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Measures of noncompactness in spaces of regulated functions with application to semilinear measure driven equations

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Abstract

We investigate the existence of mild solutions for abstract semilinear measure driven equations with nonlocal conditions. We first establish some results on Kuratowski measure of noncompactness in the space of regulated functions. Then we obtain some existence results for the abstract measure system by using the measure of noncompactness and a corresponding fixed point theorem. The usual Lipschitz-type assumptions are avoided, and the semigroup related to the linear part of the system is not claimed to be compact, which improves and generalizes some known results in the literature.

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1 Introduction

In this paper, we consider the following semilinear measure driven differential system with nonlocal condition:

$$\begin{aligned} &dx(t) = Ax(t) + f\big(t, x(t)\big)\, dg(t), \quad t \in J, \\ &x(0) = p(x), \end{aligned} \tag{1}$$

where $J = [0, a]$ with $a > 0$. The state variable $x(\cdot)$ takes values in a Banach space X. $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$, and $g : J \to \mathbb{R}$ is a nondecreasing left-continuous function; the functions $f : J \times X \to X$ and $p : G(J; X) \to X$ will be specified later, where $G(J; X)$ denotes the space of regulated functions on J in which we consider the problem. By dx and dg we denote the distributional derivatives of the solution and the function g, respectively [1, 2].

Measure driven differential equations are also called differential equations with measures or measure differential equations; they arise in many areas of applied mathematics such as nonsmooth mechanics, game theory, *etc.* (see [3–7]). This type of systems covers some well-known cases up to the difference of g. When g is an absolutely continuous function, a step function, or the sum of an absolutely continuous function with a step function, the system corresponds to ordinary differential equations, difference equations, or impulsive differential equations, respectively. On the other hand, since measure differential equations admit discontinuous paths that may exhibit infinitely many discontinuities in a finite interval, they can model some complex behaviors in dynamic systems, for example, Zeno trajectories [8].

The nonlocal problem was considered by Byszewski [9]. This type of systems is more appropriate than the classical initial value problem to describe real phenomena because it allows us to consider additional information. In the past few years theorems about the existence and controllability of differential and functional differential abstract evolution systems with nonlocal conditions have been fully studied (see [10–17] and the references therein).

Measure differential equations were investigated early by [18–21]. We can refer to the review paper [22] for a complete introduction of measure differential systems. Recently, the theory of measure differential equations in $\mathbb{R}^n$ space has been developed to some extent (see [2, 23–25]). However, to the best of our knowledge, little literatures has been devoted to measure differential equations in infinite-dimensional spaces except [1] and [26, 27]. In separable Banach spaces, by applying Hausdorff measure of noncompactness, the paper [1] discussed the existence of solutions for nonlinear measure driven system in the Kurzweil integral setting (a kind of nonabsolutely convergent integral generalizing the Lebesgue integral), in which the system can be viewed as a particular case of system (1) in this paper with $A = 0$ and $p(x) = x_0$. Although some properties of Hausdorff measure of noncompactness in the space of $G(J;X)$ were provided in [1], those properties are not intrinsic for $G(J;X)$ since the function g in the system was involved. Under Lipschitz-type conditions, [23, 24] studied the retarded version of nonlinear measure driven system by means of generalized ordinary differential equations when $X = \mathbb{R}^n$ and in the Kurzweil integral setting. The authors in [26] investigated the existence of mild solutions for abstract semilinear measure driven system without consideration of nonlocal conditions, where the compactness of the C_0-semigroup related to the linear part of the system is claimed. In this paper, for a general Banach space X, we first establish some useful properties of the Kuratowski measure of noncompactness in the space of regulated functions $G(J;X)$. Then we obtain some existence results for semilinear measure driven system with nonlocal conditions (1) by applying the Kuratowski measure of noncompactness and a corresponding fixed point theorem. The compactness of the C_0-semigroup is not demanded in this paper. In addition, without any assumptions of Lipschitz-type as those in [23, 24], a similar analysis to system (1) can lead to the existence result for nonlinear measure retarded equations in the Lebesgue integral setting.

This paper is organized as follows. In Section 2, we review some concepts and results about the Lebesgue-Stieltjes integral and regulated functions and the Kuratowski measure of noncompactness, which will be used throughout this paper. In Section 3, some results of the Kuratowski measure of noncompactness and regulated functions are established and applied to investigate the existence for the semilinear measure system (1). An example that illustrates our results is presented in Section 4. Finally, some conclusions are drawn in Section 5.

2 Preliminaries

In this section, we recall some concepts and basic results about the Lebesgue-Stieltjes integral and regulated functions and the Kuratowski measure of noncompactness. For the properties of operator semigroups, we refer the reader to [28, 29].

Let X be a Banach space with a norm $\|\cdot\|$, and $[a,b]$ be a closed interval of the real line. A function $f:[a,b]\to X$ is called regulated on $[a,b]$ if the limits

$$\lim_{s\to t^-} f(s)=f(t^-),\quad t\in(a,b]\quad\text{and}\quad \lim_{s\to t^+} f(s)=f(t^+),\quad t\in[a,b),$$

exist and are finite. The space of regulated functions $f:[a,b]\to X$ is denoted by $G([a,b];X)$. It is well known that the set of discontinuities of a regulated function is at most countable and that the space $G([a,b];X)$ is a Banach space endowed with the norm $\|f\|_\infty=\sup_{t\in[a,b]}\|f(t)\|$ (see [30]).

The finite sets $d=\{t_0,t_1,\dots,t_n\}$ of points in the closed interval $[a,b]$ such that $a=t_0<t_1<\cdots<t_n=b$ are called partitions of $[a,b]$. For $\delta>0$, we say that a partition of $[a,b]$ is δ-fine if $|t_i-t_{i-1}|<\delta$ for all $i=1,2,\dots,n$.

The following result holds by Proposition 3 in [1] since the Kurzweil integral is more general than the Lebesgue integral.

