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Exponential attractors for the strongly damped wave equations with critical exponent

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Abstract
In this paper, we prove the existence of global attractor and exponential attractor in some stronger spaces for the strongly damped nonlinear wave equation when the nonlinear term $f(u, u_t)$ depends on u_t and contains a critical exponent with respect to u and the external forcing term g merely belongs to the weak space $H^{-1}(\Omega)$.

MSC: Primary 37L30; secondary 39A14

Keywords: wave equation; critical nonlinearity; exponential attractor

1 Introduction

We study the following strongly damped nonlinear wave equation:

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + f(u, u_t) = g & t > 0, x \in \Omega, \\ u(x,t) = 0 & t > 0, x \in \partial\Omega, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) & t = 0, x \in \Omega. \end{cases} \tag{1.1}$$

Here $u = u(x,t)$ is a real-valued function defined on $\Omega \times [0,\infty)$. Ω is an open bounded set of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. $f(u,v) \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $g \in H^{-1}(\Omega)$.

In the case that $f = f(u) \in C^1(\mathbb{R}, \mathbb{R})$ with $\liminf_{|r|\to\infty} \frac{f(r)}{r} > -\lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, Webb first considered the asymptotic behavior of strongly damped wave equations in [1]. Then, in [2], Carvalho *et al.* showed the existence of the global attractor for wave equations with the critical nonlinearity. The regularity of solutions was also investigated via a bootstrapping technique in [3, 4], and we mention that a similar result has also been given by Pata *et al.* in [5, 6]. Recently, Sun and Yang in [7, 8] proved the existence of global attractor and exponential attractor for the same equation with the weaker external term $g \in H^{-1}(\Omega)$.

For another case, $f = f(u, u_t) \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, Massatt [9] and Hale [10] proved the existence of global attractor when the continuous semigroup of the mapping $S(t) : \{u_0, u_1\} \mapsto \{u, u_t\}$ is pointwise dissipative and a bounded map. Moreover, under the assumptions that $f(u, u_t)$ is subcritical with respect to u and the external force term g belongs to $L^2(\Omega)$, the author in [11] proved the existence of global attractor in the space $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$.

In this paper, we investigate the latter case with the conditions given in [8, 11]. Compared with those in [11], the nonlinear term $f(u,u_t)$ satisfies the critical exponent growth condition with respect to u (see (2.4)) and the external force $g \in H^{-1}(\Omega)$, which is weaker than the assumptions in [11]. We also remove the additional assumptions (4.26), (4.27) in [8]. Motivated by the key ideas in [8], by making a shifting on the semigroup $\{S(t)\}_{t\geq 0}$ with a (proper) fixed point $\phi(x)$, we first show the global attractor $\mathcal{A} - \phi(x)$ is bounded in a stronger topology. More precisely, $\mathcal{A} - \phi(x)$ is bounded in the space $\mathcal{H}^\sigma = D((-\Delta)^{\frac{1+\sigma}{2}}) \times D((-\Delta)^{\frac{\sigma}{2}})$, $\sigma \in [0, \frac{1}{2})$ (see Theorem 3.1). Then, by proving that the semigroup $\{S(t)\}_{t\geq 0}$ is Fréchet differential with respect to the initial value, we apply our standard method established in [12] to obtain the exponential attractor for equation (1.1) without the restrictions (4.26), (4.27) in [8]. In addition, with the regularity of solutions as in [6], we establish the existence of exponential attractor in the stronger space $H_0^1(\Omega) \times H_0^1(\Omega)$.

In order to have a comparison, we organize this paper as follows. In Section 1, we briefly review some results. Section 2 is devoting to proving that the existence of global attractor in the space $\mathcal{H}^\sigma$. In Section 3, we obtain the exponential attractor in the space $H_0^1(\Omega) \times H_0^1(\Omega)$.

2 Preliminaries

Let

$$
\begin{aligned}
&(u,v) = \int_\Omega uv\,dx, \qquad \|u\|_2 = (u,u)^{1/2}, \quad \forall u,v \in L^2(\Omega),\\
&\big((u,v)\big) = \int_\Omega \nabla u \nabla v\,dx, \qquad \|u\|_{H_0^1(\Omega)} = \big((u,v)\big)^{1/2}, \quad \forall u,v \in H_0^1(\Omega),\\
&\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega),\\
&\mathcal{H}^\sigma = \big(H_0^1(\Omega) \cap H^{1+\sigma}\big) \times H^\sigma(\Omega) = D\big((-\Delta)^{\frac{1+\sigma}{2}}\big) \times D\big((-\Delta)^{\frac{\sigma}{2}}\big), \quad \sigma \in \left[0, \frac{1}{2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&(y_1,y_2)_{\mathcal{H}} = (y_1,y_2)_{H_0^1(\Omega),L^2(\Omega)} = \big((u_1,u_2)\big) + (v_1,v_2), \qquad \|y\|_{H_0^1(\Omega)\times L^2(\Omega)} = (y,y)^{1/2}_{H_0^1(\Omega)\times L^2(\Omega)},\\
&\|y_i\|_\sigma = \|y_i\|_{\mathcal{H}^\sigma} = \big\|(u_i,v_i)^T\big\|_{H^{1+\sigma}(\Omega),H^\sigma(\Omega)},\\
&\forall y_i = (u_i,v_i)^T, \qquad y = (u,v)^T \in H_0^1(\Omega) \times L^2(\Omega) \text{ or } H^{1+\sigma}(\Omega) \times H^\sigma(\Omega), \quad i = 1,2,
\end{aligned}
$$

denotes the usual inner products and norms in $L^2(\Omega)$, $H_0^1(\Omega)$, and $H_0^1(\Omega) \times L^2(\Omega)$, $H^{1+\sigma}(\Omega) \times H^\sigma(\Omega)$, respectively.

Let $u_t = v$, then equations (1.1) are equivalent to the following initial value problem in the space $\mathcal{H}$:

$$
\begin{cases}
\dot{Y} = \mathbb{L}Y + F(Y), & x \in \Omega, t > 0,\\
Y(0) = Y_0 = (u_0,u_1)^T \in \mathcal{H}, & t = 0,
\end{cases}
\tag{2.1}
$$

where

$$Y = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbb{L} = \begin{pmatrix} 0 & I \\ -A & -A \end{pmatrix}, \qquad F(Y) = \begin{pmatrix} 0 \\ -f(u,u_t) + g \end{pmatrix},$$
$$D(\mathbb{L}) = D(A) \times D(A), \qquad D(A) = D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega). \tag{2.2}$$

Massatt in [9] proved that $\mathbb{L}$ defined in (2.2) is a sectorial operator on $\mathcal{H}$ and generates an analytic compact semigroup $e^{\mathbb{L}t}$ on $\mathcal{H}$ for $t > 0$. By the appropriate assumptions on f and the external forcing term $g \in L^2(\Omega)$, they proved that there exists a unique function $Y(\cdot) = Y(\cdot, Y_0) \in C(R_+, \mathcal{H})$ such that $Y(0, Y_0) = Y_0$ and $Y(t)$ satisfies the integral equation

$$Y(t, Y_0) = e^{\mathbb{L}t} Y_0 + \int_0^t e^{\mathbb{L}(t-s)} F\big(Y(\tau)\big)\, d\tau,$$

which is also called a mild solution of equation (2.1).

