derivative admits a continuous inverse on X^*.*

Proof Let $\{u_n\} \subset X$, $u_n \rightharpoonup u$ in X, we see that $\{u_n\}$ converges uniformly to u on $[0, M]$ for any $M \in (0, +\infty)$ and $\liminf_{n\to\infty} \|u_n\|_X \ge \|u\|_X$. Thus

$$\begin{aligned}\liminf_{n\to\infty} \Phi(u_n) &= \liminf_{n\to\infty}\left(\frac{1}{2}\|u_n\|_X^2 + \sum_{j=1}^{p}\int_0^{u_n(t_j)} I_j(s)\,ds + \int_0^{u_n(0)} h(s)\,ds\right) \\ &\ge \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)\,ds + \int_0^{u(0)} h(s)\,ds = \Phi(u).\end{aligned}$$

So Φ is sequentially weakly lower semicontinuous. Furthermore, in view of (2.3) and (A_2), we have

$$\Phi(u) = \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)\,ds + \int_0^{u(0)} h(s)\,ds \ge \frac{1}{2}\|u\|_X^2.$$

Thus, Φ is coercive.

Next we will show that Φ' admits a continuous inverse on X^*. For each $u \in X\backslash\{0\}$, by (2.5) and (A_2), we have

$$\langle \Phi'(u), u\rangle = \int_0^{+\infty}\bigl[\bigl|u'(t)\bigr|^2 + c\bigl|u(t)\bigr|^2\bigr]\,dt + \sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr)u(t_j) + h\bigl(u(0)\bigr)u(0) \ge \|u\|_X^2.$$

So $\lim_{\|u\|_X\to+\infty}\langle\Phi'(u), u\rangle/\|u\|_X = +\infty$, that is, Φ' is coercive.

For any $u, v \in X$, in view of (A_2) and (2.2), we have

$$\begin{aligned}\langle \Phi'(u) - \Phi'(v), u - v\rangle &= \|u - v\|_X^2 + \sum_{j=1}^{p}\bigl[I_j\bigl(u(t_j)\bigr) - I_j\bigl(v(t_j)\bigr)\bigr]\bigl(u(t_j) - v(t_j)\bigr) \\ &\quad + \bigl(h\bigl(u(0)\bigr) - h\bigl(v(0)\bigr)\bigr)\bigl(u(0) - v(0)\bigr) \\ &\ge \left[1 - \beta^2\left(L + \sum_{j=1}^{p} L_j\right)\right]\|u - v\|_X^2.\end{aligned}$$

Since $L + \sum_{j=1}^{p} L_j < \frac{1}{\beta^2}$, so Φ' is uniformly monotone. By [44], Theorem 26.A(d), we see that Φ' admits a continuous inverse on X^*. □

Lemma 2.3 *Assume that* (A_1) *holds, then* Φ *is sequentially weakly lower semicontinuous, coercive and its derivative admits a continuous inverse on* X^*.

Proof The proof is similar to the proof of Lemma 2.2, and we omit it here. □

Lemma 2.4 *Suppose that* (A_4) *is satisfied. If* $u_n \rightharpoonup u$ *in* E, *then* $g(t,u_n) \to g(t,u)$ *in* $L^1[0,+\infty)$.

Proof Assume that $u_n \rightharpoonup u$. In view of (A_4) and (2.2), we have

$$\begin{aligned}
\left|g(t,u_n)-g(t,u)\right| &\leq \left(a_1(t)|u_n|+a_2(t)|u_n|^{\alpha-1}+a_3(t)\right)+\left(a_1(t)|u|+a_2(t)|u|^{\alpha-1}+a_3(t)\right)\\
&\leq \left(\|u_n\|_C+\|u\|_C\right)a_1(t)+\left(\|u_n\|_C^{\alpha-1}+\|u\|_C^{\alpha-1}\right)a_2(t)+2a_3(t)\\
&\leq \beta\left(\|u_n\|_X+\|u\|_X\right)a_1(t)+\beta^{\alpha-1}\left(\|u_n\|_X^{\alpha-1}+\|u\|_X^{\alpha-1}\right)a_2(t)+2a_3(t).
\end{aligned}$$