Proposition 2.1 *Consider the functions $f:[a,b]\to X$ and $g:[a,b]\to\mathbb{R}$ such that g is regulated and $\int_a^b f\,dg$ exists. Then for every $t_0\in[a,b]$, the function $h(t)=\int_{t_0}^t f\,dg$, $t\in[a,b]$, is regulated and satisfies*

$$\begin{aligned}&h(t^+)=h(t)+f(t)\Delta^+g(t),\quad t\in[a,b),\\&h(t^-)=h(t)-f(t)\Delta^-g(t),\quad t\in(a,b],\end{aligned}$$

where $\Delta^+g(t)=g(t^+)-g(t)$ and $\Delta^-g(t)=g(t)-g(t^-)$.

The readers can refer to [31, 32] for the theory of Lebesgue-Stieltjes integral and other types of integrals together with the relations among them.

Definition 2.2 ([33]) A set $\mathcal{A}\subset G([a,b];X)$ is called equiregulated if, for every $\varepsilon>0$ and every $t_0\in[a,b]$, there is $\delta>0$ such that:

(i) If $x\in\mathcal{A}$, $t\in[a,b]$, and $t_0-\delta<t<t_0$, then $\|x(t_0{}^-)-x(t)\|<\varepsilon$.

(ii) If $x\in\mathcal{A}$, $t\in[a,b]$, and $t_0<t<t_0+\delta$, then $\|x(t)-x(t_0{}^+)\|<\varepsilon$.

Lemma 2.3 ([33]) *A set $\mathcal{A}\subset G([a,b];X)$ is equiregulated if and only if, for every $\varepsilon>0$, there is a δ-fine partition $a=t_0<t_1<\cdots<t_n=b$ such that*

$$\|x(t'')-x(t')\|<\varepsilon$$

for all $x\in\mathcal{A}$ and $t_{i-1}<t'<t''<t_i$, $i=1,2,\dots,n$.

Remark 2.4 According to Lemma 2.3, it is clear that $x:[a,b]\to X$ is a regulated function if and only if, for every $\varepsilon>0$, there is a δ-fine partition $a=t_0<t_1<\cdots<t_n=b$ such that

$$\|x(t'')-x(t')\|<\varepsilon$$

for $t_{i-1}<t'<t''<t_i$ and $i=1,2,\dots,n$. One can also refer to Lemma 1.1 in [33] for the proof.

Lemma 2.5 ([33]) *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of functions from $[a,b]$ to X. If x_n converges pointwise to x_0 as $n \to \infty$ and the sequence $\{x_n\}_{n=1}^{\infty}$ is equiregulated, then x_n converges uniformly to x_0.*

Lemma 2.6 *Let $W \subset G(J;X)$. If W is bounded and equiregulated, then the set $\overline{\text{co}}(W)$ is also bounded and equiregulated.*

Proof According to the boundedness of W, it is evident that $\overline{\text{co}}(W)$ is bounded.

For any $\varepsilon > 0$, it follows from the equiregularity of the set W and Lemma 2.3 that there exists a δ-fine partition $0 = t_0 < t_1 < \cdots < t_n = a$ such that

$$\|w(t'') - w(t')\| < \varepsilon$$

for all $w \in W$ and $t_{i-1} < t' < t'' < t_i$, $i = 1, 2, \ldots, n$.

For any $h \in \text{co}(W)$, there exist $w_1, w_2, \ldots, w_m \in W$, $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$, $\sum_{j=1}^{m} \lambda_j = 1$, such that $h = \sum_{j=1}^{m} \lambda_j w_j$. Then, for all $t_{i-1} < t' < t'' < t_i$, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} \|h(t'') - h(t')\| &= \left\| \sum_{j=1}^{m} \lambda_j (w_j(t'') - w_j(t')) \right\| \leq \sum_{j=1}^{m} \lambda_j \|w_j(t'') - w_j(t')\| \\ &< \varepsilon \sum_{j=1}^{m} \lambda_j = \varepsilon. \end{aligned}$$

Hence, the set $\text{co}(W)$ is equiregulated.

For any $h \in \overline{\text{co}}(W)$, there exists a sequence of functions $\{h_n\}_{n=1}^{\infty}$ in the set $\text{co}(W)$ such that $\lim_{n\to\infty} h_n = h$. Then for all $t_{i-1} < t' < t'' < t_i$, $i = 1, 2, \ldots, n$, we have

$$\|h(t'') - h(t')\| = \lim_{n\to\infty} \|h_n(t'') - h_n(t')\| \leq \varepsilon.$$

Therefore, the set $\overline{\text{co}}(W)$ is equiregulated. □

The Kuratowski measure of noncompactness of a bounded subset S of the Banach space X is defined by

$\alpha(S) = \inf\{\delta > 0 : S$ can be expressed as the union of a finite number of sets such that the diameter of each set does not exceed δ, *i.e.*, $S = \bigcup_{i=1}^{m} S_i$ with $\text{diam}(S_i) \leq \delta$, $i = 1, 2, \ldots, m\}$,

where $\text{diam}(S)$ denotes the diameter of a set S (see [34], Definition 1.2.1).

Lemma 2.7 ([34]) *Let S, T be bounded sets of X, and $\lambda \in \mathbb{R}$. Then*:

(i) $\alpha(S) = 0$ *if and only if S is relatively compact*;
(ii) $S \subseteq T$ *implies* $\alpha(S) \leq \alpha(T)$;
(iii) $\alpha(\overline{S}) = \alpha(S)$;
(iv) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$;
(v) $\alpha(\lambda S) = |\lambda| \alpha(S)$, *where* $\lambda S = \{x = \lambda z : z \in S\}$;
(vi) $\alpha(S + T) \leq \alpha(S) + \alpha(T)$, *where* $S + T = \{x = y + z : y \in S, z \in T\}$;
(vii) $\alpha(\text{co}\, S) = \alpha(S)$;

(viii) $|\alpha(S) - \alpha(T)| \leq 2d_h(S,T)$, *where* $d_h(S,T)$ *denotes the Hausdorff metric of* S *and* T, *that is*,

$$d_h(S,T) = \max\Big\{\sup_{x\in S} d(x,T), \sup_{x\in T} d(x,S)\Big\},$$

and $d(\cdot,\cdot)$ *is the distance from an element of* X *to a subset of* X.

Lemma 2.8 ([35]) *Let* X *be a Banach space, and* $D \subseteq X$ *a bounded set. Then there exists a countable subset* D_0 *of* D *such that* $\alpha(D) \leq 2\alpha(D_0)$.

