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Periodic solutions for a class of second order delay differential systems

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Abstract

Consider the periodic boundary value problem

$$\begin{cases} \ddot{u}(t) + ru(t - \tau) + \nabla F(t, u(t - \tau)) = 0, & \text{a.e. } t \in [0, 2\tau], \\ u(0) - u(2\tau) = \dot{u}(0) - \dot{u}(2\tau) = 0, \end{cases}$$

where $\tau > 0$ is a given constant, $r \in \mathbf{R}$ is a parameter. Periodic solutions are obtained by using linking theorems.

MSC: 34K13; 34K18; 58E50

Keywords: periodic boundary value problem; delay differential systems; linking theorems

1 Introduction

Consider the periodic boundary value problem

$$\begin{cases} \ddot{u}(t) + ru(t - \tau) + \nabla F(t, u(t - \tau)) = 0, & \text{a.e. } t \in [0, 2\tau], \\ u(0) - u(2\tau) = \dot{u}(0) - \dot{u}(2\tau) = 0, \end{cases} \quad (1.1)$$

where $\tau > 0$ is a given constant, $r \in \mathbf{R}$ is a parameter. $F : [0, 2\tau] \times \mathbf{R}^N \rightarrow \mathbf{R}$ satisfies the following assumptions:

(A) $F(t, x)$ is measurable with respect to t , for all $x \in \mathbf{R}^N$, continuously differentiable in x , for a.e. $t \in [0, 2\tau]$, and there exist $a \in C(\mathbf{R}^+; \mathbf{R}^+)$ and $b \in L^1([0, 2\tau]; \mathbf{R}^+)$ such that

$$|F(t, x)|, |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbf{R}^N$ and a.e. $t \in [0, 2\tau]$.

Variational methods are very powerful techniques in nonlinear analysis and are extensively used in many disciplines of pure and applied mathematics, including ordinary and partial differential equations, mathematical physics, and geometrical analysis. The existence and multiplicity of solutions for Hamilton systems and Schrödinger equations have been studied extensively via critical point theory; see [1–8].

In the past several years, some results on the existence of periodic solutions for the functional differential equation by the critical point theory have been obtained (see [9–18]). In

[16], the authors obtained the multiplicity results for periodic solutions to (1.1) by using critical point theory. Our proof method in this paper is different from the literature [16].

Motivated by the above observation, in this paper, we study the existence of periodic solutions to the system (1.1). The following theorems are the main results of our paper.

(H1) There exists a constant $R_1 > 0$ such that

(i) $F(t, x) \geq 0, \forall |x| \leq R_1, t \in [0, 2\tau]$; or

(ii) $F(t, x) \leq 0, \forall |x| \leq R_1, t \in [0, 2\tau]$.

(H2) $\lim_{|x| \rightarrow \infty} \inf \frac{F(t,x)}{|x|^2} > 0$ uniformly for $t \in [0, 2\tau]$.

(H3) $\lim_{|x| \rightarrow 0} \frac{|\nabla F(t,x)|}{|x|} = 0$ uniformly for $t \in [0, 2\tau]$.

(H4) There exist $\alpha > 1$ and $a > 0$ such that

$$|\nabla F(t, x)| \leq a(|x|^\alpha + 1), \quad \forall (t, x) \in [0, 2\tau] \times \mathbf{R}^N.$$

(H5) There exist $\mu > 2, 0 < \beta < 2, R > 0$, and a function $b(t) \in L^1([0, 2\tau]; \mathbf{R}^+)$ such that

$$\mu F(t, x) \leq (\nabla F(t, x), x) + b(t)|x|^\beta, \quad \forall |x| \geq R, t \in [0, 2\tau].$$

(H5') There exist $\beta > \alpha \geq 1$ and $b > 0, R > 0$ such that

$$(\nabla F(t, x), x) - 2F(t, x) \geq b|x|^\beta, \quad \forall |x| \geq R, t \in [0, 2\tau].$$

Theorem 1.1 *Assume that (H1)-(H5) hold, if 0 is an eigenvalue of L (L is defined in Section 2), BVP (1.1) has at least one nontrivial solution.*

Theorem 1.2 *Assume that (H1)-(H4) and (H5') hold, if 0 is an eigenvalue of L (L is defined in Section 2), BVP (1.1) has at least one nontrivial solution.*

Remark 1.1 The condition (H5) and the condition (H5') are appeared, respectively, in [7] and [8].