Applying the Lebesgue dominated convergence theorem, the lemma is proved. □

Lemma 2.5 *The functional* Ψ *is a sequentially weakly upper semicontinuous and its derivative is compact.*

Proof Let $\{u_n\} \subset X$, $u_n \rightharpoonup u$ in X, we see that $\{u_n\}$ converges uniformly to u on $[0,M]$ for any $M \in (0,+\infty)$. It follows from the reverse Fatou lemma that

$$\begin{aligned}
\limsup_{n\to+\infty}\Psi(u_n) &= \limsup_{n\to+\infty}\lim_{M\to+\infty}\int_0^M G(t,u_n)\,dt\\
&\leq \lim_{M\to+\infty}\int_0^M \limsup_{n\to+\infty} G(t,u_n)\,dt\\
&= \int_0^{+\infty} G(t,u)\,dt=\Psi(u).
\end{aligned}$$

So Ψ is sequentially weakly upper semicontinuous.

Next we will show that Ψ' is compact. Let $\{u_n\} \subset X$, $u_n \rightharpoonup u$ in X. By Lemma 2.4, we get

$$\begin{aligned}
\left\|\Psi'(u_n)-\Psi'(u)\right\|_{X^*} &= \sup_{\|v\|_X=1}\left\|\left(\Psi'(u_n)-\Psi'(u)\right)v\right\|\\
&= \sup_{\|v\|_X=1}\left|\int_0^{+\infty}\left(g(t,u_n)-g(t,u)\right)v\,dt\right|\\
&\leq \|v\|_C \sup_{\|v\|_X=1}\int_0^{+\infty}\left|g(t,u_n)-g(t,u)\right|dt\\
&\leq \beta\int_0^{+\infty}\left|g(t,u_n)-g(t,u)\right|dt \to 0
\end{aligned}$$

as $k\to\infty$, for any $u \in X$. Thus, Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by [44], Proposition 26.2. □

3 Proof of Theorems 1.1-1.3

Now we give the proof of Theorem 1.1.

Proof By Lemma 2.3, Φ is a sequentially weakly lower semicontinuous, continuously Gâteaux derivative and coercive functional whose Gâteaux derivative admits a continuous inverse on X^*. By Lemma 2.5, Ψ is a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.

Let $r = \frac{d^2}{2\beta^2}$, $u_0(t) = 0$, $u_1(t) = qe^{-t}$ for any $t \in [0, +\infty)$, one has $u_0, u_1 \in X$, $\Phi(u_0) = \Psi(u_0) = 0$, $\Phi(u_1) = \frac{(1+c)q^2}{4} + \sum_{j=1}^{p} \int_0^{qe^{-t_j}} I_j(s)\,ds + \int_0^q h(s)\,ds$, $\Psi(u_1) = \int_0^{+\infty} G(t, qe^{-t})\,dt$. Therefore, we get

$$\big(r - \Phi(u_0)\big)\frac{\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} = \frac{d^2}{\beta^2}\frac{\int_0^{+\infty} G(t, qe^{-t})\,dt}{\frac{(1+c)q^2}{2} + 2\sum_{j=1}^{p}\int_0^{qe^{-t_j}} I_j(s)\,ds + 2\int_0^q h(s)\,ds}, \tag{3.1}$$

and by (A_3), we obtain $\Phi(u_0) < r < \Phi(u_1)$.

On the other hand, for any $u \in X$ such that $\Phi(u) \le r$, we have $\|u\|_X \le (2r)^{\frac{1}{2}}$. Owing to (2.2), one has $\|u\|_C \le \beta\|u\|_X \le \beta(2r)^{\frac{1}{2}} = d$. Therefore,

$$\sup_{\Phi(x)\le r} \Psi(x) \le \int_0^{+\infty} \max_{|\xi|\le d} G(t,\xi)\,dt. \tag{3.2}$$

By (3.1), (3.2), and (A_3), condition (i) in Theorem 2.1 is satisfied.

