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Discreteness of the spectrum of vectorial Schrödinger operators with δ-interactions

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Abstract

This paper deals with the vectorial Schrödinger operators with δ-interactions generated by $L_{X,A,Q} := -\frac{d^2}{dx^2} + Q(x) + \sum_{k=1}^{\infty} A_k\delta(x - x_k)$, $x \in [0, +\infty)$. First, we obtain an embedding inequality. Then using standard form methods, we prove that the operator $\mathbf{H}_{X,A,Q}$ given in this paper is self-adjoint. Finally, a sufficient condition and a necessary condition are given for the spectrum of the operator $\mathbf{H}_{X,A,Q}$ to be discrete. By giving additional restrictions on the symmetric potential matrix $Q(x)$ and A_k, we also give a necessary and sufficient condition for a special case. The conditions are analogous to Molchanov's discreteness criteria.

MSC: Primary 34B24; secondary 34L05; 47e05

Keywords: vectorial Schrödinger operators; δ-interactions; self-adjointness; discrete spectrum

1 Introduction

The present paper deals with the vectorial (matrix) Schrödinger operators with δ-interactions generated by the formal differential expression

$$L_{X,A,Q} := -\frac{d^2}{dx^2} + Q(x) + \sum_{k=1}^{\infty} A_k\delta(x - x_k), \quad x \in [0, +\infty), \tag{1.1}$$

where $Q(x) = (q_{ij}(x))_{i,j=1}^m \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^{m\times m})$ and $A_k = (a^k_{i,j})_{i,j=1}^m \in \mathbb{R}^{m\times m}$ are real valued and symmetric $m \times m$ matrices. Denote $A := \{A_k\}_{k=1}^{\infty} \subset \mathbb{R}^{m\times m}$, and let $X := \{x_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ be a strictly increasing sequence such that $x_k \to +\infty$. The minimal operator $\mathbf{H}_{X,A,Q}$ can be defined as follows:

$$\begin{aligned}
&\mathbf{H}^0_{X,A,Q} Y := L(Y) = -Y'' + Q(x)Y, \\
&\mathrm{Dom}\big(\mathbf{H}^0_{X,A,Q}\big) = \big\{ Y \in L^2_{\mathrm{comp}}\big(\mathbb{R}_+, \mathbb{C}^m\big) : Y, Y' \in W^{1,1}\big(\mathbb{R}_+ \backslash X, \mathbb{C}^m\big), \\
&\qquad L(Y) \in L^2\big(\mathbb{R}_+, \mathbb{C}^m\big), Y(0) = \theta, Y(x_k+) = Y(x_k-), \\
&\qquad Y'(x_k+) - Y'(x_k-) = A_k Y(x_k), k \in \mathbb{N}\big\}.
\end{aligned} \tag{1.2}$$

Here θ is the m-dimensional zero vector. It is clear that $\mathbf{H}^0_{X,A,Q}$ is a symmetric operator, and the minimal operator $\mathbf{H}_{X,A,Q}$ is the closure of $\mathbf{H}^0_{X,A,Q}$ in $L^2(\mathbb{R}_+, \mathbb{C}^m)$, that is, $\mathbf{H}_{X,A,Q} := \overline{\mathbf{H}^0_{X,A,Q}}$.

$L^2(\mathbb{R}_+, \mathbb{C}^m)$ is the Hilbert space of vector-valued functions with the scalar product

$$(u, v) = \int_0^\infty (u, v)_m \, dx,$$

where $u = (u_1, u_2, \dots, u_m)^T$, $v = (v_1, v_2, \dots, v_m)^T$, $(u, v)_m := \overline{v}^T u = \sum_{i=1}^m \overline{v_i} u_i$, $u_i, v_i \in L^2(\mathbb{R}_+)$.

Schrödinger operators with δ-interactions can be used as solvable models in many situations. The operators $\mathbf{H}_{X,A,Q}$ in the scalar case describes δ-interactions of strength at the points x_k. Numerous results can be found in [1–5]. Also, there are some papers about a vectorial operator with δ-interactions. For example, a detailed spectral theoretic treatment of Schrödinger operators with distributional matrix-valued potentials is developed in [6]. Some results about the defect index of the matrix case $\mathbf{H}_{X,A,Q}$ when Q is missing are obtained in [7]. However, there are a few results about the spectral properties of such operators.

The main objective of the present paper is to give conditions for the spectra of the vectorial Schrödinger operators $\mathbf{H}_{X,A,Q}$ to be discrete. First, we prove that the operator $\mathbf{H}_{X,A,Q}$ given in this paper is self-adjoint. If the singular part in (1.1) is missing, then $\mathbf{H}_Q = \mathbf{H}_{X,0,Q}$ is just a classical vectorial Sturm-Liouville expression. Liu and Wang [8] gave some criterions that guarantee the operator $\mathbf{H}_Q$ to have a purely discrete spectrum. This result was proved by direct sum decomposition methods of operators and estimation of the corresponding quadratic forms. Clark and Gesztesy [9] derived Povzner-Wienholtz-type self-adjointness results for the classical matrix Sturm-Liouville operators. A generalization of this result to the case of scalar ($m = 1$) operators with point interactions was obtained by Albeverio *et al.* [1]. Second, after we derive an embedding inequality of vector-valued functions, we obtain some criterions that guarantee the operator $\mathbf{H}_{X,A,Q}$ to have a purely discrete spectrum. The condition is an analog of the classical discreteness criterion due to Molchanov [10]. Our result reads as follows:

$$\int_x^{x+\varepsilon} \mu_{\min}\bigl(Q(x)\bigr)\, dx + \sum_{x_k \in (x, x+\varepsilon)} \mu_{\min}(A_k) \to \infty \quad \text{as } x \to \infty. \tag{1.3}$$

However, condition (1.3) is no longer necessary in the case of matrix Hamiltonians $\mathbf{H}_{X,A,Q}$ (see Theorem 1). This results coincides with the corresponding result obtained by Albeverio *et al.* [1] in the scalar case ($m = 1$), whereas in this case it is also necessary. Subsequently, giving additional restrictions on the symmetric potential matrix $Q(x)$ and A_k, we obtain a sufficient and necessary condition for a special case (see Theorem 3).

This paper is organized as follows. In Section 2, we introduce some basic definitions and lemmas and a generalized embedding theorem. Section 3 contains some lemmas and quadratic forms associated with the operator $\mathbf{H}_{X,A,Q}$ for our main results. Some criterions based on the Molchanov's theorem, which guarantee that the operator has a purely discrete spectrum, are given in Section 4.

2 Preliminaries

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$, and let $\mathbf{t}$ be a densely defined quadratic form in $\mathfrak{H}$ with lower bound $-c$, that is, $\mathbf{t}[u] \geq -c\|u\|_{\mathfrak{H}}^2$, $c \in \mathbb{R}$. Let $\mathbf{t}[\cdot, \cdot]$ be the sesquilinear form associated with $\mathbf{t}$. Then the equality

$$(u, v)_{\mathbf{t}} = \mathbf{t}[u, v] + (1 + c)(u, v)$$

defines a scalar product on $\mathrm{Dom}(\mathbf{t})$ such that $\|u\|_{\mathbf{t}} \geq \|u\|_{\mathfrak{H}}$ for all $u \in \mathrm{Dom}(\mathbf{t})$, where

$$\|u\|_{\mathbf{t}}^2 := \mathbf{t}[u] + (1+c)\|u\|_{\mathfrak{H}}^2, \quad u \in \mathrm{Dom}(\mathbf{t}).$$

The form $\mathbf{t}$ is called closable if the norm $\|\cdot\|_{\mathbf{t}}$ is compatible with $\|\cdot\|_{\mathfrak{H}}$, that is, for every $\|\cdot\|_{\mathbf{t}}$-Cauchy sequence $\{u_n\}_{n=1}^{\infty}$ in $\mathrm{Dom}(\mathbf{t})$, $\|u_n\|_{\mathfrak{H}} \to 0$ implies $\|u_n\|_{\mathbf{t}} \to 0$. Let $\mathfrak{H}_{\mathbf{t}}$ be the $\|\cdot\|_{\mathbf{t}}$-completion of $\mathrm{Dom}(\mathbf{t})$. In this case, the completion $\mathfrak{H}_{\mathbf{t}}$ can be considered as a subset of $\mathfrak{H}$. The form $\mathbf{t}$ is closed if the sets $\mathfrak{H}_{\mathbf{t}}$ and $\mathrm{Dom}(\mathbf{t})$ are equal. Let A be a self-adjoint lower semibounded operator in $\mathfrak{H}$, that is, $(Au,u) \geq -c(u,u)$ for all $u \in \mathrm{Dom}(A)$ and some $c \in \mathbb{R}$. Denote by $\mathbf{t}'_A$ the densely defined quadratic form given by

$$\mathbf{t}'_A[u] = (Au,u), \quad \mathrm{Dom}(\mathbf{t}'_A) = \mathrm{Dom}(A).$$