2 Preliminaries

In order to seek 2τ -periodic orbits of (1.1), let us transform (1.1) to

$$\begin{cases} \ddot{u}(t) + r\lambda^2 u(t - \pi) + \lambda^2 \nabla F(\lambda t, u(t - \pi)) = 0, & \text{a.e. } t \in [0, 2\pi], \\ u(0) - u(2\pi) = \dot{u}(0) - \dot{u}(2\pi) = 0, \end{cases} \tag{2.1}$$

by making the change of variable $t \rightarrow \frac{\pi}{\tau}t = \lambda^{-1}t$, which implies that a 2π -periodic solution of (2.1) corresponds to a 2τ -periodic solution of (1.1). Hence we will only look for the 2π -periodic solutions of (2.1) in the sequel.

Let $L^2(S^1, \mathbf{R}^N)$ be the space of square integrable 2π -periodic vector-valued functions with dimension \mathbf{N} , and $C^\infty(S^1, \mathbf{R}^N)$ be the space of 2π -periodic C^∞ vector-valued functions with dimension \mathbf{N} . For any $u \in C^\infty(S^1, \mathbf{R}^N)$, it has the following Fourier expansion in the sense that it is convergent in the space $L^2(S^1, \mathbf{R}^N)$:

$$u(t) = \frac{a_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{+\infty} (a_j \cos jt + b_j \sin jt),$$

where $a_0, a_j, b_j \in \mathbf{R}^N$. Let $H^1(S^1, \mathbf{R}^N)$ be the closure of $C^\infty(S^1, \mathbf{R}^N)$ with respect to the Hilbert norm

$$\|u\| = \left[|a_0|^2 + \sum_{j=1}^{+\infty} (1 + j^2)(|a_j|^2 + |b_j|^2) \right]^{1/2}.$$

More specifically, $H^1(S^1, \mathbf{R}^N) = \{u \in L^2(S^1, \mathbf{R}^N) : \|u\| < +\infty\}$ with the inner product

$$\langle u, v \rangle = \int_0^{2\pi} [(u(t), v(t)) + (\dot{u}(t), \dot{v}(t))] dt$$

for any $u, v \in H^1(S^1, \mathbf{R}^N)$, where (\cdot, \cdot) denotes the usual inner product in \mathbf{R}^N . The norm on $H^1(S^1, \mathbf{R}^N)$ is defined by

$$\|u\| = \left[\int_0^{2\pi} (|u(t)|^2 + |\dot{u}(t)|^2) dt \right]^{1/2}. \tag{2.2}$$

It is well known that $H^1(S^1, \mathbf{R}^N)$ is compactly embedded in $C(S^1, \mathbf{R}^N)$ (see Proposition 1.2 of [15]). Define two operators L_0 and L_1 from $H^1(S^1, \mathbf{R}^N)$ into $H^*(S^1, \mathbf{R}^N)$ as follows: for any $u \in H^1(S^1, \mathbf{R}^N)$, which are given by

$$\begin{aligned} (L_0 u)(v) &= \int_0^{2\pi} (\dot{u}(t + \pi), \dot{v}(t)) dt, \\ (L_1 u)(v) &= -r\lambda^2 \int_0^{2\pi} (u(t), v(t)) dt, \end{aligned}$$

for all $v \in H^*(S^1, \mathbf{R}^N)$, where $\dot{u}(t)$ and $H^*(S^1, \mathbf{R}^N)$ denote the weak derivative of u and the dual space of $H^1(S^1, \mathbf{R}^N)$, respectively. By the Riesz representation theorem, we can identify $H^*(S^1, \mathbf{R}^N)$ with $H^1(S^1, \mathbf{R}^N)$. Thus, $L_i u$ can also be viewed as an element belonging to $H^1(S^1, \mathbf{R}^N)$ such that $\langle L_i u, v \rangle = (L_i u)(v)$ for all $u, v \in H^1(S^1, \mathbf{R}^N)$, where $i = 0$ or 1 . It is easy to check that L_0 and L_1 are bounded linear operators on $H^1(S^1, \mathbf{R}^N)$. Note that $H^1(S^1, \mathbf{R}^N)$ is compactly embedded in $C(S^1, \mathbf{R}^N)$, thus L_1 is compact on $H^1(S^1, \mathbf{R}^N)$.