The main purpose here is to study the case $g \in H^{-1}(\Omega)$ and to provide some weaker assumptions on $f(u,v)$ than the one in [8, 11], that is, the function $f(u,v) \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $f(0,0) = 0$ satisfies the following condition:

$$\liminf_{|s| \to +\infty} \frac{f(s,0)}{s} > -\lambda_1 \tag{2.3}$$

and its partial derivatives $f_1'(u,v), f_2'(u,v), f_{11}''(u,v), f_{12}''(u,v), f_{22}''(u,v)$ satisfy

$$\big|f_1'(u,v)\big| \leq C\big(1 + |u|^4\big), \quad \forall u, v \in \mathbb{R}, \tag{2.4}$$

$$f_1'(u,v) \geq -\ell, \quad \forall u, v \in \mathbb{R}, \tag{2.5}$$

$$f_2'(u,v) \leq \delta \text{ (small enough)}, \quad \forall u, v \in \mathbb{R}, \tag{2.6}$$

$$\big|f_{11}''(u,v)\big|, \big|f_{12}''(u,v)\big|, \big|f_{22}''(u,v)\big| \leq C\big(1 + |u|^3\big), \quad \forall u, v \in \mathbb{R}. \tag{2.7}$$

Note again that in contrast to [8], here $f = f(u,u_t)$ without the addition assumptions (4.26), (4.27) in [8], and in contrast to [11], here $f = f(u,u_t)$ is critical with respect to u, and its partial derivatives f_j', f_{ij}'' is weaker than assumptions (3), (4) in [11].

Obviously, such conditions are satisfied in particular for the nonlinearities $f(u,v) = u^5 + \delta \sin v$ (in other words, a small perturbation of u^5), etc.

As is well known, if $g \in H^{-1}(\Omega)$, the solution of the elliptic equation ($\theta > \ell$)

$$\begin{cases} -\Delta u + f(u,0) + \theta u = g \in H^{-1}(\Omega), \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.8}$$

only belongs to $H_0^1(\Omega)$. The regularity of the attractor (if it exists) is not higher than $\mathcal{H}$ in this case. However, by a decomposition as in [8], $u(t) = \hat{u}(t) + \phi(x)$ where $\phi(x)$ is the solution of equation (2.8) for some θ, and $\hat{u}(t)$ satisfies

$$\begin{cases} \hat{u}_{tt} - \Delta \hat{u}_t - \Delta \hat{u} + f(\hat{u} + \phi, \hat{u}_t) - f(\phi, 0) = \theta\phi, \\ \hat{u}|_{\partial\Omega} = 0. \end{cases} \tag{2.9}$$

Next, we will get the regularity of the solution $\hat{u}(t)$.

3 Global attractor

We first present the following asymptotic regularity by the Galerkin approximate scheme (see [8, 13]).

Theorem 3.1 *Let $f(u,v) \in C^2(\mathbb{R}\times\mathbb{R},\mathbb{R})$ with $f(0,0)=0$ satisfying the above assumptions* (2.3)-(2.7), *$g \in H^{-1}$, and $\{S(t)\}_{t\geq 0}$ be the semigroup generated by the weak solution of* (1.1) *in the space $H_0^1(\Omega)\times L^2(\Omega)$. Then, for each $0<\sigma<\frac{1}{2}$, there exist a subset $\mathcal{B}_\sigma$, a monotone increasing function $Q_\sigma(\cdot)$, and a positive constant ν (independent of σ) such that: for any bounded set $B\subset\mathcal{H}$,*

$$\operatorname{dist}_{\mathcal{H}}\big(S(t)B,\mathcal{B}_\sigma\big)\leq Q_\sigma\big(\|B\|_{\mathcal{H}}\big)e^{-\nu t},\quad \text{for all } t\geq 0,$$

where $\mathcal{B}_\sigma$ satisfies, for some constant $\Lambda_\sigma>0$,

$$\mathcal{B}_\sigma=\big\{\varsigma\in\mathcal{H}:\big\|\varsigma-\big(\phi(x),0\big)\big\|_{H^{1+\sigma}(\Omega)\times H^\sigma(\Omega)}\leq\Lambda_\sigma<\infty\big\},$$

and $\phi(x)$ is the unique solution of the above equation (2.8) *by choosing $\theta=\eta_0$ large enough, that is,*

$$\begin{cases}-\Delta\phi+f(\phi,0)+\eta_0\phi=g\in H^{-1}(\Omega), & \text{in } \Omega,\\ \phi|_{\partial\Omega}=0.\end{cases}\tag{3.1}$$

Remark 3.1 From [8], we know that

1. for each θ $(>\ell)$, equation (2.8) has a unique solution $u_\theta(x)\in H_0^1(\Omega)$ satisfying

$$\|\nabla u_\theta\|^2+2(\theta-\ell)\|u_\theta\|_2^2\leq\|g\|_{H^{-1}}^2;$$

2. $\|\nabla u_\theta\|\to 0$, $\|u_\theta\|_{L^p}\to 0$ as $\theta\to\infty$ for any fixed $p\in[2,6)$.