For any $u \in X$, in view of (A_1), (A_4), and (2.2), we obtain

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{p}\int_0^{u(t_j)} I_j(s)\,ds + \int_0^{u(0)} h(s)\,ds - \lambda\int_0^{+\infty} G\big(t,u(t)\big)\,dt \\ &\ge \left(\frac{1}{2} - \lambda\beta^2|a_1|_1\right)\|u\|_X^2 - \lambda|a_2|_1\beta^\alpha\|u\|_X^\alpha - \lambda\beta|a_3|_1\|u\|_X.\end{aligned}$$

In view of (A_5), we get

$$\frac{r - \Phi(u_0)}{\sup_{\Phi(u)\le r}\Psi(u)} = \frac{d^2}{2\beta^2\sup_{\Phi(x)\le r}\Psi(x)} \le \frac{d^2}{2\beta^2\int_0^{+\infty} G(t, me^{-t})\,dt} \le \frac{1}{2\beta^2|a_1|_1}. \tag{3.3}$$

Then, for any $\lambda \in]0, \frac{1}{2\beta^2|a_1|_1}[$ (with the conventions $\frac{1}{0} = +\infty$), we get $\lim_{\|u\|_X\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$. So condition (ii) in Theorem 2.1 is satisfied. Hence, by Theorem 2.1, for each $\lambda \in]\frac{1}{\alpha_2}, \frac{1}{\alpha_1}[$, the functional $\Phi - \lambda\Psi$ has at three distinct critical points in X. That is, for each $\lambda \in]\frac{1}{\alpha_2}, \frac{1}{\alpha_1}[$, problem (1.1) has at least three classical solutions. □

Now we give the proof of Theorem 1.2.

Proof First of all, we will show that $\Phi - \lambda\Psi$ is weakly lower semicontinuous. Let $\{u_n\} \subset X$, $u_n \rightharpoonup u$ in X, we see that $\{u_n\}$ converges uniformly to u on $[0,M]$ with $M \in (0,+\infty)$ an arbitrary constant and $\liminf_{n\to\infty}\|u_n\|_X \ge \|u\|_X$. By Lemma 2.4, we have

$$\begin{aligned}\liminf_{n\to\infty}\big(\Phi(u_n) - \lambda\Psi(u_n)\big) &\ge \liminf_{n\to\infty}\left(\frac{1}{2}\|u_n\|_X^2 + \sum_{j=1}^{p}\int_0^{u_n(t_j)} I_j(s)\,ds + \int_0^{u_n(0)} h(s)\,ds\right) \\ &\quad - \lambda\limsup_{n\to\infty}\Psi(u_n)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)\,ds + \int_0^{u(0)} h(s)\,ds \\
&\quad - \lambda \limsup_{n\to+\infty} \Psi(u_n) \\
&\geq \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)\,ds + \int_0^{u(0)} h(s)\,ds \\
&\quad - \lambda \int_0^{+\infty} G(t,u)\,dt \\
&= \Phi(u) - \lambda\Psi(u).
\end{aligned}
$$

Then $\Phi - \lambda\Psi$ is sequentially weakly lower semicontinuous.

Second, we will show that $\Phi - \lambda\Psi$ is coercive. By (A_6), (A_7), and (2.2), we obtain

$$
\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)\,ds + \int_0^{u(0)} h(s)\,ds - \lambda \int_0^{+\infty} G\bigl(t,u(t)\bigr)\,dt \\
&\geq \left(\frac{1}{2} - \lambda\beta^2|c_1|_1\right)\|u\|_X^2 - \lambda|c_2|_1\bigl(\beta^\sigma\|u\|_X^\sigma + c_3\bigr),
\end{aligned}
$$

for any $u \in X$. Since $0 < \sigma < 2$, for any $\lambda \in]0, \frac{1}{2\beta^2|c_1|_1}[$ (with the conventions $\frac{1}{0} = +\infty$), we obtain $\lim_{\|u\|\to\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$, that is, $\Phi - \lambda\Psi$ is coercive. Hence, $\Phi - \lambda\Psi$ has a minimum (Theorem 1.1 of [45]), which is a critical point of $\Phi - \lambda\Psi$. Thus, for each $\lambda \in]0, \frac{1}{2\beta^2|c_1|_1}[$, problem (1.1) has at least one classical solution. □

Now we give the proof of Theorem 1.3.