Let μ *be a regular Borel measure on* J *and* $L^1_\mu(J;X)$ *denote the set of* μ*-integrable functions from* J *to* X.

Lemma 2.9 ([36]) *Let* $W_0 \subseteq L^1_\mu(J;X)$ *be a countable set. Assume that there exists a positive function* $k \in L^1_\mu(J;\mathbb{R}^+)$ *such that* $\|w(t)\| \leq k(t)$ μ*-a.e. for all* $w \in W_0$. *Then we have*

$$\alpha\left(\int_J W_0(t)\,d\mu(t)\right) \leq 2\int_J \alpha\big(W_0(t)\big)\,d\mu(t).$$

Corollary 2.10 *Let* $W \subseteq L^1_\mu(J;X)$. *If there exists a positive function* $k \in L^1_\mu(J;\mathbb{R}^+)$ *such that* $\|w(t)\| \leq k(t)$ μ*-a.e. for all* $w \in W$, *then we have*

$$\alpha\left(\int_J W(t)\,d\mu(t)\right) \leq 4\int_J \alpha\big(W(t)\big)\,d\mu(t).$$

Proof By Lemma 2.7(ii), Lemma 2.8, and Lemma 2.9, there exists a countable set $W_0 \subseteq W$ such that

$$\begin{aligned}\alpha\left(\int_J W(t)\,d\mu(t)\right) &\leq 2\alpha\left(\int_J W_0(t)\,d\mu(t)\right) \leq 4\int_J \alpha\big(W_0(t)\big)\,d\mu(t)\\ &\leq 4\int_J \alpha\big(W(t)\big)\,d\mu(t).\end{aligned}$$

□

Remark 2.11 Since the Lebesgue-Stieltjes measure is a regular Borel measure, the result of Corollary 2.10 holds for the Lebesgue-Stieltjes measure.

Lemma 2.12 ([37]) *Let* F *be a closed convex subset of a Banach space, and the operator* $N: F \to F$ *be continuous with* $N(F)$ *bounded. For any bounded* $B \subseteq F$, *set*

$$\widetilde{N}^1(B) = N(B) \quad \text{and} \quad \widetilde{N}^n(B) = N\big(\overline{\mathrm{co}}\big(\widetilde{N}^{n-1}(B)\big)\big), \quad \forall n \geq 2, n \in \mathbb{N}.$$

If there exist a constant $0 \leq \gamma < 1$ *and* $n_0 \in \mathbb{N}$ *such that* $\alpha(\widetilde{N}^{n_0}(B)) \leq \gamma\alpha(B)$ *for every bounded* $B \subseteq F$, *then* N *has a fixed point.*

3 Main results

In this section, we show the main results of this paper that are divided into two parts. Some properties of the Kuratowski measure of noncompactness in the space of regulated functions $G(J;X)$ are established in the first part. These properties are then applied to discuss the existence for the semilinear measure system (1) in the second part.

3.1 Measure of noncompactness in $G(J;X)$

Let W be a subset of $G(J;X)$. For each fixed $t \in J$, we denote $W(t) = \{x(t) : x \in W\}$. Further, let $W(J) = \bigcup_{t\in J} W(t) = \{x(t) : x \in W, t \in J\}$. Next, we will provide some results on the Kuratowski measure of noncompactness in the space of regulated functions $G(J;X)$, which generalize those in the space of continuous functions $C(J;X)$ in [34].

Theorem 3.1 *Let $W \subset G(J;X)$ be bounded and equiregulated on J. Then $\alpha(W(t))$ is regulated on J.*

Proof Since W is equiregulated, then for every $\varepsilon > 0$, there is a δ-fine partition $0 = t_0 < t_1 < \cdots < t_n = a$ such that

$$\|x(t'') - x(t')\| < \varepsilon/2 \quad \text{for any } t_{i-1} < t' < t'' < t_i \ (i = 1, 2, \ldots, n), x \in W.$$

Note that

$$\begin{aligned}\sup_{x\in W} d\bigl(x(t'), W(t'')\bigr) &= \sup_{x\in W} \inf_{y\in W} \bigl\|x(t') - y(t'')\bigr\| \\ &\leq \sup_{x\in W} \bigl\|x(t') - x(t'')\bigr\| + \sup_{x\in W} \inf_{y\in W} \bigl\|x(t'') - y(t'')\bigr\| \leq \varepsilon/2.\end{aligned}$$

Further, the same proof as before leads to

$$\sup_{x\in W} d\bigl(x(t''), W(t')\bigr) \leq \varepsilon/2.$$

Then we have

$$d_h\bigl(W(t'), W(t'')\bigr) \leq \varepsilon/2.$$

By Lemma 2.7(viii) we have

$$\bigl\|\alpha\bigl(W(t'')\bigr) - \alpha\bigl(W(t')\bigr)\bigr\| \leq 2d_h\bigl(W(t'), W(t'')\bigr) \leq \varepsilon.$$

According to Remark 2.4, $\alpha(W(t))$ is regulated on J. □

Theorem 3.2 *Let $W \subset G(J;X)$ be bounded and equiregulated on J. Then*

$$\alpha(W) = \sup\bigl\{\alpha\bigl(W(t)\bigr) : t \in J\bigr\}.$$

Proof *Step* 1. We first prove that $\alpha(W) = \alpha(W(J))$.

(i) Let us first show that $\alpha(W(J)) \leq \alpha(W)$. For every $\varepsilon > 0$, let $W = \bigcup_{j=1}^{m} W_j$ be such that $\operatorname{diam}(W_j) < \alpha(W) + \varepsilon$, $j = 1, 2, \ldots, m$.

By the equiregularity of the set W and by Lemma 2.3 there is a δ-fine partition $0 = t_0 < t_1 < \cdots < t_n = a$ such that

$$\bigl\|x(t'') - x(t')\bigr\| < \varepsilon$$

for all $x \in W$ and $t_{i-1} < t' < t'' < t_i$, $i = 1, 2, \ldots, n$.