Lemma 2.1 [16] *The operator $L := L_0 + L_1$, that is,*

$$\langle Lu, v \rangle = (Lu)(v) = \int_0^{2\pi} (\dot{u}(t + \pi), \dot{v}(t)) dt - r\lambda^2 \int_0^{2\pi} (u(t), v(t)) dt$$

is self-adjoint on $H^1(S^1, \mathbf{R}^N)$. The operators L_0 and L_1 are also self-adjoint on $H^1(S^1, \mathbf{R}^N)$.

Lemma 2.2 [16] *The essential spectrum of L_0 on $H^1(S^1, \mathbf{R}^N)$ is just $\{-1, 1\}$.*

In view of the above facts, 0 is not in the essential spectrum of L as it is a compact perturbation of the self-adjoint operator L_0 . This implies that 0 is at most an eigenvalue of finite multiplicity of L .

Remark 2.1 $H^1(S^1, \mathbf{R}^N)$ has an orthogonal decomposition

$$H^1(S^1, \mathbf{R}^N) = H^+ \oplus H^- \oplus H^0,$$

where $H^0 = \text{Ker } L$ is finite dimensional, and H^+, H^- are L -invariant subspaces such that for some $\sigma > 0$,

$$\langle Lu, u \rangle \geq \sigma \|u\|^2 \tag{2.3}$$

for all $u \in H^+$,

$$\langle Lu, u \rangle \leq -\sigma \|u\|^2 \tag{2.4}$$

for all $u \in H^-$.

Remark 2.2 By [16], we can choose the proper r to make that the set $\{j \geq 0 \mid \frac{(-1)^j j^2 - r\lambda^2}{j^2 + 1} = 0\} \neq \emptyset$, which means $H^0 \neq \emptyset$, where $j \in \mathbf{Z}$.

Lemma 2.3 [5] *There exists $C > 0$ such that*

$$\|u\|_{L^p} \leq C \|u\|, \quad \|u\|_\infty \leq C \|u\|, \tag{2.5}$$

for $p = 1 + \alpha, 2, \|u\|_\infty = \max_{t \in [0, 2\pi]} |u(t)|, \forall u \in H^1(S^1, \mathbf{R}^N)$.

Now consider the functional φ defined on $H^1(S^1, \mathbf{R}^N)$ given by

$$\varphi(u) = \frac{1}{2} \langle Lu, u \rangle - \lambda^2 \int_0^{2\pi} F(\lambda t, u(t)) dt. \tag{2.6}$$

Since F satisfies the assumption (A), a standard argument shows the following.

Lemma 2.4 [16] *The functional φ is continuously differentiable on $H^1(S^1, \mathbf{R}^N)$ and φ' is defined by*

$$\langle \varphi'(u), v \rangle = \langle Lu, v \rangle - \lambda^2 \int_0^{2\pi} (\nabla F(\lambda t, u(t)), v(t)) dt \tag{2.7}$$

for all $v \in H^1(S^1, \mathbf{R}^N)$.

Lemma 2.5 [16] *A critical point $u \in H^1(S^1, \mathbf{R}^N)$ of functional φ is equivalent to a 2π -periodic solution of system (2.1).*

In this paper, we will use the following local linking Theorem A to prove our theorems. The following concepts appeared in [4, 7, 8].

Let X be a real Banach space with direct decomposition $X = X^1 \oplus X^2$. Consider two sequences of subspaces:

$$X_0^j \subset X_1^j \subset \dots \subset X^j$$

such that $X^j = \overline{\bigcup_{n \in \mathbf{N}} X_n^j}, j = 1, 2$. For every multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$, we denote by X_α the space $X_{\alpha_1} \oplus X_{\alpha_2}$. We say $\alpha \leq \beta$ if $\alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2$.