Clearly, this form is closable and lower semibounded, $\mathbf{t}'_A \geq -c$, and its closure $\mathbf{t}_A$ satisfies $\mathbf{t}_A \geq -c$. We set $\mathfrak{H}_A := \mathfrak{H}_{\mathbf{t}_A}$. By the first representation theorem [11], Theorem 6.2.1, for any closed lower semibounded quadratic form $\mathbf{t} \geq -c$ in $\mathfrak{H}$, there corresponds a unique self-adjoint operator $A = A^*$ in $\mathfrak{H}$ satisfying $(Au,u) \geq -c(u,u)$ for all $u \in \mathrm{Dom}(A)$ such that $\mathbf{t}$ is the closure of $\mathbf{t}'_A$. The form $\mathbf{t}$ is uniquely determined by the conditions $\mathrm{Dom}(A) \subset \mathrm{Dom}(\mathbf{t})$ and

$$(Au,v) = \mathbf{t}[u,v], \quad u \in \mathrm{Dom}(A), v \in \mathrm{Dom}(\mathbf{t}).$$

Lemma 1 *Let $A = A^*$ be a lower semibounded operator in $\mathfrak{H}$, and let $\mathbf{t}_A$ be the corresponding form. The spectrum $\sigma(A)$ of the operator A is discrete if and only if the embedding $i_A : \mathfrak{H}_A \hookrightarrow \mathfrak{H}$ is compact.*

Proof See [11]. □

Definition 1 Let the operator A be self-adjoint and positive on $\mathfrak{H}$, and let $\mathbf{t}_A$ be the corresponding form. The form $\mathbf{t}$ is called relatively form bounded with respect to $\mathbf{t}_A$ ($\mathbf{t}_A$-bounded) if $\mathrm{Dom}(\mathbf{t}_A) \subset \mathrm{Dom}(\mathbf{t})$ and there are positive constants a, b such that

$$\left|\mathbf{t}[f]\right| \leq a\mathbf{t}_A[f] + b\|f\|_{\mathfrak{H}}^2, \quad f \in \mathrm{Dom}(\mathbf{t}_A).$$

The infimum of all possible a is called the form bound of $\mathbf{t}$ with respect to $\mathbf{t}_A$. If a can be chosen arbitrary small, then $\mathbf{t}$ is called infinitesimally form bounded with respect to $\mathbf{t}_A$.

Lemma 2 *Let $\mathbf{t}_A$ be the form corresponding to the operator $A = A^* > 0$ in $\mathfrak{H}$. If the form $\mathbf{t}$ is $\mathbf{t}_A$-bounded with relative bound $a < 1$, then the form*

$$\mathbf{t}_1 := \mathbf{t}_A + \mathbf{t}, \quad \mathrm{Dom}(\mathbf{t}_1) = \mathrm{Dom}(\mathbf{t}_A),$$

is closed and lower semibounded in $\mathfrak{H}$ and hence gives rise to a self-adjoint semibounded operator. Moreover, the norms $\|\cdot\|_A$ and $\|\cdot\|_{\mathbf{t}_1}$ are equivalent.

Proof See [12]. □

We also need a generalized embedding inequality of vector-valued functions, which we state here. In [13], p.232, Theorem A.2, Muller-Pfeiffer gave an embedding theorem for scalar functions. We extend the theorem to vectorial functions. For an interval I, let $u(x) = (u_1(x), u_2(x), \dots, u_m(x))^T$, where $u_j \in W^{l,p}(I)$, $j = 1, 2, \dots, m$, and we denote by $W^{l,p}(I, \mathbb{C}^m)$ the space of all such vectorial functions. The norm is $\|u\|^2_{W^{l,p}(I,\mathbb{C}^m)} = \sum_{i=1}^m \|u_i\|^2_{W^{l,p}(I)}$, $\|u_i\|_{W^{l,p}(I)} = (\sum_{k=0}^l \int_I |u_i^{(k)}(x)|^p\,dx)^{1/p}$. If $u_j \in C^k(I)$, $j = 1, 2, \dots, m$, then we denote by $C^k(I, \mathbb{C}^m)$ the space of all such vectorial functions. The norm is $\|u\|^2_{C^k(I,\mathbb{C}^m)} = \sum_{i=1}^m \|u_i\|^2_{C^k(I)}$, $\|u_i\|_{C^k(I)} = \sum_{l=0}^k \sup_{x\in I} |u_i^{(l)}(x)|$. If $u_j \in L^k(I)$, $j = 1, 2, \dots, m$, then we denote by $L^k(I, \mathbb{C}^m)$ the space of all such vectorial functions. The norm is $\|u\|^2_{L^k(I,\mathbb{C}^m)} = \sum_{i=1}^m \|u_i\|^2_{L^k(I)}$, $\|u_i\|_{L^k(I)} = (\int_I |u_i(x)|^k)^{1/k}$. We get the following embedding inequality for vectorial functions.

Lemma 3 *If $l > k \geq 0$ and $1 < p < \infty$, $W^{l,p}((x_1, x_2), \mathbb{C}^m)$ is continuously embedded into the space $C^k((x_1, x_2), \mathbb{C}^m)$, and, for any $\varepsilon > 0$, there exists a constant C_ε such that*

$$\|u\|^2_{C^k((x_1,x_2),\mathbb{C}^m)} \leq \varepsilon \bigl\|u^{(l)}\bigr\|^2_{L^p((x_1,x_2),\mathbb{C}^m)} + C_\varepsilon \|u\|^2_{L^p((x_1,x_2),\mathbb{C}^m)} \tag{2.1}$$

for all $u \in W^{l,p}((x_1, x_2), \mathbb{C}^m)$.

Proof Using the embedding theorem in [13], for any $\varepsilon > 0$ and every component function u_i, $i = 1, 2, \dots, m$, there are constants $C_{i,\varepsilon} > 0$ such that

$$\|u_i\|^2_{C^k(x_1,x_2)} \leq \varepsilon \bigl\|u_i^{(l)}\bigr\|^2_{L^p(x_1,x_2)} + C_{i,\varepsilon} \|u_i\|^2_{L^p(x_1,x_2)}.$$

Then

$$\begin{aligned} \|u\|^2_{C^k((x_1,x_2),\mathbb{C}^m)} &= \sum_{i=1}^m \|u_i\|^2_{C^k(x_1,x_2)} \\ &\leq \sum_{i=1}^m \varepsilon \bigl\|u_i^{(l)}\bigr\|^2_{L^p(x_1,x_2)} + \sum_{i=1}^m C_{i,\varepsilon} \|u_i\|^2_{L^p(x_1,x_2)} \\ &\leq \sum_{i=1}^m \varepsilon \bigl\|u_i^{(l)}\bigr\|^2_{L^p(x_1,x_2)} + \sum_{i=1}^m C_\varepsilon \|u_i\|^2_{L^p(x_1,x_2)}, \end{aligned}$$

where $C_\varepsilon = \max_{1\leq i\leq m}\{C_{i,\varepsilon}\}$. For integer $l > 0$ and $1 < p < \infty$,

$$\bigl\|u^{(l)}\bigr\|^2_{L^p((x_1,x_2),\mathbb{C}^m)} = \left(\sum_{i=1}^m \bigl\|u_i^{(l)}\bigr\|_{L^p(x_1,x_2)}\right)^2 \geq \sum_{i=1}^m \bigl\|u_i^{(l)}\bigr\|^2_{L^p(x_1,x_2)};$$

hence, we get (2.1). □

3 Quadratic forms associated with the operator

We begin this section with the operators $\mathbf{H}_{X,A,Q}$ and their corresponding quadratic forms in the Hilbert space $L^2(\mathbb{R}_+, \mathbb{C}^m)$. First, we recall some notation for the convenience of the readers. The inequality $Q(x) \geq (>) 0$ means that for any $Y(x) = (y_1(x), y_2(x), \dots, y_m(x))^T$,

$(Q(x)Y(x), Y(x))_m \geq (>)0$, and $Q(x)$ is called bounded from below if $(Q(x)Y(x), Y(x))_m \geq -c(Y(x), Y(x))_m$, where c is a constant. Then we consider the quadratic forms

$$\mathfrak{q}[Y] := \int_0^\infty \bigl(Q(x)Y(x), Y(x)\bigr)_m \, dx, \qquad (3.1)$$
$$\mathrm{Dom}(\mathfrak{q}) = \bigl\{Y \in L^2\bigl(\mathbb{R}_+, \mathbb{C}^m\bigr) : \bigl|\mathfrak{q}[Y]\bigr| < \infty\bigr\}.$$