Let $J_i = (t_{i-1}, t_i)$, $S_{ji} = \{x(t) : x \in W_j, t \in J_i\}$, $j = 1, 2, \dots, m$, $i = 1, 2, \dots, n$. It is evident that

$$W(J) = \left(\bigcup_{j=1}^{m}\bigcup_{i=1}^{n} S_{ji}\right) \cup \left(\bigcup_{j=1}^{m}\bigcup_{i=0}^{n} W_j(t_i)\right).$$

For any $x, y \in W_j$, $t, t' \in J_i$, we have

$$\begin{aligned}\|x(t) - y(t')\| &\le \|x(t) - y(t)\| + \|y(t) - y(t')\| \le \|x - y\|_\infty + \varepsilon \\ &\le \operatorname{diam}(W_j) + \varepsilon < \alpha(W) + 2\varepsilon.\end{aligned}$$

Hence, $\alpha(\bigcup_{j=1}^{m}\bigcup_{i=1}^{n} S_{ji}) \le \alpha(W) + 2\varepsilon$.

For any $x, y \in W_j$, $t = t_i$, we have

$$\|x(t_i) - y(t_i)\| \le \|x - y\|_\infty \le \operatorname{diam}(W_j) < \alpha(W) + \varepsilon.$$

Hence, $\alpha(\bigcup_{j=1}^{m}\bigcup_{i=0}^{n} W_j(t_i)) \le \alpha(W) + \varepsilon$. Therefore, we have

$$\alpha(W(J)) = \max\left\{\alpha\left(\bigcup_{j=1}^{m}\bigcup_{i=1}^{n} S_{ji}\right), \alpha\left(\bigcup_{j=1}^{m}\bigcup_{i=0}^{n} W_j(t_i)\right)\right\} \le \alpha(W) + 2\varepsilon.$$

The arbitrariness of ε shows that $\alpha(W(J)) \le \alpha(W)$.

(ii) Let us now show that $\alpha(W) \le \alpha(W(J))$. For every $\varepsilon > 0$, by the equiregularity of the set W and by Lemma 2.3 there is a δ-fine partition $0 = t_0 < t_1 < \cdots < t_n = a$ such that

$$\|x(t'') - x(t')\| < \varepsilon$$

for all $x \in W$ and $t', t'' \in J_i = (t_{i-1}, t_i)$, $i = 1, 2, \dots, n$.

On the other hand, there is a partition $W(J) = \bigcup_{j=1}^{m} T_j$ such that

$$\operatorname{diam}(T_j) < \alpha(W(J)) + \varepsilon, \quad j = 1, 2, \dots, m.$$

Let P be the finite set of all maps $i \to \mu(i)$ of $\{1, 2, \dots, n\}$ into $\{1, 2, \dots, m\}$. Let Q be the finite set of all maps $i \to \nu(i)$ of $\{0, 1, 2, \dots, n\}$ into $\{1, 2, \dots, m\}$. Fixing arbitrarily $\tau_i \in J_i$, $i = 1, 2, \dots, n$, for $\mu \in P$, $\nu \in Q$, let $L_\mu = \{x \in W : x(\tau_i) \in T_{\mu(i)}, i = 1, 2, \dots, n\}$, $L_\nu = \{x \in W : x(t_i) \in T_{\nu(i)}, i = 0, 1, \dots, n\}$, and let $L_{\mu\nu} = L_\mu \cap L_\nu$. It is clear that $W = \bigcup_{\mu \in P, \nu \in Q} L_{\mu\nu}$.

For any $x, y \in L_{\mu\nu}$ and $t \in J$, if $t \in J_i$ for some $i = 1, 2, \dots, n$, then we have

$$\begin{aligned}\|x(t) - y(t)\| &\le \|x(t) - x(\tau_i)\| + \|x(\tau_i) - y(\tau_i)\| + \|y(\tau_i) - y(t)\| \\ &< 2\varepsilon + \operatorname{diam}(T_{\mu(i)}) < \alpha(W(J)) + 3\varepsilon.\end{aligned}$$

If $t = t_i$ for some $i = 0, 1, \dots, n$, then we have

$$\|x(t_i) - y(t_i)\| \le \operatorname{diam}(T_{\nu(i)}) < \alpha(W(J)) + \varepsilon.$$

Consequently, for any $x, y \in L_{\mu\nu}$, we have

$$\|x - y\|_\infty = \sup_{t\in J}\big\|x(t) - y(t)\big\| \le \alpha\big(W(J)\big) + 3\varepsilon,$$

which implies $\operatorname{diam}(L_{\mu\nu}) \le \alpha(W(J)) + 3\varepsilon$, and hence $\alpha(W) \le \alpha(W(J)) + 3\varepsilon$. Since ε is arbitrary, we have $\alpha(W) \le \alpha(W(J))$.

(i) and (ii) show that $\alpha(W) = \alpha(W(J))$.

Step 2. Now we prove that $\alpha(W(J)) = \sup\{\alpha(W(t)) : t \in J\}$.

We first observe that, by Theorem 3.1, $\alpha(W(t))$ is regulated on J, and hence $\sup_{t\in J}\alpha(W(t))$ exists. Since $W(t) \subset W(J)$ for any $t \in J$, we have

$$\sup_{t\in J}\alpha\big(W(t)\big) \le \alpha\big(W(J)\big).$$

On the other hand, for any given $\varepsilon > 0$, by the equiregularity of the set W and by Lemma 2.3, there is a δ-fine partition $0 = t_0 < t_1 < \cdots < t_n = a$ such that

$$\big\|x\big(t''\big) - x\big(t'\big)\big\| < \varepsilon$$

for all $x \in W$ and $t', t'' \in J_i = (t_{i-1}, t_i)$, $i = 1, 2, \ldots, n$.