Definition 2.1 [4] A sequence $\{\alpha_n\} \subset \mathbb{N}^2$ is said to be admissible if, for every $\alpha \in \mathbb{N}^2$ there is $m \in \mathbb{N}$ such that $n \geq m \Rightarrow \alpha_n > \alpha$.

Definition 2.2 [4] Let $f \in C^1(X, \mathbb{R})$. Then the function f satisfies the $(PS)^*$ condition if every sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup_n f(u_{\alpha_n}) < \infty, \quad f'_{\alpha_n}(u_{\alpha_n}) \rightarrow 0$$

contains a subsequence which converges to a critical point of f , where $f_\alpha = f|_{X_\alpha}$.

Definition 2.3 [4] Let X be a Banach space with a direct sum decomposition $X = X^1 \oplus X^2$. The function $f \in C^1(X, \mathbb{R})$ has a local linking at 0, with respect to (X^1, X^2) , if, for some $r > 0$,

$$\begin{aligned} f(x) &\geq 0, \quad \forall x \in X^1, \|x\| \leq r, \\ f(x) &\leq 0, \quad \forall x \in X^2, \|x\| \leq r. \end{aligned}$$

Theorem A [4] Suppose that $f \in C^1(X, \mathbb{R})$ satisfies the following assumptions:

- (f1) f has a local linking at 0 and $X^1 \neq \{0\}$;
- (f2) f satisfies $(PS)^*$ condition;
- (f3) f maps bounded sets into bounded sets;
- (f4) for every $m \in \mathbb{N}$, $f(x) \rightarrow -\infty$ as $\|x\| \rightarrow \infty$ on $X_m^1 \oplus X^2$.

Then f has at least one nonzero critical point.

3 Proofs of theorems

In this section, c_i stand for different positive constants for $i \in \mathbb{Z}^+$, \mathbb{Z}^+ is the set of all positive integers.

Let

$$X = X^1 \oplus X^2, \tag{3.1}$$

where $X^1 = H^+$, $X^2 = H^- \oplus H^0$.

Lemma 3.1 Under assumption (H2), (H4)-(H5), the functional φ satisfies the $(PS)^*$ condition.

Proof Let $\{u_{\alpha_n}\}$ be a sequence such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup_n \varphi(u_{\alpha_n}) < \infty, \quad \varphi'_{\alpha_n}(u_{\alpha_n}) \rightarrow 0.$$

For the sake of notational simplicity, set $u_n = u_{\alpha_n}$.

Claim 1. $\{u_n\}$ is bounded in X .

If not, passing to a subsequence if necessary, we assume that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, then $\{v_n\}$ is bounded in X . Hence, there exists a subsequence, still denoted by $\{v_n\}$. Write $v_n = v_n^+ + v_n^- + v_n^0$ and $v = v^+ + v^- + v^0$, then

$$v_n \rightharpoonup v, \quad v_n^0 \rightarrow v^0 \quad \text{in } X, \quad v_n \rightarrow v \quad \text{in } C(S^1, \mathbb{R}^N).$$

In view of (H4), we have

$$|F(t, x)| \leq c_1(|x|^{1+\alpha} + |x|), \quad \forall x \in \mathbf{R}^N \text{ and } t \in [0, 2\tau]. \tag{3.2}$$

For $|x| \leq R$ and $t \in [0, 2\tau]$, one has

$$|F(t, x)| \leq c_1(|x|^{1+\alpha} + |x|) \leq c_1(R^{1+\alpha} + |R|) := A,$$

together with (H5), one has

$$\mu F(t, x) \leq (\nabla F(t, x), x) + b(t)|x|^\beta + \mu A, \quad \forall x \in \mathbf{R}^N \text{ and } t \in [0, 2\tau]. \tag{3.3}$$