We denote by $\mu_{\min}(Q(x))$ and $\mu_{\max}(Q(x))$ respectively the minimal and maximal eigenvalues of $Q(x)$, and by $\mu_{\min}(A_k)$ and $\mu_{\max}(A_k)$ respectively the minimal and maximal eigenvalues of A_k. The quadratic form $\mathfrak{q}[Y]$ is called semibounded from below (above) if and only if so is $\mu_{\min}(Q(x))(\mu_{\max}(Q(x)))$. We denote by $\mathbf{t}_0[Y]$ the following quadratic form:

$$\mathbf{t}_0[Y] := \int_0^\infty \bigl(Y'(x), Y'(x)\bigr)_m \, dx, \quad \mathrm{Dom}\bigl(\mathbf{t}_0[Y]\bigr) = W_0^{1,2}\bigl(\mathbb{R}_+, \mathbb{C}^m\bigr). \qquad (3.2)$$

Together with the form $\mathfrak{q}$, we consider the form

$$\mathbf{t}_Q[Y] := \mathbf{t}_0[Y] + \mathfrak{q}[Y] = \int_0^\infty \bigl[\bigl(Y'(x), Y'(x)\bigr)_m + \bigl(Q(x)Y(x), Y(x)\bigr)_m\bigr] \, dx, \qquad (3.3)$$
$$\begin{aligned}\mathrm{Dom}(\mathbf{t}_Q) &= W_0^{1,2}\bigl(\mathbb{R}_+, \mathbb{C}^m\bigr) \cap \mathrm{Dom}(\mathfrak{q}) \\ &= \bigl\{Y \in L^2\bigl(\mathbb{R}_+, \mathbb{C}^m\bigr) \cap AC_{\mathrm{loc}}\bigl(\mathbb{R}_+, \mathbb{C}^m\bigr), \mathbf{t}_Q[Y] < +\infty, Y(0) = \theta\bigr\}.\end{aligned}$$

$\mathrm{Dom}(\mathbf{t}_0[Y])$ is the set of all $Y \in L^2(\mathbb{R}_+, \mathbb{C}^m)$ such that $Y(x)$ is absolutely continuous, $Y' \in L^2(\mathbb{R}_+, \mathbb{C}^m)$, and $Y(0) = \theta$. The form $\mathbf{t}_0[Y]$ is symmetric and closed. $\mathrm{Dom}(\mathfrak{q})$ is the set of all $Y \in L^2(\mathbb{R}_+, \mathbb{C}^m)$ such that $\int_0^\infty (Q(x)Y(x), Y(x))_m \, dx < +\infty$. The form $\mathfrak{q}$ is also symmetric and closed. Thus, $\mathbf{t}_Q[Y] = \mathbf{t}_0[Y] + \mathfrak{q}[Y]$ is symmetric and closed (see [11], Chapter VI, Theorem 1.31). If the potential $Q(x)$ is lower semibounded with bound $-c$, that is, $(Q(x)Y, Y) \geq -c(Y, Y)$, $\forall Y \in \mathbb{C}^m$, then $\mathrm{Dom}(\mathbf{t}_Q)$ equipped with the norm $\|Y\|^2_{\mathfrak{H}_Q} := \|Y\|^2_{W^{1,2}} + \mathbf{t}_Q[Y] + (1 + c)\|Y\|^2_{\mathfrak{H}}$ is the Hilbert space $\mathfrak{H}_Q := \mathfrak{H}_{\mathbf{t}_Q} := W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^m; Q)$. Let

$$\mathbf{t}_R[Y] = \sum_{k=1}^\infty \bigl(A_k Y(x_k), Y(x_k)\bigr)_m \qquad (3.4)$$

be defined on the domain

$$\mathrm{Dom}(\mathbf{t}_R) = \bigl\{Y \in L^2\bigl(\mathbb{R}_+, \mathbb{C}^m\bigr), \mathbf{t}_R[Y] < +\infty\bigr\}.$$

Denote by $\mathbf{t}_R^\pm[Y]$ the sum of k positive (negative) parts of $(A_k Y(x_k), Y(x_k))_m$, that is,

$$\mathbf{t}_R^\pm[Y] = \sum_{k=1}^\infty \bigl(A_k Y(x_k), Y(x_k)\bigr)_m^\pm, \qquad (3.5)$$
$$\mathrm{Dom}\bigl(\mathbf{t}_R^\pm[Y]\bigr) = \bigl\{Y \in W_0^{1,2}\bigl(\mathbb{R}_+, \mathbb{C}^m\bigr), \mathbf{t}_R^\pm[Y] < +\infty\bigr\}.$$

Similarly,

$$\mathfrak{q}^\pm[Y] := \int_0^\infty \bigl(Q(x)Y(x), Y(x)\bigr)_m^\pm \, dx, \quad \mathrm{Dom}\bigl(\mathfrak{q}^\pm\bigr) = \bigl\{Y \in AC_{\mathrm{loc}}\bigl(\mathbb{R}_+, \mathbb{C}^m\bigr), \mathfrak{q}^\pm[Y] < +\infty\bigr\}.$$

Then we define the form

$$\begin{aligned}
&\mathbf{t}_{X,A,Q} = \mathbf{t}_Q + \mathbf{t}_R, \qquad \mathbf{t}_Q = \mathbf{t}_0 + \mathbf{t}_Q = \mathbf{t}_0 + \mathfrak{q}^+ + \mathfrak{q}^-, \\
&\mathbf{t}_Q^+ = \mathbf{t}_0 + \mathfrak{q}^+, \qquad \mathbf{t}_Q^- = \mathbf{t}_0 + \mathfrak{q}^-, \\
&\operatorname{Dom}(\mathbf{t}_{X,A,Q}) = \operatorname{Dom}(\mathbf{t}_Q) \cap \operatorname{Dom}(\mathbf{t}_R),
\end{aligned}$$

which is naturally associated with the differential expression, and the form is as follows:

$$\begin{aligned}
\mathbf{t}_{X,A,Q}[Y] = {}& \int_0^\infty \bigl[\bigl(Y'(x), Y'(x)\bigr)_m + \bigl(Q(x)Y(x), Y(x)\bigr)_m\bigr]\,dx \\
&+ \sum_{k=1}^\infty \bigl(A_k Y(x_k), Y(x_k)\bigr)_m.
\end{aligned} \tag{3.6}$$

Notation *If the form* $\mathbf{t}_{X,A,Q}$ *is nonnegative, then* $\mathfrak{H}_{X,A,Q} := \mathfrak{H}_{\mathbf{t}_{X,A,Q}}$ *denotes the domain* $\operatorname{Dom}(\mathbf{t}_{X,A,Q})$ *equipped with the norm*

$$\begin{aligned}
&\|Y\|^2_{\mathfrak{H}_{X,A,Q}} = \mathbf{t}_Q[Y] + \mathbf{t}_R[Y] + \|Y\|^2_{L^2(\mathbb{R}_+,\mathbb{C}_m)}, \\
&\sup_{x>0} \int_x^{x+1} \bigl|\mu^-_{\max}\bigl(Q(x)\bigr)\bigr|\,dx < +\infty, \qquad \sup_{x>0} \sum_{x_k\in[x,x+1]} \bigl|\mu^-_{\max}(A_k)\bigr| < +\infty,
\end{aligned} \tag{3.7}$$

where $\mu^\pm_{\max}(Q(x)) := (\mu_{\max}(Q(x)) \pm |\mu_{\max}(Q(x))|)/2$, $\mu(A)^\pm := \{\mu(A_k)^\pm\}_1^\infty$, *and* $\mu^\pm_{\max}(A_k) := (\mu_{\max}(A_k) \pm |\mu_{\max}(A_k)|)/2$.

Lemma 4 *If the minimal operator* $\mathbf{H}_{\min} = \mathbf{H}_{X,A,Q}$ *is lower semibounded, then the operator* $\mathbf{H}_{X,A,Q}$ *is self-adjoint*, $\mathbf{H}_{X,A,Q} = (\mathbf{H}_{X,A,Q})^*$.

Proof Without loss of generality, we assume that $\mathbf{H}_{X,A,Q} \geq I$. It is sufficient to show that $\ker((\mathbf{H}_{X,A,Q})^*) = \{\theta\}$, that is, the equation

$$-Y''(x) + Q(x)Y(x) = \theta, \quad x \in \mathbb{R}_+\backslash X, Y \in \operatorname{Dom}\bigl((\mathbf{H}_{X,A,Q})^*\bigr) \tag{3.8}$$

has only a trivial solution (the derivative is understood in a distribution sense).