Proof Let $\varphi = \Phi - \lambda\Psi$. Obviously, $\varphi \in C^1(X,\mathbb{R})$. In the following, we first show that φ is bounded from below. By (A_8), (A_9), and (2.2), we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{p} \int_0^{u(t_j)} I_j(s)\,ds + \int_0^{u(0)} h(s)\,ds - \lambda \int_0^{+\infty} G\bigl(t,u(t)\bigr)\,dt \\
&\geq \frac{1}{2}\|u\|_X^2 - \sum_{j=1}^{p} c_j'\beta^{\delta_j+1}\|u\|_X^{\delta_j+1} - c'\beta^{\delta+1}\|u\|_X^{\delta+1} \\
&\quad - \lambda \int_0^{+\infty} \bigl(k_1(t)|u|^{\gamma_1+1} + k_2(t)|u|\bigr)\,dt \\
&\geq \frac{1}{2}\|u\|_X^2 - \sum_{j=1}^{p} c_j'\beta^{\delta_j+1}\|u\|_X^{\delta_j+1} - c'\beta^{\delta+1}\|u\|_X^{\delta+1} \\
&\quad - \lambda\beta^{\gamma_1+1}|k_1|_1\|u\|_X^{\gamma_1+1} - \lambda\beta|k_2|_1\|u\|_X.
\end{aligned}
\tag{3.4}
$$

Since $\delta_j, \delta \in (0,1)$ and $\gamma_1 \in (0,1)$, (3.4) implies that $\varphi(u) \to \infty$ as $\|u\|_X \to \infty$. Consequently, φ is bounded from below.

Next, we prove that φ satisfies the Palais-Smale condition. Suppose that $\{u_n\} \subset X$ such that $\{\varphi(u_n)\}$ be a bounded sequence and $\varphi'(u_n) \to 0$ as $n \to \infty$, it follows from (3.4) that $\{u_n\}$ is bounded in X. From the reflexivity of X, we may extract a weakly convergent subsequence, which, for simplicity, we call $\{u_n\}$, $u_n \rightharpoonup u$ in X. Next we will prove that $u_n \to u$

in X. By (2.5) and (2.6), we have

$$\begin{aligned}\big(\varphi'(u_n)-\varphi'(u)\big)(u_n-u) = {}& \|u_n-u\|_X^2+\big[h(u_n(0)-h\big(u(0)\big)\big]\big(u_n(0)-u(0)\big)\\&+\sum_{j=1}^{p}\big(I_j\big(u_n(t_j)\big)-I_j\big(u(t_j)\big)\big)\big(u_n(t_j)-u(t_j)\big)\\&-\lambda\int_0^{+\infty}\big(g\big(t,u_n(t)\big)-g\big(t,u(t)\big)\big)\big(u_n(t)-u(t)\big)\,dt.\end{aligned}\tag{3.5}$$

Obviously,

$$\big(\varphi'(u_n)-\varphi'(u)\big)(u_n-u)\to 0.\tag{3.6}$$

We claim that if $u_k \rightharpoonup u$ in E, then $g(t,u_k)\to g(t,u)$ in $L^1[0,+\infty)$. The proof is similar to that of Lemma 2.4, and we omit it here. By (2.2), we obtain

$$\begin{aligned}&\int_0^{+\infty}\big(g\big(t,u_n(t)\big)-g\big(t,u(t)\big)\big)\big(u_n(t)-u(t)\big)\,dt\\&\quad\le\big(\|u_n\|_C+\|u\|_C\big)\int_0^{+\infty}\big|g\big(t,u_n(t)\big)-g\big(t,u(t)\big)\big|\,dt\\&\quad\le\beta\big(\|u_n\|_X+\|u\|_X\big)\int_0^{+\infty}\big|g\big(t,u_n(t)\big)-g\big(t,u(t)\big)\big|\,dt\to 0\end{aligned}\tag{3.7}$$

as $n\to\infty$. Since $u_n\rightharpoonup u$ in X, for any $M>0$, we get $u_n\to u$ in $C[0,M]$. So

$$\begin{aligned}&\sum_{j=1}^{p}\big(I_j\big(u_n(t_j)\big)-I_j\big(u(t_j)\big)\big)\big(u_n(t_j)-u(t_j)\big)\to 0,\\&\big[h\big(u_n(0)\big)-h\big(u(0)\big)\big]\big(u_n(0)-u(0)\big)\to 0.\end{aligned}\tag{3.8}$$

In view of (3.5)-(3.8), we obtain $\|u_n-u\|_X\to 0$ as $n\to\infty$. Then φ satisfies the Palais-Smale condition.