Hence, by (2.6) and (2.7), we have

$$\begin{aligned} \left(\frac{\mu}{2} - 1\right) \langle Lu_n, u_n \rangle &= \mu \varphi(u_n) - \langle \varphi'(u_n), u_n \rangle + \lambda^2 \int_0^{2\pi} [\mu F(\lambda t, u_n) - (\nabla F(\lambda t, u_n), u_n)] dt \\ &\leq c_2 + c_2 \|u_n\| + \lambda^2 \int_0^{2\pi} [b(t)|u_n|^\beta + \mu A] dt \\ &\leq c_2 + c_2 \|u_n\| + c_3 \|u_n\|^\beta + c_3. \end{aligned}$$

This implies that

$$\langle Lu_n, u_n \rangle \leq c_4 + c_4 \|u_n\| + c_4 \|u_n\|^\beta \tag{3.4}$$

since $\mu > 2$. Together with (3.4), we have

$$\langle Lv_n, v_n \rangle = \frac{\langle Lu_n, u_n \rangle}{\|u_n\|^2} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

since $0 < \beta < 2$. Hence,

$$\langle Lv, v \rangle = 0.$$

That means that

$$\langle Lv^+, v^+ \rangle + \langle Lv^-, v^- \rangle = 0.$$

By using (2.3) and (2.4), we discuss two cases:

If $\|v^+\| = 0$, we have $\|v^-\| = 0$, then we obtain $\|v\| = \|v^0\| = \|v_n\| = 1$.

If $\|v^+\| \neq 0$, we have $\|v^-\| \neq 0$, then we obtain $\|v\| \neq 0$.

So we get $v(t) \neq 0$ for $t \in [0, 2\pi]$. Set

$$E_0 = \{t \in [0, 2\pi] : v(t) \neq 0\},$$

then $\text{meas}(E_0) > 0$.

From (H2), we get

$$\begin{aligned} & \lim_{|u_n| \rightarrow \infty} \inf \frac{\int_0^{2\pi} F(\lambda t, u_n) dt}{\|u_n\|^2} \\ & \geq \frac{\int_0^{2\pi} [\lim_{|u_n| \rightarrow \infty} \inf F(\lambda t, u_n)] dt}{\|u_n\|^2} \\ & \geq \int_{E_0} \left[\lim_{|u_n| \rightarrow \infty} \inf \frac{F(\lambda t, u_n)}{|u_n|^2} |v_n|^2 \right] dt \geq \int_{E_0} \left[\lim_{|u_n| \rightarrow \infty} \inf \frac{F(\lambda t, u_n)}{|u_n|^2} |v|^2 \right] dt > 0. \end{aligned} \tag{3.5}$$

By the boundedness of $\varphi(u_n)$ and (3.4), we have

$$\frac{\varphi(u_n)}{\|u_n\|^2} = \frac{\langle Lu_n, u_n \rangle}{2\|u_n\|^2} - \frac{\lambda^2 \int_0^{2\pi} F(\lambda t, u_n) dt}{\|u_n\|^2} \leq \frac{c_4 + c_4\|u_n\| + c_4\|u_n\|^\beta}{2\|u_n\|^2} - \frac{\lambda^2 \int_0^{2\pi} F(\lambda t, u_n) dt}{\|u_n\|^2},$$

which together with $0 < \beta < 2$ implies that

$$\lim_{|u_n| \rightarrow \infty} \inf \frac{\int_0^{2\pi} F(\lambda t, u_n) dt}{\|u_n\|^2} = 0.$$

This contradicts (3.5). Therefore, $\{u_n\}$ is bounded in X .

Claim 2. $\{u_n\}$ possesses a strong convergent subsequence in X .