Assume the converse, that is, let $Y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ be a solution of equation (3.8). Let $\chi_i \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^m)$ $(i = 1, 2, \ldots, m)$ be such that $0 \leq \|\chi_i\| \leq 1$ and

$$\chi_i(x) = \begin{cases} e_i, & 0 \leq x \leq 1/2, \\ \theta, & x \geq 1, \end{cases} \qquad i = 1, 2, \ldots, m,$$

where $\|\cdot\|$ is the Euclidean norm. Let the matrix $X(x)$ be the combination of $\chi_i(x)$, $i = 1, 2, \ldots, m$, that is, $X(x) = [\chi_1(x), \chi_2(x), \ldots, \chi_m(x)]$. Define

$$Y_n(x) := X_n(x)Y(x) = X(x/n)Y(x), \quad n \in \mathbb{N}.$$

Obviously, $\operatorname{sup} pY_n \subset [0, n]$. Since $Y_n(x_k-) = Y_n(x_k+)$ and $\chi_i \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^m)$ $(i = 1, 2, \ldots, m)$, we get

$$Y_n'(x_k+) - Y_n'(x_k-) = X_n(x_k)\bigl[Y'(x_k+) - Y'(x_k-)\bigr] = X_n(x_k)A_kY(x_k) = A_kY_n(x_k),$$

and hence $Y_n \in \mathrm{Dom}(\mathbf{H}_{\min})$. Furthermore,

$$\begin{aligned}
(\mathbf{H}_{\min} Y_n, Y_n) &= \int_0^\infty \bigl(\bigl[-Y_n''(x) + Q(x)Y_n(x)\bigr], Y_n(x)\bigr)_m \, dx \\
&= -\int_0^\infty \bigl(2X_n'(x)Y'(x) + X_n''(x)Y(x), X_n(x)Y(x)\bigr)_m \, dx \\
&= -\int_0^\infty \bigl[\bigl(2X_n'(x)Y'(x), X_n(x)Y(x)\bigr)_m + \bigl(X_n''(x)Y(x), X_n(x)Y(x)\bigr)_m\bigr] \, dx. \quad (3.9)
\end{aligned}$$

By $\mathbf{H}_{X,A,Q} \geq I$,

$$(\mathbf{H}_{\min} Y_n, Y_n) \geq (Y_n, Y_n) = \int_0^\infty \bigl(X_n(x)Y(x), X_n(x)Y(x)\bigr)_m \, dx. \quad (3.10)$$

On the other hand, transforming the first part of (3.9), integrating by parts, and noting that $X_n(x)$ has a compact support, we get

$$\begin{aligned}
\int_0^\infty \bigl(2X_n'(x)Y'(x), X_n(x)Y(x)\bigr)_m \, dx &= \int_0^\infty 2Y(x)^T X_n(x)^T X_n'(x)Y'(x) \, dx \\
&= \sum_{i=1}^m \int_0^\infty 2y_i(x)\chi_{i,i}(x/n)\chi_{i,i}'(x/n)y_i'(x) \, dx \\
&= \sum_{i=1}^m \int_0^\infty \frac{1}{2}\bigl(y_i^2(x)\bigr)'\bigl(\chi_{i,i}^2(x/n)\bigr)' \, dx \\
&= -\sum_{i=1}^m \int_0^\infty y_i^2(x)\bigl[\chi_{i,i}''(x/n)\chi_{i,i}(x/n) \\
&\quad + \bigl(\chi_{i,i}'(x/n)\bigr)^2\bigr] \, dx, \quad (3.11)
\end{aligned}$$

where $\chi_{i,i}(x/n)$ denotes the ith component function of $\chi_i(x/n)$, which is the ith column of the matrix $X_n(x)$ ($i = 1, 2, \ldots, m$). The second part of (3.9) is as follows:

$$\int_0^\infty \bigl(X_n''(x)Y(x), X_n(x)Y(x)\bigr)_m \, dx = \sum_{i=1}^m \int_0^\infty y_i^2(x)\chi_{i,i}''(x/n)\chi_{i,i}(x/n) \, dx. \quad (3.12)$$

By (3.9), (3.11), and (3.12) we get

$$(\mathbf{H}_{\min} Y_n, Y_n) = \sum_{i=1}^m \int_0^\infty y_i^2(x)\bigl(\chi_{i,i}'(x/n)\bigr)^2 \, dx = \int_0^\infty \bigl(X_n'(x)Y(x), X_n'(x)Y(x)\bigr)_m \, dx. \quad (3.13)$$

Therefore, by (3.10) and (3.13) we obtain

$$\begin{aligned}
\int_0^{n/2} \bigl(Y(x), Y(x)\bigr)_m \, dx &= \int_0^{n/2} Y(x)^T Y(x) \, dx \leq \int_0^\infty Y(x)^T X_n(x)^T X_n(x)Y(x) \, dx \\
&\leq \int_0^\infty Y(x)^T X_n'(x)^T X_n'(x)Y(x) \, dx \\
&\leq \frac{C}{n^2} \int_{n/2}^n Y(x)^T Y(x) \, dx = \frac{C}{n^2} \int_{n/2}^n \bigl(Y(x), Y(x)\bigr)_m \, dx, \quad (3.14)
\end{aligned}$$

where $C := \sup_{x\in[0,1]} |X'(x)|$. Noting that $Y \in L^2(\mathbb{R}_+, \mathbb{C}^m)$, inequality (3.14) yields $Y = \theta$. This contradiction completes the proof. □

Before proceeding further, we need the following fact.

Lemma 5 *If*

$$C_0 := \sup_{n\in\mathbb{N}} \int_n^{n+1} \left|\mu_{\max}\big(Q(t)\big)\right| dt < \infty, \tag{3.15}$$

$$C_1 := \sup_{n\in\mathbb{N}} \sum_{x_k\in[n,n+1]} \left|\mu_{\max}(A_k)\right| < \infty, \tag{3.16}$$

then the forms $\mathfrak{q}$ *and* $\mathbf{t}_R := \mathbf{t}_R^+ + \mathbf{t}_R^-$ *are infinitesimally* $\mathbf{t}_0$*-bounded, and hence the form* $\mathbf{t}_{X,A,Q}$ *is closed and lower semibounded, and* $\operatorname{Dom}(\mathbf{t}_{X,A,Q}) = W^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$ *algebraically and topologically.*

Proof By Lemma 3, for any $\varepsilon > 0$, we have the following inequality:

$$\begin{aligned}\|Y\|^2_{C(\mathbb{R}_+,\mathbb{C}^m)} &\le \varepsilon \int_n^{n+1} \big(Y'(t), Y'(t)\big)_m\, dt + C_\varepsilon \int_n^{n+1} \big(Y(t), Y(t)\big)_m\, dt \\ &\le \varepsilon \left\|Y'\right\|^2_{W^{1,2}([n,n+1],\mathbb{C}^m)} + C_\varepsilon \|Y\|^2_{L^2([n,n+1],\mathbb{C}^m)},\end{aligned} \tag{3.17}$$

where $x \in [n, n+1]$, and the constant $C_\varepsilon > 0$ does not depend on Y and $n \in \mathbb{N}$. Combining (3.15) and (3.16) with (3.17), we obtain, for $Y \in \operatorname{Dom}(\mathbf{t}_0) = W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$,

$$\begin{aligned}&\int_{\mathbb{R}_+} \big(Q(x)Y(x), Y(x)\big)_m\, dx + \sum_{k=1}^{\infty} \big(A_k Y(x_k), Y(x_k)\big)_m \\ &\quad= \sum_{n=0}^{\infty}\left(\int_n^{n+1} \big(Q(x)Y(x), Y(x)\big)_m\, dx + \sum_{x_k\in[n,n+1]} \big(A_k Y(x_k), Y(x_k)\big)_m\right) \\ &\quad\le \sum_{n=0}^{\infty} \int_n^{n+1} \left|\mu_{\max}\big(Q(x)\big)\right| \|Y\|^2\, dx + \sum_{x_k\in[n,n+1]} \left|\mu_{\max}(A_k)\right| \|Y\|^2 \\ &\quad\le (C_0 + C_1) \sum_{n=0}^{\infty} \|Y\|^2_{C[n,n+1]} \\ &\quad\le (C_0 + C_1)\varepsilon \|Y\|^2_{W^{1,2}(\mathbb{R}_+,\mathbb{C}^m)} + (C_0 + C_1) C_\varepsilon \|Y\|^2_{L^2(\mathbb{R}_+,\mathbb{C}^m)}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the forms $\mathfrak{q}$ and $\mathbf{t}_R$ are infinitesimally form bounded with respect to $\mathbf{t}_0$. It remains to apply Lemma 2. □

We refer to $A := \{A_k\}_{k=1}^{\infty} \subset \mathbb{R}^{m\times m}$ again. We denote by $l_m^2(\|A\|)$ the weighted space of m-dimensional vector sequences $f = \{f_k\}_{k=1}^{\infty} \subset \mathbb{C}^m$, where $\|A\|$ is any norm of a matrix A, and every element f_k of the sequence is an m-dimensional column vector. Let $f_k = (f_{k,1}, f_{k,2}, \ldots, f_{k,m})^T$; then $\{f_{k,i}\}_{k=1}^{\infty} \in l^2$, $i = 1, 2, \ldots, m$, and hence $\|f_{k,i}\|^2 = \sum_{k=1}^{\infty} |f_{k,i}|^2 < \infty$ and $\|f\|^2 = \sum_{i=1}^{m} \|f_{k,i}\|^2$. The space $l_m^2(\|A\|)$ equipped with the weighted norm $\|f\|_{l_m^2(\|A\|)} = (\sum_{i=1}^{m} \sum_{k=1}^{\infty} \|A_k\| |f_{k,i}|^2)^{\frac{1}{2}}$ is a Banach space.