Now, denote $h_\theta(u,u_t)=f(u,u_t)+\theta u$. From (2.4)-(2.6) and the mean value theorem, one has, for any $v\in C^1((0,\infty),\mathcal{H})$,

$$\begin{aligned}&\frac{1}{2}\|\nabla v\|^2+\frac{1}{2}\|v_t\|^2+2\big\langle h_\theta(v+\phi,v_t+\phi_t)-h_\theta(\phi,\phi_t),v\big\rangle-\big\langle h'_{1\theta}(\phi,0)v,v\big\rangle\\ &\quad=\frac{1}{2}\|\nabla v\|^2+\frac{1}{2}\|v_t\|^2+2\big\langle h_\theta(v+\phi,v_t)-h_\theta(\phi,0),v\big\rangle-\big\langle h'_{1\theta}(\phi,0)v,v\big\rangle\\ &\quad=\frac{1}{2}\|\nabla v\|^2+\frac{1}{2}\|v_t\|^2+2\big\langle h_\theta(v+\phi,v_t)-h_\theta(\phi,v_t)+h_\theta(\phi,v_t)-h_\theta(\phi,0),v\big\rangle\\ &\qquad-\big\langle h'_{1\theta}(\phi,0)v,v\big\rangle\\ &\quad=\frac{1}{2}\|\nabla v\|^2+\frac{1}{2}\|v_t\|^2+2\big\langle h'_{1\theta}(\vartheta_1v+\phi,v_t)v,v\big\rangle+2\big\langle h'_{2\theta}(\phi,\vartheta_2v_t)v_t,v\big\rangle-\big\langle h'_{1\theta}(\phi,0)v,v\big\rangle\\ &\quad\geq\frac{1}{2}\|\nabla v\|^2+\frac{1}{2}\|v_t\|^2+2(\theta-\ell)\|v\|^2-\theta\|v\|^2-2\delta\int_\Omega|v_tv|\,dx-C\int_\Omega\big(1+|\phi|^4\big)|v|^2\,dx\\ &\quad\geq\frac{1}{2}\|\nabla v\|^2+\frac{1}{2}\|v_t\|^2+(\theta-2\ell-C-\delta)\|v\|^2-\delta\|v_t\|^2-C\|\nabla\phi\|^4\|\nabla v\|^2,\end{aligned}\tag{3.2}$$

where the constants C, δ, and ℓ come from (2.4)-(2.6), respectively, and $\vartheta_1,\vartheta_2\in(0,1)$, ϕ is the solution of (3.1).

Hence, by choosing θ large enough in (3.2) with the assertion 2 in Remark 3.1, we know that

$$\frac{1}{2}\|\nabla v\|^2 + \frac{1}{2}\|v_t\|^2 + 2\big\langle h_\theta(v+\phi, v_t+\phi_t) - h_\theta(\phi,\phi_t), v\big\rangle - \big\langle h'_{1\theta}(\phi,0)v, v\big\rangle \geq 0,$$
$$\text{for all } v \in C^1\big((0,\infty),\mathcal{H}\big). \tag{3.3}$$

3.1 Decomposition of the equations

Let

$$h(u,u_t) = f(u,u_t) + \eta_0 u,$$

where the positive constant η_0 is large enough and such that (2.8) and (3.3) holds when $\theta = \eta_0$.

Now, we first decompose the solution $S(t)(u_0,v_0) = (u(t),u_t(t))$ into the sum

$$\big(u(t),u_t(t)\big) = S(t)\xi_u(0) = K(t)\xi_u(0) + D(t)\xi_u(0) = \big(w(t),w_t(t)\big) + \big(z(t),z_t(t)\big),$$

where $K(t)\xi_u(0) = (w(t),w_t(t))$ and $D(t)\xi_u(0) = (z(t),z_t(t))$ solve the following equations, respectively:

$$\begin{cases} w_{tt} - \Delta w_t - \Delta w + f(u,u_t) - f(z,z_t) = \eta_0 z \quad \text{in } \Omega\times\mathbb{R}^+, \\ w|_{\partial\Omega} = 0, \\ (w(x,0),w_t(x,0)) = (0,0), \end{cases} \tag{3.4}$$

and

$$\begin{cases} z_{tt} - \Delta z_t - \Delta z + h(z,z_t) = g(x) \quad \text{in } \Omega\times\mathbb{R}^+, \\ z|_{\partial\Omega} = 0, \\ (z(x,0),z_t(x,0)) = \xi_u(0). \end{cases} \tag{3.5}$$

Then we decompose further the solution $z(x,t)$ of (3.5) as $z(x,t) = v(x,t) + \phi(x)$, where $\phi(x)$ is the unique solution of (2.8) and $v(x,t)$ solves the following equation:

$$\begin{cases} v_{tt} - \Delta v_t - \Delta v + h(z,z_t) - h(\phi,0) = 0 \quad \text{in } \Omega\times\mathbb{R}^+, \\ v|_{\partial\Omega} = 0, \\ (v(x,0),v_t(x,0)) = \xi_u(0) - (\phi(x),0). \end{cases} \tag{3.6}$$

Hence,

$$\begin{aligned} \big(u(t),u_t(t)\big) &= \big(w(t),w_t(t)\big) + \big(z(t),z_t(t)\big) \\ &= \big(w(t),w_t(t)\big) + \big(v(t)+\phi, v_t(t)+\phi_t\big) \\ &= \big(w(t),w_t(t)\big) + \big(v(t)+\phi, v_t(t)\big), \quad \text{due to } \phi_t = 0. \end{aligned} \tag{3.7}$$

Hereafter, we always assume the assumptions in Theorem 3.1 hold and denote the unique solution of (2.8) by $\phi(x)$.

3.2 The prior estimates in spaces $\mathcal{H}, \mathcal{H}^\sigma$ ($\sigma \in [0, \frac{1}{2})$)

Now, we will give the prior estimates in space $\mathcal{H}$ or regular space $\mathcal{H}^\sigma$ for the above decompositions of the solutions z, v, w, u, respectively.

First of all, we have the following estimate (*e.g.*, see [5, 8]) for the solution z of (3.5).

Lemma 3.1 *There exists an increasing function $Q_1(\cdot)$ such that, for any bounded set $B \subset \mathcal{H}$, one gets, for any $t \geq 0$,*

$$\left\|\nabla z(t)\right\|^2 + \int_0^t \left\|\nabla z_t(s)\right\|^2 dx \leq Q_1\left(\|B\|_{\mathcal{H}} + \|g\|_{H^{-1}}\right), \quad \forall \xi_u(0) \in B. \tag{3.8}$$

Proof Indeed, we consider the functional (by choosing $\hat{\phi}(y) = f(y,0) + \eta_0 y$ in [5])

$$\mathcal{F}(t) = \mathcal{F}\left(z(t)\right) = 2\int_\Omega \int_0^{z(x,t)} \left(f(s,0) + \eta_0 s\right) ds\, dx. \tag{3.9}$$

We set $\xi(t) = z_t + \epsilon z$ with $\epsilon \in (0, \epsilon_0)$, for some $\epsilon_0 \leq 1$ to be determined later. Multiplying equation (3.5) by ξ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}E + \epsilon(1-\epsilon)\|\nabla z\|^2 + \|\nabla \xi\|^2 \\ &\quad = \epsilon\|\xi\|^2 - \epsilon^2\langle z, \xi\rangle + \epsilon\langle g, z\rangle - \epsilon\left\langle f(z,0) + \eta_0 z, z\right\rangle + \left\langle f(z,0) - f(z,z_t), z_t + \epsilon z\right\rangle, \end{aligned} \tag{3.10}$$

where the energy functional E is defined as

$$E(t) = E\left(z(t)\right) = (1-\epsilon)\|\nabla z\|^2 + \left\|\xi(t)\right\|^2 + \mathcal{F}(t) - 2\langle g, z\rangle. \tag{3.11}$$