Write $u_n = u_n^+ + u_n^- + u_n^0$ and $u = u^+ + u^- + u^0$, then

$$u_n^\pm \rightharpoonup u^\pm, \quad u_n^0 \rightarrow u^0 \quad \text{in } X, \quad u_n^\pm \rightarrow u^\pm \quad \text{in } C(S^1, \mathbf{R}^N).$$

In view of (3.2) and $u_n^- \rightarrow u^-$ in $C(S^1, \mathbf{R}^N)$, it is easy to verify

$$\int_0^{2\pi} (\nabla F(\lambda t, u_n) - \nabla F(\lambda t, u), u_n^- - u^-) dt \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Note that

$$\langle \varphi'(u_n) - \varphi'(u), u_n^- - u^- \rangle \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Thus,

$$\begin{aligned} & -\langle \varphi'(u_n) - \varphi'(u), u_n^- - u^- \rangle \\ & = -\langle L(u_n^- - u^-), u_n^- - u^- \rangle + \lambda^2 \int_0^{2\pi} (\nabla F(\lambda t, u_n) - \nabla F(\lambda t, u), u_n^- - u^-) dt \\ & \geq \sigma \|u_n^- - u^-\|^2 + \lambda^2 \int_0^{2\pi} (\nabla F(\lambda t, u_n) - \nabla F(\lambda t, u), u_n^- - u^-) dt. \end{aligned}$$

This yields $u_n^- \rightarrow u^-$ in X . Similarly, $u_n^+ \rightarrow u^+$ in X . Hence $u_n \rightarrow u$ in X . Hence, $\{u_n\}$ possesses a strong convergent subsequence in X . The proof of Lemma 3.1 is complete. \square

Lemma 3.2 *Under assumption (H2), (H4), and (H5'), the functional φ satisfies the (PS)* condition.*

Proof Let $\{u_{\alpha_n}\}$ be a sequence such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup_n \varphi(u_{\alpha_n}) < \infty, \quad \varphi'_{\alpha_n}(u_{\alpha_n}) \rightarrow 0.$$

For the sake of notational simplicity, set $u_n = u_{\alpha_n}$.

Claim 1. $\{u_n\}$ is bounded in X .

If not, passing to a subsequence if necessary, we assume that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. In view of (H5'), there exists $c_5 > 0$ such that

$$(\nabla F(t, x), x) - 2F(t, x) \geq b|x|^\beta - c_5, \quad \forall (t, x) \in [0, 2\tau] \times \mathbf{R}^N. \tag{3.6}$$

Hence, we have

$$\begin{aligned} 2\varphi(u_n) - \langle \varphi'(u_n), u_n \rangle &= \lambda^2 \int_0^{2\pi} [(\nabla F(\lambda t, u_n), u_n) - 2F(\lambda t, u_n)] dt \\ &\geq \lambda^2 \int_0^{2\pi} [b|u_n|^\beta - c_5] dt \\ &= b\lambda^2 \int_0^{2\pi} |u_n|^\beta dt - 2\pi c_5 \lambda^2. \end{aligned}$$

This implies that

$$\frac{\int_0^{2\pi} |u_n|^\beta dt}{\|u_n\|} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.7}$$

Let $u_n = u_n^+ + u_n^- + u_n^0 \in H^+ \oplus H^- \oplus H^0$. By (H4), we have

$$\begin{aligned} \langle \varphi'(u_n), u_n^+ \rangle &= \langle Lu_n^+, u_n^+ \rangle - \lambda^2 \int_0^{2\pi} (\nabla F(\lambda t, u_n), u_n^+) dt \\ &\geq \sigma \|u_n^+\|^2 - a\lambda^2 \int_0^{2\pi} (|u_n|^\alpha |u_n^+| + |u_n^+|) dt \\ &\geq \sigma \|u_n^+\|^2 - a\lambda^2 (2\pi)^{\frac{\beta-\alpha}{\beta}} \left(\int_0^{2\pi} |u_n|^{\alpha \times \frac{\beta}{\alpha}} dt \right)^{\frac{\alpha}{\beta}} \|u_n^+\|_\infty - 2\pi a\lambda^2 \|u_n^+\|_\infty \\ &\geq \sigma \|u_n^+\|^2 - a\lambda^2 (2\pi)^{\frac{\beta-\alpha}{\beta}} C \|u_n\|_{L^\beta}^\alpha \|u_n^+\| - 2\pi a\lambda^2 C \|u_n^+\| \end{aligned}$$

for all n , where we use the Hölder inequality and (2.5). We have

$$\frac{\|u_n^+\|}{\|u_n\|} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.8}$$