Lemma 6 *Assume that* $Q(x) \geq 0$ *and* $A_k \geq 0$ *for* $k = 1, 2, \ldots$. *Then the form* $\mathbf{t}_{X,A,Q}$ *is nonnegative and closed.*

Proof It is obvious that the form $\mathbf{t}_{X,A,Q}$ is nonnegative if $Q(x) \geq 0$ and $A_k \geq 0$. Now we prove the closeness of the form $\mathbf{t}_{X,A,Q}$. Let us equip $\mathfrak{H}_{X,A,Q} = \operatorname{Dom}(\mathbf{t}_{X,A,Q})$ with the norm

$$\|Y\|^2_{\mathfrak{H}_{X,A,Q}} = \mathbf{t}_Q[Y] + \mathbf{t}_R[Y] + \|Y\|^2_{L^2(\mathbb{R}_+,\mathbb{C}^m)}.$$

Let $\{Y_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathfrak{H}_{X,A,Q}$. Since $W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$ and $l^2_m(\|A\|)$ are Hilbert spaces, there exist $Y \in W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$ and

$$\{\xi_k\}_{k=1}^{\infty} \in l^2_m\big(\|A\|\big)$$

such that

$$\lim_{n\to\infty} \|Y_n - Y\|_{W_0^{1,2}(\mathbb{R}_+,\mathbb{C}^m)} = 0$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \|A_k\| \big|Y_n(t_k) - \xi_k\big|^2 = 0,$$

where ξ_i and Y_n are m-dimensional vectors. Since the space $W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$ is continuously embedded into $C(\mathbb{R}_+, \mathbb{C}^m)$, the Banach space of bounded continuous functions on $\mathbb{R}_+$. Therefore,

$$\lim_{n\to\infty} Y_n(x) = Y(x),$$

and then $Y \in \mathfrak{H}_{X,A,Q}$ and

$$\lim_{n\to\infty} \|Y_n - Y\|_{\mathfrak{H}_{X,A,Q}} = 0.$$

In addition, since $Q(x) \geq 0$ and $\{A_k\}_{k=1}^{\infty} \geq 0$, $\mathfrak{H}_{X,A,Q}$ is a Hilbert space with the inner product

$$\begin{aligned}(Y,Z)_{\mathfrak{H}_{X,A,Q}} &= \int_0^{\infty} \big(Y', Z'\big)_m \, dx + \int_0^{\infty} \big(\big(Q(x) + E\big)Y, Z\big)_m \, dx \\ &\quad + \sum_{k=1}^{\infty} \big(A_k Y(x_k), Z(x_k)\big)_m,\end{aligned}$$

where E is the identity matrix. Then the form $\mathbf{t}_{X,A,Q}$ is closed. □

Lemma 7 *If the form* $\mathbf{t}_{X,A,Q}$ *is lower semibounded, then the set* $\operatorname{Dom}(H^0_{X,A,Q})$ *is a core of the form* $\mathbf{t}_{X,A,Q}$.

Proof We need to show that $\operatorname{Dom}(H^0_{X,A,Q})$ is dense in $\operatorname{Dom}(\mathbf{t}_{X,A,Q})$ with respect to the norm $\|Y\|^2_{\mathfrak{H}_{X,A,Q}} = \mathbf{t}_Q[Y] + \mathbf{t}_R[Y] + \|Y\|^2_{L^2(\mathbb{R}_+,\mathbb{C}^m)}$. Let $D'_{\min}$ be the linear span of C^∞ functions with compact support in a single interval (x_{i-1}, x_i), $i \in \mathbb{N}$. Each function $f_i \in C_0^\infty((x_{i-1}, x_i), \mathbb{C}^m)$ can be extended to $[0, \infty)$, and the extended function

$$\tilde{f}_i(x) = \begin{cases} f_i(x), & x \in (x_{i-1}, x_i), \\ \theta, & x \in [0, \infty) \backslash (x_{i-1}, x_i) \end{cases}$$

belongs to $D'_{\min} \subset \mathrm{Dom}(H^0_{X,A,Q})$.

We need to prove that for $u \in \mathrm{Dom}(\mathbf{t}_{X,A,Q})$ and for all $f \in \mathrm{Dom}(H^0_{X,A,Q})$,

$$(u,f) = \int_0^\infty \left(u', f'\right)_m dx + \int_0^\infty \left(\left(Q(x) + E\right)u, f\right)_m dx + \sum_{k=1}^\infty \left(A_k u(x_k), f(x_k)\right)_m = 0 \tag{3.18}$$

implies that $u = 0$. Equation (3.18) holds for all $f \in \mathrm{Dom}(H^0_{X,A,Q})$, and thus, for each interval (x_{i-1}, x_i), the equation

$$\int_{x_{i-1}}^{x_i} \left(u', f_i'\right)_m + \int_{x_{i-1}}^{x_i} \left((Q + E)u, f_i\right)_m = 0$$

holds for all $f_i \in C_0^\infty((x_{i-1}, x_i), \mathbb{C}^m)$. Then $u'' = (Q+E)u$ on each interval (x_{i-1}, x_i) in the sense of distributions.

Since equation (3.18) holds for all $f \in \mathrm{Dom}(H^0_{X,A,Q})$, integrating by parts, we get $u \in \mathrm{Dom}((H^0_{X,A,Q})^*)$. Then by a similar method as in Lemma 4, the only function $u \in \mathrm{Dom}(\mathbf{t}_{X,A,Q})$ satisfying equation (3.18) is $u = 0$. So we obtain that the set $\mathrm{Dom}(H^0_{X,A,Q})$ is a core of the form $\mathbf{t}_{X,A,Q}$. □

Lemma 8 *If the form $\mathbf{t}_{X,A,Q}$ is closed and $\mathbf{H}_{X,A,Q}$ is self-adjoint, then the operator associated with the form $\mathbf{t}_{X,A,Q}$ coincides with $\mathbf{H}_{X,A,Q} = (\mathbf{H}_{X,A,Q})^*$.*

Proof Integrating by parts, we can get from (1.2) that $\mathrm{Dom}(\mathbf{H}^0_{X,A,Q}) \subset \mathrm{Dom}(\mathbf{t}_{X,A,Q})$. For $u, v \in \mathrm{Dom}(\mathbf{H}^0_{X,A,Q})$,

$$\begin{aligned} \mathbf{t}_{X,A,Q}[u,v] &= \int_0^\infty \left[\left(u', v'\right)_m + \left(Q(x)u, v\right)_m\right] dx + \sum_{k=1}^\infty \left(A_k u(x_k), v(x_k)\right)_m \\ &= \left(\mathbf{H}^0_{X,A,Q} u, v\right). \end{aligned}$$

Assume that T is the self-adjoint operator associated with $\mathbf{t}_{X,A,Q}$. Since $\mathrm{Dom}(H^0_{X,A,Q})$ is a core of the form $\mathbf{t}_{X,A,Q}$ by Lemma 7, by the first representation theorem we have $u \in \mathrm{Dom}(T)$ and $Tu = \mathbf{H}^0_{X,A,Q} u$, and hence $T = \mathbf{H}_{X,A,Q}$. □

Lemma 9 *Assume that $Q(x) \geq 0$, $x \in \mathbb{R}_+$. Let the forms $\mathbf{t}_Q$ and $\mathbf{t}_R^+$ be defined by* (3.3) *and* (3.5). *Then the form $\mathbf{t}_{X,A^+,Q} = \mathbf{t}_Q + \mathbf{t}_R^+$ is nonnegative and closed.*

Proof The proof is similar to that of Lemma 6. □

Lemma 10 *Let $q_{ij} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, and let the forms $\mathbf{t}_Q$ and $\mathbf{t}_R^\pm$ be given by* (3.3) *and* (3.4).