Obviously, from (2.4), we know that here the function $\hat{\phi}(y) = f(y,0) + \eta_0 y$ satisfies the assumptions (8), (9), (11), (12) in [5], and due to the mean value theorem, we have

$$\begin{aligned} \left\langle f(z,z_t) - f(z,0), z_t + \epsilon z\right\rangle &= \left\langle f_2'(z, \vartheta z_t) z_t, z_t + \epsilon z\right\rangle \\ &\leq \delta\|z_t\|^2 + \delta\epsilon \int_\Omega |z_t z|\, dx, \end{aligned} \tag{3.12}$$

where $\vartheta \in (0,1)$.

As to the assumption (2.6), if δ is small enough, the term in (3.12) can be controlled by the left-hand side of (3.10). Therefore, with the application of the same argument as in [5], it is easy to get the inequality (3.8). It finishes the proof of Lemma 3.1. □

Then, for the solution v of (3.6), we have the following.

Lemma 3.2 *There exist an increasing function $Q_2(\cdot)$ and some constant $k_1 > 0$, such that, for any bounded set $B \subset \mathcal{H}$,*

$$\left\|\left(v(x,t), v_t(x,t)\right)\right\|_{\mathcal{H}} \leq Q_2\left(\|B\|_{\mathcal{H}}\right) e^{-k_1 t}, \quad \forall t \geq 0, \xi_v(0) \in B,$$

that is,

$$\left\|\left(z(x,t), z_t(x,t)\right) - \left(\phi(x), 0\right)\right\|_{\mathcal{H}} \leq Q_2\left(\|B\|_{\mathcal{H}}\right) e^{-k_1 t}, \quad \forall t \geq 0, \xi_v(0) \in B.$$

Proof As in [8, 14], for $\epsilon \in (0,1)$ to be determined later, we define the functional

$$\Lambda(t) = \|\nabla v(t)\|^2 + \|v_t(t)\|^2 + \epsilon\|\nabla v(t)\|^2 + 2\langle h(z,0) - h(\phi,0), v\rangle + 2\epsilon\langle v_t, v\rangle - \langle h_1'(\phi,0)v, v\rangle.$$

Then, from (3.3) and by taking ϵ small enough, we have

$$\Lambda(t) \geq \frac{1}{4}\|\xi_v(t)\|_{\mathcal{H}}^2 \quad \text{for all } t \geq 0, \xi_0 \in B.$$

Multiplying (3.6) by $v_t + \epsilon v(t)$ we have (note that $z_t = v_t$ and $\phi_t = 0$)

$$\begin{aligned}&\frac{d}{dt}\Lambda(t) + \epsilon\Lambda(t) + \Gamma + \frac{\epsilon}{2}\|\nabla v(t)\|^2 \\ &\quad = 2\langle\big(h_1'(z,0) - h_1'(\phi,0)\big)z_t, v\rangle + 2\langle\big(h(z,0) - h(z,z_t)\big), v_t + \epsilon v\rangle, \end{aligned} \tag{3.13}$$

where

$$\Gamma = 2\|\nabla v_t(t)\|^2 + \frac{\epsilon}{2}\|\nabla v(t)\|^2 - 3\epsilon\|v_t\|^2 - 2\epsilon^2\langle v_t, v\rangle - \epsilon\|\nabla v\|^2 + \epsilon\langle h_1'(\phi,0), v^2\rangle.$$

It is easy to see that $\Gamma \geq 0$ as ϵ small enough, and from (2.7), we have

$$\begin{aligned} 2\langle\big(h_1'(z,0) - h_1'(\phi,0)\big)z_t, v\rangle &= 2\langle h_{11}''\big(rz + (1-r)\phi, 0\big)z_t, v^2\rangle \\ &\leq C\int_\Omega \big(1 + |z|^3 + |\phi|^3\big)|z_t||v|^2\,dx \\ &\leq c_2\|\nabla z_t\|\|\nabla v\|^2 \leq \frac{\epsilon}{2}\|\nabla v\|^2 + \frac{c_2}{\epsilon}\|\nabla z_t\|^2\Lambda, \end{aligned}$$

where $r \in (0,1)$ and the constant c_2 depends only on $\|B\|_{\mathcal{H}} + \|\nabla\phi\|$.

By the mean value theorem, for the last term in the right-hand side of (3.13), we get

$$\begin{aligned} 2\langle\big(h(z,0) - h(z,z_t)\big), v_t + \epsilon v\rangle &= 2\langle f(z,z_t) - f(z,0), z_t + \epsilon z\rangle \\ &= \langle f_2'(z,\vartheta z_t)z_t, z_t + \epsilon v\rangle \\ &\leq \delta\|z_t\|^2 + \delta\epsilon\int_\Omega |z_t v|\,dx. \end{aligned}$$

Since δ is small enough, from Lemma 3.1 and by noticing $\Lambda(0) \leq Q(\|B\|_{\mathcal{H}} + \|\nabla\phi\|)$ and by applying Lemma 2.2 [15], we can finish the proof of Lemma 3.2. □

Second, for the solution $w(t)$ in (3.4), we have the following result.

Lemma 3.3 *For each bounded subset $B \subset \mathcal{H}$ and any $\sigma \in [0, \frac{1}{2})$, there exists an increasing function $Q_\sigma(\cdot)$ such that*

$$\|K(t)\xi_u(0)\|_{\mathcal{H}^\sigma} = \|\big(w(t), w_t(t)\big)\|_{\mathcal{H}^\sigma} \leq Q_\sigma\big(\|B\|_{\mathcal{H}}\big)e^{\nu_\sigma t} \quad \forall t \geq 0, \xi_u(0) \in B, \tag{3.14}$$

where the positive constant ν_σ depends only on $\|B\|_{\mathcal{H}}$ and σ.