Take $\tau_i \in J_i$ ($i = 1, 2, \ldots, n$) arbitrarily. It is easy to see that, for any i ($i = 1, 2, \ldots, n$), there exists a partition $W = \bigcup_{j=1}^m W_j^{(i)}$ (m is independent of i) such that $W(\tau_i) = \bigcup_{j=1}^m W_j^{(i)}(\tau_i)$ satisfies

$$\operatorname{diam}\big(W_j^{(i)}(\tau_i)\big) < \alpha\big(W(\tau_i)\big) + \varepsilon, \quad j = 1, 2, \ldots, m.$$

Let $B_{ij} = W_j^{(i)}(J_i)$. It is evident that $W(J) = (\bigcup_{i=1}^n \bigcup_{j=1}^m B_{ij}) \cup (\bigcup_{i=0}^n W(t_i))$. For $x, y \in W_j^{(i)}$ and $t, t' \in J_i$, we have

$$\begin{aligned}\big\|x(t) - y\big(t'\big)\big\| &\le \big\|x(t) - x(\tau_i)\big\| + \big\|x(\tau_i) - y(\tau_i)\big\| + \big\|y(\tau_i) - y\big(t'\big)\big\| \\ &< \operatorname{diam}\big(W_j^{(i)}(\tau_i)\big) + 2\varepsilon,\end{aligned}$$

and therefore,

$$\operatorname{diam}(B_{ij}) \le \operatorname{diam}\big(W_j^{(i)}(\tau_i)\big) + 2\varepsilon < \alpha\big(W(\tau_i)\big) + 3\varepsilon \le \sup_{t\in J}\alpha\big(W(t)\big) + 3\varepsilon.$$

Hence,

$$\alpha\left(\bigcup_{i=1}^n\bigcup_{j=1}^m B_{ij}\right) \le \sup_{t\in J}\alpha\big(W(t)\big) + 3\varepsilon.$$

Since ε is arbitrary, we have

$$\alpha\left(\bigcup_{i=1}^n\bigcup_{j=1}^m B_{ij}\right) \le \sup_{t\in J}\alpha\big(W(t)\big).$$

Therefore,

$$\begin{aligned}\alpha\big(W(J)\big) &= \max\left\{\alpha\left(\bigcup_{i=1}^{n}\bigcup_{j=1}^{m}B_{ij}\right),\alpha\big(W(t_0)\big),\alpha\big(W(t_1)\big),\dots,\alpha\big(W(t_n)\big)\right\}\\ &\le \sup_{t\in J}\alpha\big(W(t)\big).\end{aligned}$$

Thus, we have $\alpha(W) = \alpha(W(J)) = \sup\{\alpha(W(t)) : t \in J\}$. □

3.2 Existence for semilinear measure driven equations

In this part, we provide existence results for the abstract measure system (1). We first give the definition of mild solutions for system (1).

Definition 3.3 The function $x \in G(J;X)$ is called a mild solution of system (1) on J if it satisfies the following measure integral equation:

$$x(t) = T(t)p(x) + \int_0^t T(t-s)f\big(s,x(s)\big)\,dg(s), \quad t \in J.$$

Denote by $\mathcal{LS}_g(J;X)$ the space of all functions $f : J \to X$ that are Lebesgue-Stieltjes integrable with respect to g. We introduce the following assumptions.

(H) The C_0-semigroup $T(t)$ generated by A is equicontinuous, that is, $\{T(t)x : x \in B\}$ is equicontinuous at any $t > 0$ for any bounded subset $B \subset X$ (*cf.* [38]). Let $M = \sup_{t\in J}\|T(t)\|$.

(cf1) For every $x \in G(J;X)$, the function $f(\cdot,x(\cdot)) \in \mathcal{LS}_g(J;X)$.

(cf2) The map $x \mapsto f(\cdot,x(\cdot))$ from $G(J;X)$ to $\mathcal{LS}_g(J;X)$ is continuous.

(cf3) There exist a function $m \in \mathcal{LS}_g(J;\mathbb{R}^+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\big\|f(t,x)\big\| \le m(t)\Phi\big(\|x\|\big)$$

for all $x \in X$ and almost all $t \in J$. In addition,

$$\liminf_{l\to+\infty}\frac{\Phi(l)}{l} = \gamma < +\infty.$$

(cf4) There exists a function $L \in \mathcal{LS}_g(J;\mathbb{R}^+)$ such that

$$\alpha\big(f(t,B)\big) \le L(t)\alpha(B)$$

for almost all $t \in J$ and every bounded set $B \subseteq X$.

(cp) $p : G(J;X) \to X$ is continuous and compact, and there exist positive constants c and d such that

$$\big\|p(x)\big\| \le c\|x\|_\infty + d \quad \text{for all } x \in G(J;X).$$

Theorem 3.4 *Assume that hypotheses* (H), (cf1)-(cf4), *and* (cp) *are satisfied and*

$$cM + M\gamma\int_0^a m(s)\,dg(s) < 1. \tag{2}$$

Then system (1) *has a mild solution on* J.

Proof Define the operator $N : G(J;X) \to G(J;X)$ by

$$N(x)(t) = T(t)p(x) + \int_0^t T(t-s)f\big(s,x(s)\big)\,dg(s).$$

As a result of assumptions (H) and (cf1), the integral in the last formula is well defined.

Let $l > 0$ be a constant, and $B_l = \{x \in G(J;X) : \|x\|_\infty \le l\}$. For every positive number l, B_l is clearly a bounded closed convex set in $G(J;X)$. Write $N(B_l) = \{N(x) : x(\cdot) \in B_l\}$.

Step 1. There exists a positive number l such that $\{N(B_l)\} \subseteq B_l$.

If this statement is not true, then for each positive constant l, there exists a function $x_l(\cdot) \in B_l$ such that $N(x_l)(\cdot) \notin B_l$, that is, $\|N(x_l)(t)\| > l$ for some $t(l) \in J$, where $t(l)$ denotes t that is dependent on l. We have

$$\begin{aligned} l &< \big\|N(x_l)(t)\big\| = \left\|T(t)p(x_l) + \int_0^t T(t-s)f\big(s,x_l(s)\big)\,dg(s)\right\| \\ &\le \big\|T(t)p(x_l)\big\| + \int_0^t \big\|T(t-s)f\big(s,x_l(s)\big)\big\|\,dg(s) \\ &\le M(cl+d) + M\Phi(l)\int_0^a m(s)\,dg(s). \end{aligned}$$

Dividing both sides by l and taking the lower limit as $l \to +\infty$, we get

$$cM + M\gamma \int_0^a m(s)\,dg(s) \ge 1.$$

It is a contradiction to (2). Hence, for some positive number l, $N(B_l) \subseteq B_l$.

Step 2. $N(B_l)$ is an equiregulated set of functions.