It is easy to see that φ is even and $\varphi(0)=0$. In order to apply Theorem 2.2, we prove now that

$$\text{for each } n\in\mathbb{N},\ \text{there exists } \varepsilon>0 \text{ such that } \gamma\big(\varphi^{-\varepsilon}\big)\ge n.\tag{3.9}$$

For each $n\in\mathbb{N}$, we take n disjoint open sets B_i such that

$$\bigcup_{i=1}^{n}B_i\subset J.$$

For $i=1,2,\dots,n$, let $u_i\in(W_0^{1,2}(B_i)\cap X)$ and $\|u_i\|_X=1$, and

$$E_n=\operatorname{span}\{u_1,u_2,\dots,u_n\},\qquad J_n=\big\{u\in E_n:\|u\|_X=1\big\}.$$

For any $u \in E_n$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, such that

$$u(t) = \sum_{i=1}^{n} \lambda_i u_i(t) \quad \text{for } t \in [0, +\infty). \tag{3.10}$$

Then

$$|u|_{\gamma_2} = \left(\int_0^{+\infty} \left|u(t)\right|^{\gamma_2} dt \right)^{\frac{1}{\gamma_2}} = \left(\sum_{i=1}^{n} |\lambda_i|^{\gamma_2} \int_{B_i} \left|u_i(t)\right|^{\gamma_2} dt \right)^{\frac{1}{\gamma_2}} \tag{3.11}$$

and

$$\begin{aligned} \|u\|_X^2 &= \int_0^{+\infty} \left(\left|u_i'(t)\right|^2 + c\left|u(t)\right|^2 \right) dt \\ &= \sum_{i=1}^{n} \lambda_i^2 \int_{B_i} \left(\left|u_i'(t)\right|^2 + c\left|u_i(t)\right|^2 \right) dt \\ &= \sum_{i=1}^{n} \lambda_i^2 \int_0^{+\infty} \left(\left|u_i'(t)\right|^2 + c\left|u_i(t)\right|^2 \right) dt \\ &= \sum_{i=1}^{n} \lambda_i^2 \|u_i\|_X^2 \\ &= \sum_{i=1}^{n} \lambda_i^2. \end{aligned} \tag{3.12}$$

Since all norms of any finite dimensional normed space are equivalent, so there exists $M_0 > 0$ such that

$$M_0 \|u\|_X \le |u|_{\gamma_2} \quad \text{for } u \in E_n. \tag{3.13}$$

In view of (A_8), (A_{10}), (2.2), (3.11), (3.12), and (3.13), we get

$$\begin{aligned} \varphi(\rho u) &= \frac{\rho^2}{2} \|u\|_X^2 + \sum_{j=1}^{p} \int_0^{\rho u(t_j)} I_j(s)\, ds + \int_0^{\rho u(0)} h(s)\, ds - \lambda \int_0^{+\infty} G\big(t, \rho u(t)\big)\, dt \\ &= \frac{\rho^2}{2} \|u\|_X^2 + \sum_{j=1}^{p} \int_0^{\rho u(t_j)} I_j(s)\, ds + \int_0^{\rho u(0)} h(s)\, ds \\ &\quad - \lambda \sum_{i=1}^{n} \int_{B_i} G\big(t, \rho u(t)\big)\, dt \\ &\le \frac{\rho^2}{2} \|u\|_X^2 + \sum_{j=1}^{p} c_j'(\rho\beta)^{\delta_j+1} \|u\|_X^{\delta_j+1} + c'(\rho\beta)^{\delta+1} \|u\|_X^{\delta+1} \\ &\quad - \lambda\eta\rho^{\gamma_2} \sum_{i=1}^{n} |\lambda_i|^{\gamma_2} \int_{B_i} \left|u_i(t)\right|^{\gamma_2} dt \\ &= \frac{\rho^2}{2} \|u\|_X^2 + \sum_{j=1}^{p} c_j'(\rho\beta)^{\delta_j+1} \|u\|_X^{\delta_j+1} + c'(\rho\beta)^{\delta+1} \|u\|_X^{\delta+1} - \lambda\eta\rho^{\gamma_2} |u|_{\gamma_2}^{\gamma_2} \end{aligned}$$