Similarly for u_n^- , we also get

$$\frac{\|u_n^-\|}{\|u_n\|} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.9}$$

Again, by (H5'), we have

$$(\nabla F(t, x), x) - 2F(t, x) \geq b|x| - c_6, \quad \forall |x| \geq R > 1, t \in [0, 2\tau],$$

since $\beta > \alpha \geq 1$. Since $\dim H^0 < \infty$,

$$\begin{aligned} 2\varphi(u_n) - \langle \varphi'(u_n), u_n \rangle &= \lambda^2 \int_0^{2\pi} [(\nabla F(\lambda t, u_n), u_n) - 2F(\lambda t, u_n)] dt \\ &\geq \lambda^2 \int_0^{2\pi} [b|u_n| - c_6] dt \\ &\geq \lambda^2 \int_0^{2\pi} (b|u_n^0| - b|u_n^+| - b|u_n^-| - c_6) dt \\ &\geq c_7 \|u_n^0\| - c_7 \|u_n^+\| - c_7 \|u_n^-\| - c_7. \end{aligned}$$

So we get

$$\frac{\|u_n^0\|}{\|u_n\|} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.10}$$

Hence by (3.8)-(3.10), one has

$$1 = \frac{\|u_n\|}{\|u_n\|} \leq \frac{\|u_n^0\| + \|u_n^+\| + \|u_n^-\|}{\|u_n\|} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

a contradiction. Therefore, $\{u_n\}$ is bounded in X .

Using similar arguments to the proof of Claim 2 in Lemma 3.1, $\{u_n\}$ possesses a strong convergent subsequence in X . The proof of Lemma 3.2 is complete. \square

Lemma 3.3 *Under assumption (H1)-(H4), the function φ satisfies the conditions (f1), (f3), and (f4) of Theorem A.*

Proof We only consider the case (i) in (H1). The other case is similar.

(1) We claim that φ has a local linking at 0 with respect to (X^1, X^2) .

By (H3), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|F(t, x)| \leq \varepsilon|x|^2, \quad \forall t \in [0, 2\tau], |x| \leq \delta. \tag{3.11}$$

By (3.2) and (3.11), there exists $c_8 > 0$ such that

$$|F(t, x)| \leq \varepsilon|x|^2 + c_8|x|^{1+\alpha}, \quad \forall t \in [0, 2\tau], x \in \mathbf{R}^N. \tag{3.12}$$

Hence, for $u \in X^1$, we have

$$\varphi(u) \geq \frac{1}{2} \langle Lu, u \rangle - \lambda^2 \int_0^{2\pi} (\varepsilon|u|^2 + c_8|u|^{1+\alpha}) dt \geq \frac{\sigma}{2} \|u\|^2 - \varepsilon\lambda^2 C \|u\|^2 - c_8\lambda^2 C \|u\|^{1+\alpha}.$$

Taking $\varepsilon = \sigma/(4\lambda^2 C)$ and noting that $\alpha > 1$ in (H4), we can find a constant $\rho_1 > 0$ such that

$$\varphi(u) \geq 0, \quad \forall u \in X^1 \text{ with } \|u\| \leq \rho_1.$$

On the other hand, let $u = u^0 + u^- \in X^2$ satisfying $\|u\| \leq \rho_0 = R_1/(2C)$ and let

$$\Omega_1 = \{t \in [0, 2\pi] \mid |u^-| \leq R_1/2\}, \quad \Omega_2 = \{t \in [0, 2\pi] \mid |u^-| > R_1/2\}.$$

Then we have

$$|u^0| \leq \|u^0\|_\infty \leq C\|u^0\| \leq R_1/2$$

for all $t \in [0, 2\pi]$. Consequently,

$$|u| \leq |u^0| + |u^-| \leq R_1, \quad \text{for all } t \in \Omega_1.$$

Hence, from (H1)(i), we obtain

$$\int_{\Omega_1} F(t, u) dt \geq 0.$$

On the one hand, one obtains

$$|u| \leq |u^0| + |u^-| \leq R_1/2 + |u^-| \leq 2|u^-|, \quad \text{for all } t \in \Omega_2.$$