For $t_0 \in [0,a)$, we have

$$\begin{aligned} &\big\|N(x)(t) - N(x)\big(t_0^+\big)\big\| \\ &\quad\le \big\|\big(T(t) - T\big(t_0^+\big)\big)p(x)\big\| + \int_0^{t_0^+} \big\|\big(T(t-s) - T\big(t_0^+ - s\big)\big)f\big(s,x(s)\big)\big\|\,dg(s) \\ &\qquad + M\int_{t_0^+}^t \big\|f\big(s,x(s)\big)\big\|\,dg(s) \\ &\quad= I_1 + I_2 + I_3. \end{aligned}$$

Hypotheses (cp) and (cf3) show that the sets $\{p(x) : x \in B_l\}$ and $\{f(s,x(s)) : s \in J, x \in B_l\}$ are bounded, respectively. On the other hand, by condition (H) the C_0-semigroup $T(t)$ is equicontinuous on J. Thus, $I_1 \to 0$ and $I_2 \to 0$ as $t \to t_0^+$, independently on particular choices of $x(\cdot)$. Let $h(t) = \int_0^t m(s)\,dg(s)$; by Proposition 2.1, $h(t)$ is a regulated function on J. Hence,

$$I_3 \le M\Phi(l)\int_{t_0^+}^t m(s)\,dg(s) = M\Phi(l)\big(h(t) - h\big(t_0^+\big)\big) \to 0 \quad \text{as } t \to t_0^+,$$

also independently on $x(\cdot)$.

We can use a similar procedure to show that $\|N(x)(t_0^-) - N(x)(t)\| \to 0$ as $t \to t_0^-$ for each $t_0 \in (0,a]$. Therefore, $N(B_l)$ is equiregulated on J in terms of Definition 2.2.

Step 3. N is a continuous operator on B_l.

Let $\{x_n, n \in \mathbb{N}\}$ be a convergent sequence in B_l and $x_n \to x$ as $n \to \infty$. In view of assumptions (cf2) and (cp) and the strong continuity of $T(t)$, we have, for each $t \in J$,

$$\begin{aligned}
&\|N(x_n)(t) - N(x)(t)\| \\
&\quad = \left\| T(t)\bigl(p(x_n) - p(x)\bigr) + \int_0^t T(t-s)\bigl[f\bigl(s, x_n(s)\bigr) - f\bigl(s, x(s)\bigr)\bigr]\,dg(s) \right\| \\
&\quad \le M\|p(x_n) - p(x)\| + M\int_0^a \|f\bigl(s, x_n(s)\bigr) - f\bigl(s, x(s)\bigr)\|\,dg(s) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Step 2 implies that $\{N(x_n)\}_{n=1}^{\infty}$ is equiregulated. This property and the above verification, together with Lemma 2.5, show that $N(x_n)$ converges uniformly to $N(x)$ as $n \to \infty$, namely,

$$\|N(x_n) - N(x)\|_\infty = \sup_{t\in J}\|N(x_n)(t) - N(x)(t)\| \to 0 \quad \text{as } n \to \infty.$$

Therefore, N is a continuous operator.

Step 4. There exist a constant $0 \le \gamma < 1$ and a positive integer n_0 such that for any $B \subseteq F$, we have $\alpha(\widetilde{N}^{n_0}(B)) \le \gamma\alpha(B)$, where $F = \overline{\mathrm{co}}(N(B_l))$ and $\widetilde{N}^n$, $n \ge 1$, are defined as in Lemma 2.12.

First, since $p(B_l)$ is relatively compact in X and $T(\cdot)$ is strongly continuous, applying the Arzelà-Ascoli theorem, we infer that the set $\{T(\cdot)p(x) : x \in B_l\}$ is relatively compact in $C(J; X)$. Therefore, we have $\alpha(\{T(\cdot)p(x) : x \in B_l\}) = 0$.

Since $N(B_l) \subseteq B_l$, we have $F \subseteq \overline{\mathrm{co}}(B_l) = B_l$. Hence, $N(F) \subseteq N(B_l) \subseteq F$. This implies that $N : F \to F$ and $N(F)$ is a bounded set in $G(J; X)$.

For each bounded set $B \subseteq F$, we have $\widetilde{N}^1(B) = N(B) \subseteq N(F) \subseteq N(B_l)$. Suppose that $\widetilde{N}^{n-1}(B) \subseteq N(B_l)$ $(n \ge 2)$. Then

$$\widetilde{N}^n(B) = N\bigl(\overline{\mathrm{co}}\bigl(\widetilde{N}^{n-1}(B)\bigr)\bigr) \subseteq N\bigl(\overline{\mathrm{co}}\bigl(N(B_l)\bigr)\bigr) \subseteq N\bigl(\overline{\mathrm{co}}(B_l)\bigr) = N(B_l).$$

Hence, by mathematical induction as before, we get $\widetilde{N}^n(B) \subseteq N(B_l)$ for every $n \ge 1$. On the other hand, Step 1 and Step 2 show that $N(B_l)$ is bounded and equiregulated on J; therefore, $\widetilde{N}^n(B)$ is bounded and equiregulated on J for every $n \ge 1$. By Theorem 3.2, $\alpha(\widetilde{N}^n(B)) = \sup_{t\in J}\alpha(\widetilde{N}^n(B)(t))$.

Since $N(B_l)$ is bounded and equiregulated on J, by Lemma 2.6, F is bounded and equiregulated on J. By hypotheses (cf3) and (cf4) and Theorem 3.2 together with Corollary 2.10, we have, for each bounded set $B \subseteq F$,

$$\begin{aligned}
\alpha\bigl(N(B)(t)\bigr) &\le \alpha\bigl(T(t)p(B)\bigr) + \alpha\left(\int_0^t T(t-s)f\bigl(s, B(s)\bigr)\,dg(s)\right) \\
&= \alpha\left(\int_0^t T(t-s)f\bigl(s, B(s)\bigr)\,dg(s)\right) \\
&\le 4\int_0^t \alpha\bigl(T(t-s)f\bigl(s, B(s)\bigr)\bigr)\,dg(s) \\
&\le 4M\int_0^t L(s)\alpha\bigl(B(s)\bigr)\,dg(s) \\
&\le 4M\int_0^t L(s)\,dg(s)\alpha(B).
\end{aligned}$$

Then

$$\alpha\big(\widetilde{N}^1(B)(t)\big) = \alpha\big(N(B)(t)\big) \le 4M\int_0^t L(s)\,dg(s)\alpha(B).$$