(i) *If $\mu^-_{\max}(Q(x))$ and $\mu^-_{\max}(A_k)$ satisfy* (3.7), *then the form $\mathbf{t}_{X,A,Q}$ is closed and lower semibounded. Moreover,* $\mathrm{Dom}(\mathbf{t}_{X,A,Q}) = \mathrm{Dom}(\mathbf{t}_{X,A^+,Q^+})$, *and the operator associated with $\mathbf{t}_{X,A,Q}$ coincides with $\mathbf{H}_{X,A,Q}$.*

(ii) *If* $\mu_{\max}(Q(x)) = \mu^-_{\max}(Q(x))$ *and* $\mu_{\max}(A_k) = \mu^-_{\max}(A_k)$, *then conditions* (3.7) *are necessary and sufficient for the form* $\mathbf{t}_{X,A,Q}$ *to be lower semibounded.*

Proof (i) By the Lemma 9 the form $\mathbf{t}_{X,A^+,Q}$is closed. Moreover, by Lemma 5, $\mathbf{t}^-_Q$ and $\mathbf{t}^-_R$ are infinitesimally $\mathbf{t}^+_Q$-bounded and hence infinitesimally $\mathbf{t}_{X,A^+,Q^+}$-bounded. Using Lemma 2, we complete the proof.

(ii) Sufficiency is implied by (i). Let us prove necessity. Assume the converse, that is, the second condition in (3.7) does not hold. Then there exists $\{n_j\}_1^\infty \subset \mathbb{N}$ such that

$$\sum_{x_k\in[n_j,n_j+1]} \mu^-_{\max}(A_k) < -n_j, \quad j \in \mathbb{N}. \tag{3.19}$$

Define $\varphi_j(x) := \varphi(x - n_j)$, $j \in \mathbb{N}$, where $\varphi \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^m)$ is such that $\operatorname{sup} p\varphi \in (-\frac{1}{2}, \frac{3}{2})$, $0 \le \|\varphi\| \le 1$, and $\varphi(x) = e_1$, $x \in [0,1]$, where e_1 denotes the m-dimensional column unit vector whose first component is 1. By (3.19) and the form (3.6) we obtain $\mathbf{t}_{X,A,Q}[\varphi_j] \le -n_j + \|\varphi\|^2_{W^{1,2}(\mathbb{R}_+,\mathbb{C}^m)}$, and we note that when $n_j \to \infty$, $\mathbf{t}_{X,A,Q}$ is not lower semibounded. Hence, the contradiction finishes the proof. □

Corollary 1 *If*

$$\inf_{k\in\mathbb{N}} \frac{\mu^-_{\max}(A_k)}{d_k} > -\infty, \qquad d_k := x_k - x_{k-1}, \tag{3.20}$$

then the form $\mathbf{t}_{X,A} = \mathbf{t}_{X,A,0}$ *is closed, and lower semibounded, and* $\operatorname{Dom}(\mathbf{t}_{X,A}) = W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$. *Moreover, the operator associated with the form* $\mathbf{t}_{X,A}$ *coincides with* $\mathbf{H}_{X,A}$.

Proof Clearly, (3.20) yields (3.15) and (3.16). Lemma 10 completes the proof. □

4 Discrete spectrum

4.1 Sufficient conditions and necessary conditions

In this section we find conditions under which the vectorial Schrödinger operator $\mathbf{H}_{X,A,Q}$ has a purely discrete spectrum. The discrete spectrum of a self-adjoint operator T, $\sigma_d(T)$, is the set of all isolated eigenvalues of T with finite multiplicity, and the essential spectrum of T is the complement in $\sigma(T)$ of $\sigma_d(T)$ (see [14, 15]).

Theorem 1 *Assume that* $q_{ij} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *and* (3.7) *is satisfied. Then the operator* $\mathbf{H}_{X,A,Q}$ *is lower semibounded and self-adjoint, and its spectrum* $\sigma(\mathbf{H}_{X,A,Q})$ *is discrete if, for every* $\varepsilon > 0$,

$$\int_x^{x+\varepsilon} \mu_{\min}\bigl(Q(x)\bigr)\,dx + \sum_{x_k\in(x,x+\varepsilon)} \mu_{\min}(A_k) \to \infty \quad \text{as } x \to \infty. \tag{4.1}$$

If the spectrum $\sigma(\mathbf{H}_{X,A,Q})$ *is discrete, then*

$$\int_x^{x+\varepsilon} \mu_{\max}\bigl(Q(x)\bigr)\,dx + \sum_{x_k\in(x,x+\varepsilon)} \mu_{\max}(A_k) \to \infty \quad \text{as } x \to \infty. \tag{4.2}$$

Remark 1 Before giving the proof, we note the following facts about matrices.

(i) If a matrix-valued function $Q(x) \geq 0$, then its eigenvalues are all positive.
(ii) If $Q(x) \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^{m\times m})$, then the eigenvalue functions are also integrable.

Proof We divide the proof into two steps.

(i) First, assume that the potential $Q(x)$ is lower semibounded. Let $Y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$.

Sufficiency. Without loss of generality we assume that $(Q(x)Y(x), Y(x))_m \geq (Y(x), Y(x))_m$. By Lemma 6 the form $\mathbf{t}_{X,A,Q}$ is closed in $\mathfrak{H} = L^2(\mathbb{R}_+, \mathbb{C}^m)$. Let $\mathfrak{H}_{X,A,Q}$ be the Hilbert space generated by $\mathbf{t}_{X,A,Q}$. Let us show that the unit ball in $\mathfrak{H}_{X,A,Q}$,

$$\begin{aligned} \mathbb{U}_{X,A,Q} := \Biggl\{ & Y \in W_0^{1,2}(\mathbb{R}_+, \mathbb{C}^m) : \|Y\|^2_{W^{1,2}(\mathbb{R}_+,\mathbb{C}^m)} \\ & + \bigl\| \bigl(Q(x)Y(x), Y(x)\bigr)^{1/2} \bigr\|^2_{L^2(\mathbb{R}_+,\mathbb{C}^m)} + \sum_{k=1}^{\infty} \bigl(A_k Y(x_k), Y(x_k)\bigr)_m \leq 1 \Biggr\}, \end{aligned}$$

is compact in $L^2(\mathbb{R}_+, \mathbb{C}^m)$. Since the embedding $W^{1,2}([0,a], \mathbb{C}^m) \hookrightarrow L^2([0,a], \mathbb{C}^m)$ is compact for any $a > 0$, it suffices to show that the tails $\int_N^\infty (Y(x), Y(x))_m \, dx$ uniformly tend to zero in $\mathbb{U}_{X,A,Q}$.

Let us divide the semiaxis $\mathbb{R}_+$ into intervals $\Omega_n := \Omega_n(2\varepsilon)$ of length 2ε, $\Omega_k \cap \Omega_j = \varnothing$. Clearly, for any $Y \in W^{1,2}(\mathbb{R}_+, \mathbb{C}^m)$ and any $x, z \in \Omega_n$, we have

$$\begin{aligned} \bigl| \bigl(Y(x), Y(x)\bigr)_m - \bigl(Y(z), Y(z)\bigr)_m \bigr| &= 2 \left| \int_z^x \bigl(Y(t), Y'(t)\bigr)_m \, dt \right| \\ &\leq 2 \int_z^x \bigl| \bigl(Y(t), Y'(t)\bigr)_m \bigr| \, dt \\ &\leq \bigl\| Y(x) \bigr\|^2_{W^{1,2}(\Omega_n, C^m)}. \end{aligned} \tag{4.3}$$

Since Y is continuous on $\mathbb{R}_+$, there exists $t_n \in \Omega_n$ such that

$$\begin{aligned} & \int_{\Omega_n} \bigl(Q(x)Y(x), Y(x)\bigr)_m \, dx + \sum_{x_k \in \Omega_n} \bigl(A_k Y(x_k), Y(x_k)\bigr)_m \\ &\quad = \int_{\Omega_n} \sum_{i=1}^m \sum_{j=1}^m q_{ij} y_j y_i \, dx + \sum_{x_k \in \Omega_n} \left[\sum_{i=1}^m \sum_{j=1}^m a_{ij}^k y_j(x_k) y_i(x_k) \right] dx \\ &\quad \geq \int_{\Omega_n} \mu_{\min}\bigl(Q(x)\bigr) \sum_{i=1}^m |y_i|^2 \, dx + \sum_{x_k \in \Omega_n} \mu_{\min}(A_k) \sum_{i=1}^m \bigl| y_i(x_k) \bigr|^2 \\ &\quad = \sum_{i=1}^m \bigl| y_i(t_n) \bigr|^2 \left(\int_{\Omega_n} \mu_{\min}\bigl(Q(x)\bigr) \, dx + \sum_{x_k \in \Omega_n} \mu_{\min}(A_k) \right). \end{aligned} \tag{4.4}$$