It follows from (3.12) that

$$|F(\lambda t, u)| \leq \varepsilon|u|^2 + c_8|u|^{1+\alpha} \leq 4\varepsilon|u^-|^2 + 2^{1+\alpha}c_8|u^-|^{1+\alpha},$$

for all $t \in \Omega_2$, which implies that

$$\begin{aligned} \left| \int_{\Omega_2} F(\lambda t, u) dt \right| &\leq 4\varepsilon \int_{\Omega_2} |u^-|^2 dx + 2^{1+\alpha}c_8 \int_{\Omega_2} |u^-|^{1+\alpha} dt \\ &\leq 4\varepsilon \int_0^{2\pi} |u^-|^2 dx + 2^{1+\alpha}c_8 \int_0^{2\pi} |u^-|^{1+\alpha} dt \\ &\leq 4\varepsilon C^2 \|u^-\|^2 + (2C)^{1+\alpha}c_8 \|u^-\|^{1+\alpha}. \end{aligned}$$

Setting $\varepsilon = \sigma/(16C^2)$, we have

$$\varphi(u) \leq -\frac{\sigma}{2}\langle u^-, u^- \rangle - \lambda^2 \int_0^{2\pi} F(\lambda t, u) dt \leq -\frac{\sigma}{4}\|u^-\|^2 + (2C)^{1+\alpha}c_8 \|u^-\|^{1+\alpha}.$$

Consequently,

$$\varphi(u) \leq 0, \quad \forall u \in X^2 \text{ with } \|u\| \leq \rho_2,$$

where $\rho_2 < \rho_0$ is small enough.

(2) We claim that for every $m \in \mathbf{N}$, $\varphi(u) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$ on $X_m^1 \oplus X^2$.

Since X_m^1 and H^0 are finite dimensional, we can choose $\sigma_m > 0$, $C_m > 0$ such that

$$\langle Lu^+, u^+ \rangle \leq \sigma_m \|u^+\|, \quad \text{for all } u^+ \in X_m^1,$$

$$\|u\| \leq C_m \|u\|_{L^2}, \quad \text{for all } u \in X_m^1 \oplus H^0.$$

By (H2), there exists $c_9 > 0$ such that

$$F(t, x) \geq \frac{\sigma_m}{C_m} |x|^2 - c_9, \quad \forall t \in [0, 2\tau], \forall x \in \mathbf{R}^N.$$

So for $u \in X_m^1 \oplus X^2$,

$$\begin{aligned} \varphi(u) &= \frac{1}{2} \langle Lu^+, u^+ \rangle + \frac{1}{2} \langle Lu^-, u^- \rangle - \lambda^2 \int_0^{2\pi} F(\lambda t, u(t)) dt \\ &\leq \frac{\sigma_m}{2} \|u^+\|^2 - \frac{\sigma}{2} \|u^-\|^2 - \frac{\sigma_m}{C_m} \int_0^{2\pi} |u(t)|^2 dt + c_{10} \\ &= \frac{\sigma_m}{2} \|u^+\|^2 - \frac{\sigma}{2} \|u^-\|^2 - \frac{\sigma_m}{C_m} (\|u^+\|_{L^2} + \|u^-\|_{L^2} + \|u^0\|_{L^2}) + c_{10} \\ &\leq -\frac{\sigma_m}{2} \|u^+\|^2 - \frac{\sigma}{2} \|u^-\|^2 - \sigma_m \|u^0\| + c_{10} \rightarrow -\infty, \end{aligned}$$

for $\|u\| \rightarrow \infty$.

(3) By (3.2), we see that φ maps bounded sets into bounded sets.

The proof of Lemma 3.3 is complete. □

Proof of Theorem 1.1 By Lemmas 3.1 and 3.3, all conditions of Theorem A are satisfied.

Thus, problem (2.1) has at least one nonzero critical point. □

Proof of Theorem 1.2 By Lemmas 3.2 and 3.3, all conditions of Theorem A are satisfied.

Thus, problem (2.1) has at least one nonzero critical point. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors read and approved the final manuscript.

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