For $n \ge 2$, $n \in \mathbb{N}$, suppose that

$$\alpha\big(\widetilde{N}^{n-1}(B)(t)\big) \le \frac{(4M)^{n-1}}{(n-1)!}\left(\int_0^t L(s)\,dg(s)\right)^{n-1}\alpha(B).$$

Then

$$\begin{aligned}\alpha\big(\widetilde{N}^n(B)(t)\big) &= \alpha\big(N\big(\overline{\mathrm{co}}\big(\widetilde{N}^{n-1}(B)\big)(t)\big)\big)\\ &\le 4M\int_0^t L(s)\alpha\big(\widetilde{N}^{n-1}(B)(s)\big)\,dg(s)\\ &\le \frac{(4M)^n}{(n-1)!}\int_0^t L(s)\left(\int_0^s L(\tau)\,dg(\tau)\right)^{n-1}dg(s)\alpha(B)\\ &= \frac{(4M)^n}{n!}\left(\int_0^t L(s)\,dg(s)\right)^n\alpha(B).\end{aligned}$$

Therefore, by mathematical induction as before and by Theorem 3.2 we get

$$\alpha\big(\widetilde{N}^n(B)\big) = \sup_{t\in J}\alpha\big(\widetilde{N}^n(B)(t)\big) \le \frac{(4M)^n}{n!}\left(\int_0^a L(s)\,dg(s)\right)^n\alpha(B).$$

Since $\frac{(4M)^n}{n!}(\int_0^a L(s)\,dg(s))^n \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\frac{(4M)^{n_0}}{n_0!} \times (\int_0^a L(s)\,dg(s))^{n_0} = \gamma < 1$, and applying Lemma 2.12, it follows that the operator N has a fixed point in F. This fixed point is a mild solution of the measure driven system (1). □

In the special case of $A = 0$ and $p(x) = x_0$, system (1) degenerates to the nonlinear measure driven system

$$\begin{aligned}&dx = f(t,x)\,dg, \quad t \in J,\\ &x(0) = x_0,\end{aligned} \tag{3}$$

which was investigated by [1] in the Kurzweil integral setting.

The function $x \in G(J;X)$ is a solution of system (3) if it satisfies the measure integral equation

$$x(t) = x_0 + \int_0^t f\big(s,x(s)\big)\,dg(s), \quad t \in J.$$

For the nonlinear measure driven system (3), we have the following existence result. The proof of it is similar to that of Theorem 3.4 (in view of $T(t) = I$ and $p(x) = x_0$ in this case). So we omit it.

Theorem 3.5 *Assume that hypotheses* (cf1)-(cf4) *are fulfilled and*

$$\gamma\int_0^a m(s)\,dg(s) < 1.$$

Then system (3) *has a solution on* J.

Remark 3.6 In the Kurzweil integral setting, paper [1] investigated the nonlinear measure equation (3) in separable Banach spaces. For system (3), Theorem 3.5 can be applied to general Banach spaces in the Lebesgue integral setting. On the other hand, Theorem 3.5 provides existence criteria different from those of Theorem 10 in [1] due to the different results on the measure of noncompactness in the space of regulated functions $G(J;X)$.

For the special case $X = \mathbb{R}^n$, the papers [23, 24] studied nonlinear measure retarded differential equations in the Kurzweil integral setting, where the Lipschitz-type conditions are demanded (see Theorem 5.3 in [23] and Theorem 6.1 in [24], respectively). Theorem 3.5 can be generalized to the retarded version with suitable modifications. Thus, the Lipschitz-type conditions in the literature cited are unnecessary here. On the other hand, it is well known that the compactness condition (cf4) on f in Theorem 3.5 is much weaker than the Lipschitz condition on f in Theorem 5.3 in [23]. From this point of view, Theorem 3.5 is less restrictive than that in [23].

Remark 3.7 Let $PC(J;X)$ denote the space of piecewise continuous functions on J (see [39]). Since $C(J;X) \subset PC(J;X) \subset G(J;X)$, in view of the relation of differential equations with impulses and measure driven equations [24], our results generalize those in [12, 17], where existence criteria were provided for abstract semilinear differential equations with or without impulsive effects, and correspondingly, the solutions belong to $C(J;X)$ or $PC(J;X)$.

4 An example

As an application of Theorem 3.4, we consider the following partial differential system driven by a measure:

$$\begin{cases} d_t x(t,\omega) = \frac{\partial}{\partial\omega} x(t,\omega) + f(t,x(t,\omega))\,dg(t) & \text{for } t \in [0,1], \omega \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \quad t \in [0,1], \\ x(0,\omega) = \int_0^1 h(s)\log(1 + |x(s,\omega)|)\,ds, \quad \omega \in [0,\pi]. \end{cases} \tag{4}$$

Let $X = L^2([0,\pi])$. Define $A : X \to X$ by $Az = z'$ with the domain $D(A) = \{z \in X : z \text{ is absolutely continuous}, z' \in X, z(0) = z(\pi) = 0\}$. It is well known that A is an infinitesimal generator of the C_0-semigroup $T(t)$ defined by $T(t)z(s) = z(t+s)$ for each $z \in X$. The semigroup $T(t)$ is not a compact semigroup on X but equicontinuous. In addition, $M = \sup_{0\le t\le 1}\|T(t)\| \le 1$ and $\alpha(T(t)D) \le \alpha(D)$, where α is the Kuratowski measure of noncompactness (see [40, 41]).

Take

$$g(t) = \begin{cases} 1 - \frac{1}{2}, & 0 \le t \le 1 - \frac{1}{2}, \\ \dots, \\ 1 - \frac{1}{n}, & 1 - \frac{1}{n-1} < t \le 1 - \frac{1}{n} \text{ for } n > 2 \text{ and } n \in \mathbb{N}, \\ \dots, \\ 1, & t = 1. \end{cases}$$

It is evident that $g : [0,1] \to \mathbb{R}$ is a left-continuous and nondecreasing function on $[0,1]$.