Proof Rewriting equation (1.1) as follows:

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + f(u,0) = g + f(u,0) - f(u,u_t) & t > 0, x \in \Omega, \\ u(x,t) = 0 & t > 0, x \in \partial\Omega, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) & t = 0, x \in \Omega, \end{cases}$$

and applying the same argument as in the proof procedure of Lemma 3.1 with the assumptions (2.4)-(2.6), and combining with (3.8), it is easy to show that

$$\|\nabla u(t)\| + \|\nabla z(t)\| \le c(\|B\|_{\mathcal{H}}), \quad \forall t \ge 0.$$

Now, rewrite equation (3.4) as follows:

$$\begin{cases} w_{tt} - \Delta w_t - \Delta w + f(u,0) + \eta_0 u - (f(z,0) + \eta_0 z), \\ = \eta_0 u + f(u,0) - f(u,u_t) - (f(z,0) - f(z,z_t)) \quad \text{in } \Omega \times \mathbb{R}^+, \\ w|_{\partial\Omega} = 0, \\ (w(x,0), w_t(x,0)) = (0,0). \end{cases} \tag{3.15}$$

Denoting $\hat{\phi}(u) = f(u,0) + \eta_0 u$, $\hat{\phi}(z) = f(z,0) + \eta_0 z$ like the one in [5], and testing equation (3.15) with $A^\sigma w_t$, we are led to the identity (denote $\gamma(t) = (w(t), w_t(t))$)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\gamma(t)\|_\sigma^2 + \|A^{(1+\sigma)/2} w_t\|^2 \\ &\quad = -\langle \hat{\phi}(u) - \hat{\phi}(z), A^\sigma w_t \rangle + \langle g, A^\sigma w_t \rangle \\ &\qquad + \langle f(u,0) - f(u,u_t) - (f(z,0) - f(z,z_t)), A^\sigma w_t \rangle. \end{aligned} \tag{3.16}$$

Due to (2.4), we get

$$\begin{aligned} -\langle \hat{\phi}(u) - \hat{\phi}(z), A^\sigma w_t \rangle &\le c(1 + \|u\|_{L^6}^4 + \|z\|_{L^6}^4)\|w\|_{L^{6/(1-2\sigma)}} \|A^\sigma w_t\|_{L^{6/(1+2\sigma)}} \\ &\le c(1 + \|A^{1/2}u\|^4 + \|A^{1/2}v\|^4)\|A^{(1+\sigma)/2}w\| \|A^{(1+\sigma)/2}w_t\| \\ &\le c\|\gamma(t)\|_\sigma^2 + \frac{1}{3}\|A^{(1+\sigma)/2}w_t\|^2. \end{aligned} \tag{3.17}$$

By virtue of (2.6), we have

$$\begin{aligned} &\langle f(u,0) - f(u,u_t) - (f(z,0) - f(z,z_t)), A^\sigma w_t \rangle \\ &\quad = \langle -f_2'(u, \vartheta_2 u_t)u_t + f_2'(z, \vartheta_2 z_t)z_t, A^\sigma w_t \rangle \\ &\quad \le \delta(\|u_t\|_{L^{6/(5-2\sigma)}} + \|z_t\|_{L^{6/5-2\sigma}}) \|A^\sigma w_t\|_{L^{6/(1+2\sigma)}} \\ &\quad \le \delta(\|u_t\|_{L^{6/(5-2\sigma)}} + \|z_t\|_{L^{6/5-2\sigma}}) \|A^{(1+\sigma)/2} w_t\| \\ &\quad \le c + \frac{1}{3}\|A^{(1+\sigma)/2}w_t\|^2, \end{aligned} \tag{3.18}$$

where $\vartheta_2 \in (0,1)$.

Additionally,

$$\langle g, A^{\sigma} w_t \rangle \leq \|A^{-1/2} g\| \|A^{(1+\sigma)/2} w_t\| \leq c + \frac{1}{3} \|A^{(1+\sigma)/2} w_t\|^2. \tag{3.19}$$

Plugging (3.17)-(3.19) into (3.16), we obtain

$$\frac{d}{dt} \|\gamma(t)\|_{\sigma}^2 \leq c \|\gamma(t)\|_{\sigma}^2 + c, \tag{3.20}$$

and the Gronwall lemma entails

$$\|\gamma(t)\|_{\sigma}^2 \leq e^{kt} - 1,$$

which concludes the proof. □

Now, based on Lemmas (3.2) and (3.3), one can also decompose the solution $u(t)$ as follows.

Lemma 3.4 *For any* $\epsilon > 0$,

$$u(t) = v_1(t) + w_1(t), \quad \text{for all } t \geq 0, \tag{3.21}$$

where $v_1(t)$ *and* $w_1(t)$ *satisfy the following*:

$$\int_s^t \|\nabla v_1(\tau)\|^2 \, d\tau \leq \epsilon(t-s) + C_{\epsilon} \quad \text{for all } t \geq s \geq 0, \tag{3.22}$$

and

$$\|A^{\frac{1+\sigma}{2}} w_1(t)\|^2 \leq K_{\epsilon} \quad \text{for all } t \geq 0, \tag{3.23}$$

with the constants C_{ϵ} *and* K_{ϵ} *depending on* ϵ, *the initial value* $\|\xi_u(0)\|_{\mathcal{H}}$ *and* $\|g\|_{H^{-1}}$.

Due to (3.7) and Lemma 4.5 in [8], one can easily deduce Lemma 3.4.

Next, we will show further that the estimate w in (3.14) can be chosen independent of the time t.

Lemma 3.5 *For every* $\sigma \in [0, \frac{1}{2})$, *there exists a constant* $J_{B,\sigma}$ *which depends only on the* $\mathcal{H}$*-bound of* B ($\subset \mathcal{H}$) *and* σ, *such that*

$$\|K(t)\xi_u(0)\|_{\mathcal{H}^{\sigma}}^2 = \|(w(t), w_t(t))\|_{\mathcal{H}^{\sigma}}^2 \leq J_{B,\sigma} \quad \text{for all } t \geq 0 \text{ and } \xi_u(0) \in B.$$

Proof The idea comes from [8, 16, 17] but with different details.

Multiplying (3.15) by $A^{\sigma}(w_t(t) + \epsilon w(t))$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_{\Omega} |A^{\frac{\sigma}{2}}(w_t + \epsilon w)|^2 - \langle \epsilon w_t, A^{\sigma}(w_t + \epsilon w) \rangle \\ &\quad - \langle A w_t, A^{\sigma}(w_t + \epsilon w) \rangle - \langle A w, A^{\sigma}(w_t + \epsilon w) \rangle \end{aligned}$$

$$\begin{aligned}&= -\langle f(u,0) - f(z,0), A^{\sigma}(w_t + \epsilon w)\rangle + \langle \eta_0 z, A^{\sigma}(w_t + \epsilon w)\rangle \\ &\quad + \langle f(u,0) - f(u,u_t) - (f(z,0) - f(z,z_t)), A^{\sigma} w_t\rangle,\end{aligned}$$

where ϵ (> 0) is small enough to be determined later.