$$\leq \frac{\rho^2}{2}\|u\|_X^2 + \sum_{j=1}^{p} c_j'(\rho\beta)^{\delta_j+1}\|u\|_X^{\delta_j+1} + c'(\rho\beta)^{\delta+1}\|u\|_X^{\delta+1} - \lambda\eta(M_0\rho)^{\gamma_2}\|u\|_X^{\gamma_2}$$

$$= \frac{\rho^2}{2} + \sum_{j=1}^{p} c_j'(\rho\beta)^{\delta_j+1} + c'(\rho\beta)^{\delta+1} - \lambda\eta(M_0\rho)^{\gamma_2}, \tag{3.14}$$

for $\forall u \in J_n$, $0 < \rho \leq \frac{T}{\beta}$.

Since $\gamma_2 \in (1,2)$ with $\gamma_2 < \min\{\min_{1\leq j\leq p}\{\delta_j\}, \delta\} + 1$, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\varphi(\delta u) < -\varepsilon \quad \text{for } u \in J_n. \tag{3.15}$$

Let

$$J_n^\delta = \{\delta u : u \in J_n\}, \qquad \Omega = \left\{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R} : \sum_{l=1}^{n} \lambda_l^2 < \delta^2\right\},$$

then it follows from (3.13) that

$$\varphi(u) < -\varepsilon \quad \text{for } u \in J_n^\delta.$$

Together with the fact that $\varphi \in C^1(X,\mathbb{R})$ and is even, it implies that

$$J_n^\delta \subset \varphi^{-\varepsilon} \in \Sigma. \tag{3.16}$$

By virtue of (3.10) and (3.12), there exists an odd homeomorphism mapping $f \in C(J_n^\delta, \partial\Omega)$. By some properties of the genus (see 3° of Propositions 7.5 and 7.7 in [42]), one has

$$\gamma\left(\varphi^{-\varepsilon}\right) \geq \gamma\left(J_n^\delta\right) = n, \tag{3.17}$$

so the proof of (3.9) follows. Let

$$d_n := \inf_{J\in\Sigma_n} \sup_{u\in J} \varphi(u).$$

It follows from (3.17) and the fact that φ is bounded from below on X that $-\infty < d_n \leq -\varepsilon < 0$, that is, for any $n \in \mathbb{N}$, d_n is a real negative number. By Theorem 2.2, φ has infinitely many critical points, and so problem (1.1) has infinitely many solutions. □

4 Examples

In order to illustrate our results, we give two examples.

Example 4.1 Consider the following problem:

$$\begin{aligned} &-u''(t) + u(t) = \lambda g\bigl(t, u(t)\bigr), \quad \text{a.e. } t \in [0,+\infty), \\ &\Delta u'(t_j) = I_j\bigl(u(t_j)\bigr), \quad j = 1, \\ &u'\bigl(0^+\bigr) = h\bigl(u(0)\bigr), \qquad u'(+\infty) = 0, \end{aligned} \tag{4.1}$$

where $h(u) = u$, $I_j(u) = u$.

Compared to problem (1.1), $c = 1$. It is clear that (A_1) is satisfied. β is defined in (2.2). When β lies in different intervals, we can choose different g satisfies the conditions. So we only consider one case. If $\beta < \frac{\sqrt{10}}{12}$, we take

$$g(t,u) = \begin{cases} \sqrt{\beta}e^{-t}, & u \le \beta, \\ e^{-t}(\frac{u}{100} + 600u^{\frac{1}{2}} - 599\sqrt{\beta} - \frac{\beta}{100}), & u > \beta. \end{cases}$$

Then

$$G(t,u) = \begin{cases} \sqrt{\beta}e^{-t}u, & u \le \beta, \\ e^{-t}[\frac{u^2}{200} + 400u^{\frac{3}{2}} - (599\sqrt{\beta} + \frac{\beta}{100})u + 200\beta^{\frac{3}{2}} + \frac{\beta^2}{200}], & u > \beta. \end{cases}$$