(H1) Assume that $f : [0,1] \times X \to X$ is a continuous function defined by

$$f(t,x)(\omega) = F\bigl(t, x(\omega)\bigr), \quad t \in [0,1], \omega \in [0,\pi].$$

Take $F(t,x(\omega)) = c_0 \sin(x(\omega))$, where c_0 is a real constant. We claim that conditions (cf1)-(cf4) in Theorem 3.4 can be satisfied:

(cf1) clearly holds;

(cf2) is satisfied because

$$\begin{aligned}
&\|f(t,x_n) - f(t,x)\| \\
&\quad = \int_0^1 \left(\int_0^\pi c_0^2 \bigl|\sin\bigl(x_n(\omega)\bigr) - \sin\bigl(x(\omega)\bigr)\bigr|^2 \, d\omega \right)^{1/2} dg(t) \\
&\quad \le c_0\bigl(g(1) - g(0)\bigr) \left(\int_0^\pi \bigl|x_n(\omega) - x(\omega)\bigr|^2 \, d\omega \right)^{1/2} \\
&\quad = \frac{1}{2} c_0 \|x_n - x\|;
\end{aligned}$$

(cf3) is checked with $m(t) = c_0$, $\Phi(l) = l$, and hence, $\gamma = 1$ since

$$\begin{aligned}
\|f(t,x)\| &= \left(\int_0^\pi c_0^2 \sin^2\bigl(x(\omega)\bigr) \, d\omega \right)^{1/2} \\
&\le \left(\int_0^\pi c_0^2 x^2(\omega) \, d\omega \right)^{1/2} = c_0 \|x\|;
\end{aligned}$$

(cf4) is checked with $L(t) = c_0$ since $\alpha(f(t,B)) \le c_0 \alpha(B)$ for any bounded subset B of X.

(H2) Define $p : G([0,1];X) \to X$ by

$$p(\phi)(\omega) = \int_0^1 h(s) \log\bigl(1 + \bigl|\phi(s)(\omega)\bigr|\bigr) \, ds, \quad \phi \in G\bigl([0,1];X\bigr),$$

with $\phi(s)(\omega) = x(s,\omega)$. Suppose $h \in L^2([0,1];\mathbb{R})$ and

$$\left(\int_0^1 \bigl|h(s)\bigr|^2 \, ds \right)^{\frac{1}{2}} + \frac{1}{2} c_0 < 1.$$

Let $\{\phi_n, n \in \mathbb{N}\}$ be a convergent sequence in $G([0,1];X)$, and $\phi_n \to \phi$ as $n \to \infty$, that is,

$$\begin{aligned}
\|\phi_n - \phi\|^2 &= \sup_{s \in [0,1]} \bigl\|\phi_n(s) - \phi(s)\bigr\|_X^2 \\
&= \sup_{s \in [0,1]} \int_0^\pi \bigl|\phi_n(s)(\omega) - \phi(s)(\omega)\bigr|^2 \, d\omega \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Then

$$\begin{aligned}
&\bigl\|p(\phi_n) - p(\phi)\bigr\|_X^2 \\
&\quad = \int_0^\pi \bigl|p(\phi_n)(\omega) - p(\phi)(\omega)\bigr|^2 \, d\omega \\
&\quad = \int_0^\pi \left| \int_0^1 h(s)\bigl[\log\bigl(1 + \bigl|\phi_n(s)(\omega)\bigr|\bigr) - \log\bigl(1 + \bigl|\phi(s)(\omega)\bigr|\bigr)\bigr] \, ds \right|^2 d\omega \\
&\quad \le \int_0^\pi \left(\int_0^1 \bigl|h(s)\bigr| \cdot \bigl|\log\bigl(1 + \bigl|\phi_n(s)(\omega)\bigr|\bigr) - \log\bigl(1 + \bigl|\phi(s)(\omega)\bigr|\bigr)\bigr| \, ds \right)^2 d\omega
\end{aligned}$$

$$\leq \int_0^1 |h(s)|^2\,ds \cdot \int_0^1 \int_0^\pi \left|\log\bigl(1+|\phi_n(s)(\omega)|\bigr) - \log\bigl(1+|\phi(s)(\omega)|\bigr)\right|^2 d\omega\,ds$$
$$\leq \int_0^1 |h(s)|^2\,ds \cdot \int_0^1 \int_0^\pi \left|\phi_n(s)(\omega) - \phi(s)(\omega)\right|^2 d\omega\,ds \to 0 \quad \text{as } n \to \infty.$$

Therefore, p is a continuous operator. Analogously, we can obtain that, for any $\phi \in G([0,1];X)$,

$$\bigl\|p(\phi)(\omega+\eta) - p(\phi)(\omega)\bigr\|_X \to 0 \quad \text{as } \eta \to 0$$

and

$$\bigl\|p(\phi)\bigr\|_X \leq \left(\int_0^1 |h(s)|^2\,ds\right)^{\frac{1}{2}} \|\phi\|_\infty.$$

According to versions of the Árzela-Ascoli theorem for L^2 space (see [42]), p is a compact operator and satisfies hypothesis (cp) in Theorem 3.4.

Moreover, we have

$$\begin{aligned} &cM + M\gamma \int_0^a m(s)\,dg(s) \\ &\quad = \left(\int_0^1 |h(s)|^2\,ds\right)^{\frac{1}{2}} + c_0\bigl(g(1) - g(0)\bigr) \\ &\quad = \left(\int_0^1 |h(s)|^2\,ds\right)^{\frac{1}{2}} + \frac{1}{2}c_0 < 1, \end{aligned}$$

which tests inequality (2).

Hence, under these assumptions, the partial differential system (4) can be reformulated as the abstract measure system (1) and there exists at least one mild solution for system (4) by Theorem 3.4.

5 Conclusions

In this paper, the issue on abstract semilinear measure driven equations in Banach spaces with nonlocal conditions has been addressed for the first time, which can model a large class of hybrid systems with Zeno behavior. We first establish some useful results on the Kuratowski measure of noncompactness in the space of regulated functions. Then the existence criteria of mild solutions for the discussed measure system are obtained by using the tools of measure of noncompactness and a corresponding fixed point theorem. The results obtained in this paper are also applicable to abstract semilinear dynamic equations on time scales. As shown in [23, 24], this type of equations can be transformed to abstract measure driven equations. Moreover, the issue on the existence for abstract semilinear measure driven equations is relatively new, and we can further develop its investigation inspired, for example, by [43].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

YC completed the proof and wrote the initial draft. JS provided the problem and gave some suggestions of amendment. YC then finalized the manuscript. Correspondence was mainly done by JS. All authors read and approved the final manuscript.

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