We only need to deal with the right-hand side term, and the others can be estimated easily as those Lemma 4.4 in [18].

From (2.4), we first deal with the first dual product,

$$\left|\langle f(u,0) - f(z,0), A^{\sigma}(w_t + \epsilon w)\rangle\right| \le C \int_{\Omega} (1 + |u|^4 + |z|^4)|w|\left|A^{\sigma}(w_t + \epsilon w)\right| dx.$$

Applying Lemma 3.4, we have

$$\int_{\Omega} |u|^4 |w| \left|A^{\sigma} w\right| dx \le C \int_{\Omega} (|v_1|^4 + |w_1|^4)\left|w(t)\right|\left|A^{\sigma} w(t)\right| dx \tag{3.24}$$

and

$$\begin{aligned}\left|\langle f(u,0) - f(z,0), A^{\sigma} w\rangle\right| &\le c_4 Q_4(\|B\|_{\mathcal{H}}) \|\nabla v_1(t)\|^2 \|A^{\frac{1+\sigma}{2}} w(t)\|^2 \\ &\quad + c_{\sigma}(K_{\epsilon} + \|\phi\|_{H^2}) Q_5(\|B\|_{\mathcal{H}}) + C + \frac{1}{4}\|A^{\frac{1+\sigma}{2}} w(t)\|^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\langle f(u,0) - f(z,0), A^{\sigma} w_t\rangle\right| &\le c_4 Q_4(\|B\|_{\mathcal{H}}) \|\nabla v_1(t)\|^2 \|A^{\frac{1+\sigma}{2}} w(t)\|^2 \\ &\quad + c_{\sigma}(K_{\epsilon} + \|\phi\|_{H^2}) Q_5(\|B\|_{\mathcal{H}}) + C + \frac{1}{4}\|A^{\frac{1+\sigma}{2}} w_t(t)\|^2.\end{aligned}$$

By the mean value theorem, similar to (3.18), we have

$$\begin{aligned}&\langle f(u,0) - f(u,u_t) - (f(z,0) - f(z,z_t)), A^{\sigma} w_t\rangle \\ &\quad = \langle -f_2'(u, \vartheta_2 u_t)u_t + f_2'(z, \vartheta_2 z_t)z_t, A^{\sigma} w_t\rangle \\ &\quad \le \delta(\|u_t\|_{L^{6/(5-2\sigma)}} + \|z_t\|_{L^{6/5-2\sigma}}) \|A^{\sigma} w_t\|_{L^{6/(1+2\sigma)}} \\ &\quad \le \delta(\|u_t\|_{L^{6/(5-2\sigma)}} + \|z_t\|_{L^{6/5-2\sigma}}) \|A^{(1+\sigma)/2} w_t\| \\ &\quad \le c + \frac{1}{3}\|A^{(1+\sigma)/2} w_t\|^2\end{aligned}$$

and

$$\begin{aligned}&\langle f(u,0) - f(u,u_t) - (f(z,0) - f(z,z_t)), A^{\sigma} w\rangle \\ &\quad = \langle -f_2'(u, \vartheta_2 u_t)u_t + f_2'(z, \vartheta_2 z_t)z_t, A^{\sigma} w\rangle \\ &\quad \le \delta(\|u_t\|_{L^{6/(5-2\sigma)}} + \|z_t\|_{L^{6/5-2\sigma}}) \|A^{\sigma} w\|_{L^{6/(1+2\sigma)}} \\ &\quad \le \delta(\|u_t\|_{L^{6/(5-2\sigma)}} + \|z_t\|_{L^{6/5-2\sigma}}) \|A^{(1+\sigma)/2} w\| \\ &\quad \le c + \frac{1}{3}\|A^{(1+\sigma)/2} w\|^2.\end{aligned}$$

Therefore, we can finish the proof by using the Gronwall-type inequality as was done in [18], Lemma 4.4. □

Finally, for $u(t)$, the following decomposition is valid, which will be used later to construct an exponential attractor.

Lemma 3.6 *For each* $\sigma \in [0, \frac{1}{2})$ *and for any bounded (in* $\mathcal{H}^\sigma$*) subset* $B_1 \subset \mathcal{H}^\sigma$*, if the initial data* $\xi_u(0) \in \phi(x) + B_1$*, then*

$$\begin{aligned}&\left\| S(t)\xi_u(0) - \big(\phi(x), 0\big)\right\|^2_{\mathcal{H}^\sigma} = \left\| \big(u(t), u_t(t)\big) - \big(\phi(x), 0\big)\right\|^2_{\mathcal{H}^\sigma} \le K_{B_1,\sigma}\\ &\quad \forall t \ge 0, \xi_u(0) \in \phi(x) + B_1,\end{aligned}$$

where the constant $K_{B_1,\sigma}$ *depends only on the* $\mathcal{H}^\sigma$*-bound of* B_1 *and* σ.

Proof By taking the following decomposition: $u(t) = \hat{u}(t) + \phi(x)$, where $\phi(x)$ is the unique solution of (3.1) and $\hat{u}(t)$ solves the following equation:

$$\begin{cases} \hat{u}_{tt} - \Delta\hat{u}_t - \Delta\hat{u} + f(u,0) - f(\phi,0) = \eta_0\phi + f(u,0) - f(u,u_t) & \text{in } \Omega \times \mathbb{R}^+, \\ \hat{u}|_{\partial\Omega} = 0, \\ (\hat{u}(x,0), \hat{u}_t(x,0)) = \xi_u(0) - (\phi, 0), \end{cases}$$

by applying Lemma 3.4, we get similar estimates to those in Lemma 3.5. Noting that the initial value data $(\hat{u}(x,0), \hat{u}_t(x,0)) = \xi_u(0) - (\phi, 0) \in \mathcal{H}^\sigma$, the conclusion can be obtained. □

Hence, the proof of Theorem 3.1 follows from the above lemmas as in [8].

4 Exponential attractor

In this section, based on the asymptotic regularity obtained above, we will construct an exponential attractor by the abstract method devised in [12]. Here it is different from [8] to prove the asymptotic smooth property (as it was called by EMS 2000 in [19]) under the additional assumptions (4.26), (4.27) in that paper.

By our abstract method devised in [12], one defines here S as the map induced by Poincaré sections of a Lipschitz continuous semigroup $\{S(t)\}_{t\ge 0}$ at the time $t = T^*$ for some $T^* > 0$; that is, $S := S(T^*)$ and $S : B_{\epsilon_0}(\mathcal{A}) \to B_{\epsilon_0}(\mathcal{A})$ is a C^1 map. $\mathcal{L}(X) = \{L | L : X \to X$ bounded linear maps$\}$, $\mathcal{L}_\lambda(X) = \{L | L \in \mathcal{L}(X)$ and $L = K + C$ with K compact, $\|C\| < \lambda\}$. For the discrete semigroup $\{S^n\}_{n=1}^\infty$ generated by S, we have the following lemmas.