Take $t_1 = \ln\sqrt{2}$, $a_1(t) = \frac{e^{-t}}{200}$, $\alpha = \frac{3}{2}$, $a_2(t) = 400e^{-t}$, $a_3 = \frac{1}{\sqrt{\beta}}$, $b_3(t) = \frac{e^{-t}}{100}$, $b_4(t) = 600e^{-t}$, $b_5(t) = \sqrt{\beta}e^{-t}$, and choose constants $d, q > 0$ and m satisfying $6\beta^2 q < d < \min\{\beta, \frac{\sqrt{10}}{2}\beta q\}$ and $\frac{5}{2}m^2 = \frac{d^2}{\beta^2}$. A simple calculation shows that (A_3), (A_4), and (A_5) are satisfied. Applying Theorem 1.1, then, for each $\lambda \in]\frac{1}{\alpha_2}, \frac{1}{\alpha_1}[$, problem (4.1) has at least three classical solutions.

Example 4.2 Consider the following problem:

$$\begin{aligned} &-u''(t) + u(t) = \lambda g(t, u(t)), \quad \text{a.e. } t \in [0, +\infty), \\ &\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, \\ &u'(0^+) = h(u(0)), \qquad u'(+\infty) = 0, \end{aligned} \tag{4.2}$$

where $\lambda > 0$, $I_j(u) = -u^{\frac{3}{5}}$, $h(u) = -u^{\frac{3}{5}}$, and $g(t,u) = (\frac{1}{(1+t^2)^2} - \frac{1}{(1+t)^2})u^{\frac{1}{3}}$.

Compared to problem (1.1), $c = 1$. By simple calculations, all conditions in Theorem 1.3 are satisfied. Applying Theorem 1.3, then (4.2) has infinitely many classical solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Aronson, D, Crandall, MG, Peletier, LA: Stabilization of solutions of a degenerate nonlinear diffusion problem. Nonlinear Anal. **6**, 1001-1022 (1982)
2. Kawano, N, Yanagida, E, Yotsutani, S: Structure theorems for positive radial solutions to $\Delta u + K(|x|)u^p = 0$ in $\mathbb{R}^n$. Funkc. Ekvacioj **36**, 557-579 (1993)
3. Iffland, G: Positive solutions of a problem Emden-Fowler type with a type free boundary. SIAM J. Math. Anal. **18**, 283-292 (1987)
4. Agarwal, RP, O'Regan, D: Infinite Interval Problems for Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht (2001)
5. Chen, S, Zhang, T: Singular boundary value problems on a half-line. J. Math. Anal. Appl. **195**, 449-468 (1995)