Integrating (4.3) over Ω_n and taking (4.4) into account, we obtain

$$\begin{aligned} & \int_{\Omega_n} \bigl(Y(x), Y(x)\bigr) \, dx \\ &\quad \leq 2\varepsilon \bigl(Y(t_n), Y(t_n)\bigr) + 2\varepsilon \bigl\| Y(x) \bigr\|^2_{W^{1,2}(\Omega_n, \mathbb{C}^m)} \end{aligned}$$

$$\begin{aligned}&\leq 2\varepsilon\biggl(\int_{\Omega_n}\bigl(Q(x)Y(x),Y(x)\bigr)_m\,dx\\&\quad+\sum_{x_k\in\Omega_n}\bigl(A_kY(x_k),Y(x_k)\bigr)_m\biggr)\biggl(\int_{\Omega_n}\mu_{\min}\bigl(Q(x)\bigr)\,dx+\sum_{x_k\in\Omega_n}\mu_{\min}(A_k)\biggr)^{-1}\\&\quad+2\varepsilon\bigl\|Y(x)\bigr\|^2_{W^{1,2}(\Omega_n,\mathbb{C}^m)}.\end{aligned}\tag{4.5}$$

According to condition (4.1), there exists $N\in\mathbb{N}$ such that

$$\int_x^{x+\varepsilon}\mu_{\min}\bigl(Q(x)\bigr)\,dx+\sum_{x_k\in(x,x+\varepsilon)}\mu_{\min}(A_k)>1\quad\text{for } n\geq N.\tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\begin{aligned}\int_{t_N}^{\infty}\bigl(Y(x),Y(x)\bigr)_m\,dx&\leq\sum_{n=N}^{\infty}\int_{\Omega_n}\bigl(Y(x),Y(x)\bigr)_m\,dx\\&\leq 2\varepsilon\sum_{n=1}^{\infty}\biggl(\int_{\Omega_n}\sum_{i=1}^m\sum_{j=1}^m q_{ij}y_jy_i\,dx\\&\qquad+\sum_{x_k\in\Omega_n}\biggl[\sum_{i=1}^m\sum_{j=1}^m a_{ij}^k y_j(x_k)y_i(x_k)\biggr]\biggr)\\&\qquad+2\varepsilon\bigl\|Y(x)\bigr\|^2_{W^{1,2}(\mathbb{R}_+,\mathbb{C}^m)}\\&\leq 2\varepsilon.\end{aligned}$$

Hence, $\int_N^\infty(Y(x),Y(x))_m\,dx$ uniformly tends to zero in $\mathbb{U}_{X,A,Q}$. By Lemma 1 the spectrum $\sigma(\mathbf{H}_{X,A,Q})$ is discrete.

Necessity. Assume that condition (4.2) is violated. Then there exist $\varepsilon>0$ and a sequence $x_k\to\infty$ such that the inequality

$$\int_{x_n}^{x_n+\varepsilon}\mu_{\max}\bigl(Q(x)\bigr)\,dx+\sum_{x_k\in(x_n,x_n+\varepsilon)}\mu_{\max}(A_k)\leq C_1<\infty,\quad n\in\mathbb{N},\tag{4.7}$$

holds with some $C_1>0$. Let $\varphi\in W^{1,2}(\mathbb{R}_+)$ with $\|\varphi\|_{W^{1,2}}=1$ and $\sup p\varphi\subset(0,\varepsilon)$. Let

$$\varphi_n(x):=\varphi(x-x_n),\quad n\in\mathbb{N}.\tag{4.8}$$

Clearly, $\|\varphi_n\|_{W^{1,2}}=\|\varphi\|_{W^{1,2}}=1$. Since $\sup_{x\in\mathbb{R}_+}|\varphi(x)|=C_2<\infty$, we get

$$\begin{aligned}\mathbf{t}_{X,A,Q}[\varphi_n]+\|\varphi_n\|^2_{L^2}&=\int_{\mathbb{R}_+}\bigl[\bigl(\varphi_n'(x),\varphi_n'(x)\bigr)_m+\bigl(\varphi_n(x),\varphi_n(x)\bigr)_m\\&\qquad+\bigl(Q(x)\varphi_n(x),\varphi_n(x)\bigr)_m\bigr]\,dx+\sum_{k=1}^{\infty}\bigl(A_k\varphi_n(x_k),\varphi_n(x_k)\bigr)_m\\&\leq 1+\int_{\mathbb{R}_+}\mu_{\max}\bigl(Q(x)\bigr)\sum_{i=1}^m|\varphi_{n,i}|^2\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{x_k \in (x_n, x_n+\varepsilon)} \mu_{\max}(A_k) \sum_{i=1}^{m} \left|\varphi_{n,i}(x_k)\right|^2 \\
&\leq 1 + C_2^2 \int_{x_n}^{x_n+\varepsilon} \mu_{\max}\bigl(Q(x)\bigr)\,dx + C_2^2 \sum_{x_k \in (x_n, x_n+\varepsilon)} \mu_{\max}(A_k) \\
&\leq 1 + C_2^2 C_1.
\end{aligned}
$$

Thus, the sequence $\{\varphi_n\}_1^\infty$ is bounded in $\mathfrak{H}_{X,A,Q}$ but is not compact in $L^2(\mathbb{R}_+, \mathbb{C}^m)$. By Lemma 1 the spectrum $\sigma(\mathbf{H}_{X,A,Q})$ is not discrete.

(ii) Assume now that $Q(x)$ satisfies condition (3.7). Then by Lemma 10(i) the operator $\mathbf{H}_{X,A,Q}$ is self-adjoint and lower semibounded if A_k ($k = 1, 2, \ldots$) satisfy (3.7). Furthermore, by (3.7), $\mu_{\max}(Q(x))$ and $\mu_{\max}^+(Q(x))$ satisfy (4.1) simultaneously. By the above considerations, the spectrum of the operator $\mathbf{H}_{X,A,Q^+}$ is discrete if $\mu_{\max}^+(Q(x))$ satisfies (4.1). Note that $\mathfrak{q}^-$ is infinitesimally $\mathbf{t}_{Q^+}$-bounded if $Q(x)$ satisfies condition (3.7). This yields

$$
W^{1,2}\bigl(\mathbb{R}_+, \mathbb{C}^m; Q\bigr) = \mathfrak{H}_{\mathbf{t}_Q} = \mathfrak{H}_{\mathbf{t}_{Q^+}} = W^{1,2}\bigl(\mathbb{R}_+, \mathbb{C}^m; Q^+\bigr),
$$

and the proof of Lemma 1 is finished. □

Remark 2 This result coincides with the corresponding result obtained in [1] in the scalar case ($m = 1$), and in this case it is also necessary. But in the vectorial case, condition (4.1) is no longer necessary. In the next subsection, by giving additional restrictions on symmetric potential matrix $Q(x)$ and A_k we obtain a sufficient and necessary condition.

Corollary 2 *Let $Q(x)$ and A_k satisfy* (3.7). *Then the spectrum $\sigma(\mathbf{H}_{X,A,Q})$ is discrete whenever*

$$
\sum_{x_k \in (x, x+\varepsilon)} \mu_{\min}(A_k) = \infty
$$

for every $\varepsilon > 0$. If, in addition,

$$
\sup_{x>0} \int_x^{x+\varepsilon} \mu_{\min}\bigl(Q(t)\bigr)\,dt < \infty
$$

for some $\varepsilon > 0$, then

$$
\sum_{x_k \in (x, x+\varepsilon)} \mu_{\max}(A_k) = \infty
$$

is necessary for $\sigma(\mathbf{H}_{X,A,Q})$ to be discrete.

Proof The proof is immediate from Theorem 1. □

Corollary 3 *Let the symmetric potential function $Q(x)$ and the matrix sequence $\{A_k\}$ satisfy* (3.7). *Assume also that $d_* = \inf_{k\in\mathbb{N}} d_k > 0$. Then the spectrum $\sigma(\mathbf{H}_{X,A,Q})$ is discrete if, for every $\varepsilon > 0$,*

$$
\int_x^{x+\varepsilon} \mu_{\min}\bigl(Q(x)\bigr)\,dx \to \infty \quad \text{as } x \to \infty. \tag{4.9}
$$

Proof Sufficiency is immediate from Theorem 1. □

4.2 Sufficient and necessary conditions for a special case

In this subsection, we give some restrictions on $Q(x)$ and A_k to get a sufficient and necessary condition for the spectrum $\sigma(\mathbf{H}_{X,A,Q})$ to be discrete:

(1) $\sum_{i\neq j} q_{ij}(x) \le a q_{jj}(x)$, $j = 1,2,\dots,m$, $0 \le a < 1$;

(2) $\sum_{i\neq j} a_{ij}^k \le b a_{jj}^k$, $j = 1,2,\dots,m$, $k = 1,2,\dots$, $0 \le b < 1$.