Lemma 4.1 (see Theorem 1.2 [12]) *If there exists* $\lambda \in (0, 1)$ *such that* $D_x S(x) \in \mathcal{L}_\lambda(X)$ *for all* $x \in B_{\epsilon_0}(\mathcal{A})$ *then* $\{S^n\}_{n=1}^\infty$ *possesses an exponential attractor* $\mathcal{M}_d$.

Lemma 4.2 (see Theorem 1.4 [12]) *Suppose that there is* $T^* > 0$ *such that* $S = S(T^*)$ *satisfies the condition of above lemma* 4.1 *and the map* $F(x,t) = S(t)x$ *is Lipschitz from* $[0, T] \times X$ *into* X *for any* $T > 0$. *Then the flow* $\{S(t)\}_{t\ge 0}$ *admits an exponential attractor* $\mathcal{M}_c$.

As regards the Fréchet differential of semigroup, we have the following crucial lemma.

Lemma 4.3 *Consider the linearized equation of* (1.1),

$$\begin{cases} U_{tt} - \Delta U_t - \Delta U + f_1'(u, u_t)U + f_2'(u, u_t)U_t = 0, \\ U(x,t)|_{\partial\Omega} = 0, \\ (U(x,0), U_t(x,0))^T = (\xi, \eta)^T. \end{cases} \tag{4.1}$$

If the function $f(u,v)$ satisfies conditions (2.3)-(2.7), *then* (4.1) *is a well-posed problem in E, the mapping $S(t)$ defined in* (1.1) *is Fréchet differentiable on E for any $t > 0$, its differential at $\varphi_0 = (u_0, u_1)^T$ is the linear operator on $E : (\xi, \eta)^T \mapsto (U(t), V(t))^T$, where U is the solution of* (4.1) *and $V = U_t$.*

Proof According to assumptions (2.4)-(2.6), (4.1) is a well-posed problem in $\mathcal{H}$.

In the sequel, we first consider the Lipschitz property of the semigroup $S(t)$ on the bounded sets B ($\subset \mathcal{H}$). Letting $\varphi_0 = (u_0, u_1)^T \in D(\mathbb{L})$, $\tilde{\varphi}_0 = \varphi_0 + (\xi, \eta)^T = (u_0 + \xi, u_1 + \eta)^T \in D(\mathbb{L})$, it follows from the above estimate that the solutions $S(t)\varphi_0 = \varphi(t) = (u(t), u_t(t))^T \in D(\mathbb{L})$, $S(t)\tilde{\varphi}_0 = \tilde{\varphi}(t) = (\tilde{u}(t), \tilde{u}_t(t))^T \in D(\mathbb{L})$.

Obviously, the difference $\psi = \tilde{u} - u$ satisfies

$$\psi_{tt} - \Delta\psi_t - \Delta\psi = -\bigl[f(\tilde{u}, \tilde{u}_t) - f(u, u_t)\bigr]. \tag{4.2}$$

Taking the scalar product of (4.2) with $\psi_t = \tilde{u}_t - u_t$ in $L^2(\Omega)$ and by the mean value theorem, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\bigl(\|\psi_t\|^2 + \|\nabla\psi\|^2\bigr) + \|\nabla\psi_t\|^2 \\ &\quad = \bigl\langle -\bigl[f(\tilde{u}, \tilde{u}_t) - f(u, \tilde{u}_t)\bigr] - \bigl[f(u, \tilde{u}_t) - f(u, u_t)\bigr], \psi_t\bigr\rangle \\ &\quad = \bigl\langle -f_1'\bigl(u + \vartheta_1(\tilde{u} - u), u_t\bigr)\psi - f_2'\bigl(u, u_t + \vartheta_2(\tilde{u}_t - u_t)\bigr)\psi_t, \psi_t\bigr\rangle \\ &\quad \bigl(\text{by (2.4), (2.6) and the Poincaré inequality}\bigr) \\ &\quad \le \int_\Omega C\bigl(1 + |u|^4 + |\tilde{u}|^4\bigr)|\psi||\psi_t|\,dx + \delta\|\psi_t\|_{L^2(\Omega)}^2 \\ &\quad \le C\bigl(1 + \|u\|_{L^6}^4 + \|\tilde{u}\|_{L^6}^4\bigr)\|\psi\|_{L^6}\|\psi_t\|_{L^6} + \delta\|\psi_t\|_{L^2(\Omega)}^2 \\ &\quad (\text{due to Lemma 3.6 and the Poincaré inequality}) \\ &\quad \le C(\delta)\|\nabla\psi\|_{L^2(\Omega)}^2 + 2\delta\|\nabla\psi_t\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.3}$$

Since δ is small enough, applying the Gronwall inequality to (4.3), it is easy to show the semigroup $\{S(t)\}_{t\ge 0}$ is Lipschitz, *i.e.*,

$$\begin{aligned} \bigl\|\tilde{\psi}(t) - \psi(t)\bigr\|_{H_0^1\times L^2}^2 &= \bigl\|\tilde{u}(t) - u(t)\bigr\|^2 + \bigl\|\nabla\tilde{u}(t) - \nabla u(t)\bigr\|^2 \\ &\le e^{ct}\bigl(\|\eta\|^2 + \|\nabla\xi\|^2\bigr), \quad \forall t \ge 0. \end{aligned} \tag{4.4}$$

Integrating (4.3) in $d\tau$ on $[0,t]$, this, on account of (4.4), yields

$$\int_0^t \|\nabla\psi\|^2\,d\tau \le e^{ct}\big(\|\eta\|^2 + \|\nabla\xi\|^2\big), \quad \forall t\ge 0. \tag{4.5}$$

Furthermore, applying the same argument as in [6] with the assumptions (2.4)-(2.6), we can obtain the same estimates for $\|\psi_t(t)\|$ and $\|\nabla\psi_t(t)\|$, that is,

$$\begin{aligned}\big\|\tilde{\psi}_t(t)-\psi_t(t)\big\|^2_{H_0^1\times L^2} &= \big\|\tilde{u}_t(t)-u_t(t)\big\|^2 + \big\|\nabla\tilde{u}_t(t)-\nabla u_t(t)\big\|^2\\ &\le e^{ct}\big(\|\eta\|^2+\|\nabla\xi\|^2\big), \quad \forall t\ge 0.\end{aligned} \tag{4.6}$$

Next, consider the difference $\theta = \tilde{u}-u-U$, with U the solution of the linearized equation (4.1). Obviously,

$$\theta(0)=\theta(0)=0, \qquad \theta_t(0)=\theta_t(0)=0; \tag{4.7}$$

and

$$\theta_{tt}-\Delta\theta_t-\Delta\theta = -\big[f(\tilde{u},\tilde{u}_t)-f(u,u_t)-f_1'(u,u_t)U-f_2'(u,u_t)U_t\big] = h, \tag{4.8}$$

where $h=-[f(\tilde{u},\tilde{u}_t)-f(u,u_t)-f_1'(u,u_t)U-f_2'(u,u_t)U_t]$.