6. Gomes, JM, Sanchez, L: A variational approach to some boundary value problems in the half-line. Z. Angew. Math. Phys. **56**, 192-209 (2005)
7. Lian, H, Ge, W: Calculus of variations for a boundary value problem of differential system on the half line. Comput. Math. Appl. **58**, 58-64 (2009)
8. Liu, X: Solutions of impulsive boundary value problems on the half-line. J. Math. Anal. Appl. **222**, 411-430 (1998)
9. Liu, Y: Existence and unboundedness of positive solutions for singular boundary value problems on half-line. Appl. Math. Comput. **144**, 543-556 (2003)
10. Yan, B: Multiple unbounded solutions of boundary value problems for second-order differential equations on half-line. Nonlinear Anal. **51**, 1031-1044 (2002)
11. Zima, M: On positive solutions of boundary value problems on the half-line. J. Math. Anal. Appl. **259**, 127-136 (2001)
12. Gao, S, Chen, L, Nieto, JJ, Torres, A: Analysis of a delayed epidemic model with pulse vaccination and saturation incidence. Vaccine **24**, 6037-6045 (2006)
13. He, Z: Impulsive state feedback control of a predator-prey system with group defense. Nonlinear Dyn. **79**, 2699-2714 (2015)
14. Liu, X, Willms, AR: Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft. Math. Probl. Eng. **2**, 277-299 (1996)
15. Nenov, S: Impulsive controllability and optimization problems in population dynamics. Nonlinear Anal. **36**, 881-890 (1999)
16. D'Onofrio, A: On pulse vaccination strategy in the SIR epidemic model with vertical transmission. Appl. Math. Lett. **18**, 729-732 (2005)
17. Xiao, Q, Dai, B: Dynamics of an impulsive predator-prey logistic population model with state-dependent. Appl. Math. Comput. **259**, 220-230 (2015)
18. Ahmad, B, Nieto, JJ: Existence and approximation of solutions for a class of nonlinear impulsive functional differential equations with anti-periodic boundary conditions. Nonlinear Anal. **69**, 3291-3298 (2008)
19. Chen, D, Dai, B: Periodic solution of second order impulsive delay differential systems via variational method. Appl. Math. Lett. **38**, 61-66 (2014)
20. Hernandez, E, Henriquez, HR, McKibben, MA: Existence results for abstract impulsive second-order neutral functional differential equations. Nonlinear Anal. **70**, 2736-2751 (2009)
21. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
22. Liang, RX: Existence of solutions for impulsive Dirichlet problems with the parameter inequality reverse. Math. Methods Appl. Sci. **36**, 1929-1939 (2013)
23. Luo, Z, Nieto, JJ: New results for the periodic boundary value problem for impulsive integro-differential equations. Nonlinear Anal. **70**, 2248-2260 (2009)
24. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
25. He, Z, He, X: Monotone iterative technique for impulsive integro-differential equations with periodic boundary conditions. Comput. Math. Appl. **48**, 73-84 (2004)
26. Li, J, Shen, J: Positive solutions for three-point boundary value problems for second-order impulsive differential equations on infinite intervals. J. Comput. Appl. Math. **235**, 2372-2379 (2011)
27. Liang, RX, Liu, ZM: Nagumo type existence results of Sturm-Liouville BVP for impulsive differential equations. Nonlinear Anal. **74**, 6676-6685 (2011)
28. Qian, D, Li, X: Periodic solutions for ordinary differential equations with sublinear impulsive effects. J. Math. Anal. Appl. **303**, 288-303 (2005)
29. Chen, H, Li, J: Variational approach to impulsive differential equations with Dirichlet boundary conditions. Bound. Value Probl. **2010**, Article ID 325415 (2010)
30. Chen, H, He, Z: New results for perturbed Hamiltonian systems with impulses. Appl. Math. Comput. **218**, 9489-9497 (2012)
31. Chen, H, He, Z: Variational approach to some damped Dirichlet problems with impulses. Math. Methods Appl. Sci. **36**, 2564-2575 (2013)
32. Chen, P, Tang, X: New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects. Math. Comput. Model. **55**, 723-739 (2012)
33. Nieto, JJ, O'Regan, D: Variational approach to impulsive differential equations. Nonlinear Anal., Real World Appl. **10**, 680-690 (2009)
34. Sun, J, Chen, H, Yang, L: The existence and multiplicity of solutions for an impulsive differential equation with two parameters via a variational method. Nonlinear Anal. **73**, 440-449 (2010)
35. Tian, Y, Ge, W: Applications of variational methods to boundary-value problem for impulsive differential equations. Proc. Edinb. Math. Soc. **51**, 509-527 (2008)
36. Zhang, D, Dai, B: Existence of solutions for nonlinear impulsive differential equations with Dirichlet boundary conditions. Math. Comput. Model. **53**, 1154-1161 (2011)
37. Zhang, L, Ge, W: Solvability of a kind of Sturm-Liouville boundary value problems with impulses via variational methods. Acta Appl. Math. **110**, 1237-1248 (2010)
38. Chen, H, Sun, J: An application of variational method to second-order impulsive differential equation on the half-line. Appl. Math. Comput. **217**, 1863-1869 (2010)
39. Dai, B, Zhang, D: The existence and multiplicity of solutions for second-order impulsive differential equation on the half-line. Results Math. **63**, 135-149 (2013)
40. Bonanno, G, Marano, SA: On the structure of the critical set of non-differentiable functions with a weak compactness condition. Appl. Anal. **89**, 1-10 (2010)
41. Bonanno, G, Riccobono, G: Multiplicity results for Sturm-Liouville boundary value problems. Appl. Math. Comput. **210**, 294-297 (2009)
42. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Reg. Conf. Ser. in Math., vol. 65. Am. Math. Soc., Providence (1986)
43. Salvatore, A: Homoclinic orbits for a special class of nonautonomous Hamiltonian systems. Nonlinear Anal. **30**, 4849-4857 (1997)

44. Zeidler, E: Nonlinear Functional Analysis and Its Applications, vol. 2. Springer, Berlin (1990)
45. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Springer, Berlin (1989)