Let $Q(x) = Q_1(x) + Q_2(x)$ and $A_k = A_{k1} + A_{k2}$, where $Q_1(x) = (q_{ij}(x)\delta_{ij})_{i,j=1}^m$ and $A_{k1} = (a_{ij}^k\delta_{ij})_{i,j=1}^m$. By (3.1) and (3.4) we denote the following forms:

$$\mathfrak{q}[Y] = \mathfrak{q}_1[Y] + \mathfrak{q}_2[Y] = \int_0^\infty \bigl(Q_1(x)Y(x), Y(x)\bigr)_m\,dx + \int_0^\infty \bigl(Q_2(x)Y(x), Y(x)\bigr)_m\,dx,$$

$$\mathbf{t}_R[Y] = \mathbf{t}_{R_1}[Y] + \mathbf{t}_{R_2}[Y] = \sum_{k=1}^\infty \bigl(A_{k,1}Y(x_k), Y(x_k)\bigr)_m + \sum_{k=1}^\infty \bigl(A_{k,2}Y(x_k), Y(x_k)\bigr)_m.$$

Theorem 2 *Under conditions* (1)-(2), $\mathfrak{q}_2[Y]$ *is* $\mathfrak{q}_1[Y]$ *relatively form bounded with bound* $a < 1$, *and* $\mathbf{t}_{R_2}[Y]$ *is* $\mathbf{t}_{R_1}[Y]$ *relatively form bounded with bound* $b < 1$.

Proof

$$\begin{aligned}\mathfrak{q}_2[Y] &= \int_0^\infty \bigl(Q_2(x)Y(x), Y(x)\bigr)_m\,dx = \int_0^\infty \sum_{i,j=1;i\neq j}^m q_{ij}y_iy_j\,dx \\ &\le \int_0^\infty a\sum_{j=1}^m q_{jj}y_j^2\,dx = a\int_0^\infty \bigl(Q_1(x)Y(x), Y(x)\bigr)_m\,dx = a\mathfrak{q}_1[Y] + c\|Y\|^2.\end{aligned}$$

Similarly, $\mathbf{t}_{R_2}[Y] \le b\mathbf{t}_{R_1}[Y] + c\|Y\|^2$, where c is a positive constant. Then by Definition 1 the theorem is proved. □

We denote the operator corresponding to the differential expression

$$L_1 = -d^2/dx^2 + Q_1(x) + \sum A_{k1}\delta(x - x_k)$$

by $\mathbf{H}$. Obviously, $\mathbf{H}$ equals the direct sum of scalar operators $\mathbf{H}_i$ on $L^2(\mathbb{R}_+)$ that are defined by the expressions

$$L_{1i} = -d^2/dx^2 + q_{ii}(x) + \sum a_{ii}^k\delta(x - x_k).$$

Then $\sigma(\mathbf{H}) = \cup_{i=1}^m \sigma(\mathbf{H}_i)$, which is obtained by the direct sum decomposition method of operators (see [16]). Denote $\mathbf{t}_\mathbf{H} = \mathfrak{q}_1[Y] + \mathbf{t}_{R_1}[Y] + \mathbf{t}_0[Y]$. Note that $\mathbf{t}_{X,A,Q} = \mathbf{t}_\mathbf{H} + \mathfrak{q}_2[Y] + \mathbf{t}_{R_2}[Y]$, $\mathfrak{q}_2[Y] + \mathbf{t}_{R_2}[Y]$ is $\mathbf{t}_\mathbf{H}$ relatively form bounded, and $\mathbf{t}_\mathbf{H}$ is associated with the operator $\mathbf{H}$. By Lemma 2 the norms $\|\cdot\|_{\mathbf{t}_{X,A,Q}}$ and $\|\cdot\|_\mathbf{H}$ are equivalent. If $i_\mathbf{H} : \mathfrak{H}_\mathbf{H} \hookrightarrow L^2(\mathbb{R}_+, \mathbb{C}^m)$ is compact, then $i_\mathbf{H} : \mathfrak{H}_{\mathbf{H}_{X,A,Q}} \hookrightarrow L^2(\mathbb{R}_+, \mathbb{C}^m)$ is also compact. Thus, if $\sigma(\mathbf{H})$ is discrete, then $\sigma(\mathbf{H}_{X,A,Q})$ is also discrete.

It is evident that $\mathbf{H}$ has a purely discrete spectrum if and only if the operator $\mathbf{H}_i$ has such a spectrum for all $i = 1,2,\dots,m$. But the operator $\mathbf{H}_i$ is the ordinary 'scalar' Schrödinger operator with positive potential and δ-interactions, and the theorem of discrete criterion in [1] holds. Hence. we get the following theorem.

Theorem 3 *Let $Q(x)$ and A_k satisfy conditions* (1)-(2). *Then the spectrum $\sigma(\mathbf{H}_{X,A,Q})$ is discrete if and only if, for each $i = 1, 2, \dots, m$ and for every $\varepsilon > 0$,*

$$\int_x^{x+\varepsilon} q_{ii}(t)\,dt + \sum_{x_k \in (x,x+\varepsilon)} a_{ii}^k \to \infty \quad \text{as } x \to \infty.$$

Example 1 Let $m = 2$, $Y(x) = (y_1(x), y_2(x))^T$, $Q(x) = \begin{pmatrix} x^2-2 & 0 \\ 0 & x-4 \end{pmatrix}$, $A_k = \begin{pmatrix} k+1 & 1 \\ 1 & 2^k \end{pmatrix}$, and $x_k = \sqrt{k}$. Then by Theorem 3 we have that the spectrum of such an operator is discrete.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Albeverio, S, Kostenko, A, Malamud, M: Spectral theory of semibounded Sturm-Liouville operators with local interactions on a discrete set. J. Math. Phys. **51**, 102102 (2010)
2. Kostenko, A, Malamud, M: 1-dimensional Schrödinger operators with local point interactions on a discrete set. J. Differ. Equ. **249**(2), 253-304 (2010)
3. Yan, J, Shi, G: Spectra properties of Sturm-Liouville operators with local interactions on a discrete set. J. Differ. Equ. **257**, 3423-3447 (2014)
4. Shubin, C, Christ Stolz, G: Spectral theory of one-dimensional Schrödinger operators with point interactions. J. Math. Anal. Appl. **184**, 491-516 (1994)
5. Albeverio, S, Gesztesy, F, Hoegh-Krohn, R, Holden, H: Solvable Models in Quantum Mechanics, 2nd edn. Chelsea, New York (2005)
6. Eckhardt, J, Gesztesy, F, et al.: Supersymmetry and Schrödinger-type operators with distributional matrix-valued potentials. J. Spectr. Theory **4**, 715-768 (2014)
7. Mirzoev, KA, Safonova, TA: Singular Sturm-Liouville operators with distribution potential on spaces of vector functions. Dokl. Akad. Nauk SSSR **441**(2), 165-168 (2011)
8. Liu, X, Wang, Z: Conditions for the discreteness of spectra of two-item self-adjoint vector differential operators. J. Inn. Mong. Norm. Univ. **2**, 153-156 (2009)
9. Clark, S, Gesztesy, F: On Povzner-Wienholtz-type self-adjointness results for matrix-valued Sturm-Liouville operators. Proc. R. Soc. Edinb. A **133**, 747-758 (2003)
10. Molčanov, AM: On conditions for the spectrum of a second order self-adjoint differential equation to be discrete. Trans. Mosc. Math. Soc. **2**, 169-200 (1953)
11. Kato, T: Perturbation Theory for Linear Operators. Springer, Berlin (1966)
12. Reed, M, Simon, B: Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness. Academic Press, New York (1975)
13. Muller-Pfeiffer, E: Spectral Theory of Ordinary Differential Operators. Ellis Horwood, Chichester (1981)
14. Glazman, IM: Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators. Israel Program for Scientific Translations, Jerusalem (1965)
15. Weidmann, J: Linear Operators in Hilbert Spaces. Springer, Berlin (1980)
16. Muller-Pferiffer, E, Sun, J: On the discrete spectrum of ordinary differential operators in weighted function spaces. Z. Anal. Anwend. **14**(5), 637-646 (1995)