By the mean value theorem, we have

$$\begin{aligned}h &= -\big[f_1'\big(u+\vartheta_3(\tilde{u}-u),\tilde{u}_t\big)-f_1'(u,\tilde{u}_t)+f_1'(u,\tilde{u}_t)-f_1'(u,u_t)\big](\tilde{u}-u)\\ &\quad -\big[f_2'\big(u,u_t+\vartheta_4(\tilde{u}_t-u_t)\big)-f_2'(u,u_t)\big](\tilde{u}_t-u_t)\\ &\quad +f_1'(u,u_t)\theta+f_2'(u,u_t)\theta_t,\end{aligned} \tag{4.9}$$

where $\vartheta_i\in(0,1)$, $i=3,4$.

Taking the scalar product of each side of (4.8) with θ_t in $L^2(\Omega)$ and by (4.7), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\theta_t\|^2+\|\nabla\theta\|^2\big)+\|\nabla\theta_t\|^2\\ &\quad = (h,\theta_t)\\ &\quad \big(\text{by assumptions (2.6), (2.7)}\big)\\ &\quad \le \int_\Omega |\theta_t|\big(C_1\big(1+|\tilde{u}|^3+|u|^3\big)\vartheta_3|\tilde{u}-u|^2 + C_2\big(1+|u|^3\big)(\tilde{u}_t-u_t)(\tilde{u}-u)\\ &\qquad + C_3\big(1+|u|^3\big)\vartheta_4|\tilde{u}_t-u_t|^2 + C_4\big(1+|u|^4\big)|\theta|+\delta|\theta_t|\big)\,dx,\end{aligned} \tag{4.10}$$

where $\vartheta_3,\vartheta_4\in(0,1)$.

We will deal with every term in the right-hand side of inequality (4.10); we have

$$\begin{aligned}&\int_\Omega |\theta_t|C_1\big(1+|\tilde{u}|^3+|u|^3\big)\vartheta_3|\tilde{u}-u|^2\,dx\\ &\quad \le C\bigg(\int_\Omega\big(1+|\tilde{u}|+|u|^3\big)^2\,dx\bigg)^{1/2}\bigg(\int_\Omega\big(|\tilde{u}-u|^2|\theta_t|\big)^2\,dx\bigg)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\int_\Omega |\tilde{u}-u|^4|\theta_t|^2\,dx\right)^{1/2}\\
&\le C\left(\int_\Omega \left[|\tilde{u}-u|^4\right]^{3/2}dx\right)^{1/3}\left(\int_\Omega \left(|\theta_t|^2\right)^3dx\right)^{1/6}\\
&\le \frac{1}{4}\|\nabla\theta_t\|^2 + C\|\nabla\tilde{u}-\nabla u\|^4; \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega \left(C_2\left(1+|u|^3\right)(\tilde{u}_t-u_t)(\tilde{u}-u)|\theta_t|\right)dx\\
&\le C\left(\int_\Omega \left(1+|u|^3\right)^2dx\right)^{1/2}\left(\int_\Omega \left(|\tilde{u}_t-u_t||\tilde{u}-u||\theta_t|\right)^2dx\right)^{1/2}\\
&\le C\left(\int_\Omega \left(|\theta_t|^2\right)^3dx\right)^{1/6}\left(\int_\Omega \left(|\tilde{u}_t-u_t|^2|\tilde{u}-u|^2\right)^{3/2}dx\right)^{1/3}\\
&\le C\|\theta_t\|_{L^6}^2 + \|\tilde{u}_t-u_t\|_{L^6}^2\|\tilde{u}-u\|_{L^6}^2\\
&\le \frac{1}{4}\|\nabla\theta_t\|^2 + C\|\nabla\tilde{u}_t-\nabla u_t\|^2\|\nabla\tilde{u}-\nabla u\|^2; \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega \left(1+|u|^3\right)\vartheta_4|\tilde{u}_t-u_t|^2|\theta_t|\,dx\\
&\le C\left(\int_\Omega \left(1+|u|^3\right)^2dx\right)^{1/2}\left(\int_\Omega \left(|\tilde{u}_t-u_t|^2|\theta_t|\right)^2dx\right)^{1/2}\\
&\le C\left(\int_\Omega |\tilde{u}_t-u_t|^4|\theta_t|^2\,dx\right)^{1/2}\\
&\le C\left(\int_\Omega \left(|\tilde{u}_t-u_t|^4\right)^{3/2}dx\right)^{1/3}\left(\int_\Omega \left(|\theta_t|^2\right)^3dx\right)^{1/6}\\
&\le C\|\tilde{u}_t-u_t\|_{L^6}^4 + \left\|\theta_t^N\right\|_{L^6}^2\\
&\le C\|\nabla\tilde{u}_t-\nabla u_t\|^4 + \frac{1}{4}\|\nabla\theta_t\|^2; \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
&\int_\Omega C_4\left(1+|u|^4\right)|\theta||\theta_t|\,dx\\
&\le C_4\left(\int_\Omega \left(1+|u|^4\right)^{3/2}dx\right)^{2/3}\left(\int_\Omega |\theta|^3|\theta_t|^3\,dx\right)^{1/3}\\
&\le C\left(\int_\Omega |\theta|^6\,dx\right)^{1/6}\left(\int_\Omega |\theta_t|^6\,dx\right)^{1/6}\\
&\le C\frac{1}{4}\|\nabla\theta\|_2^2 + \frac{1}{4}\|\nabla\theta_t\|_2^2. \qquad (4.14)
\end{aligned}
$$

Plugging (4.11)-(4.14) into (4.10), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\theta_t\|^2 + \|\nabla\theta\|^2\right)\\
&\le C_1\left(\|\theta_t\|^2 + \|\nabla\theta\|^2\right)\\
&\quad + C_2\left(\|\nabla\tilde{u}-\nabla u\|^4 + \|\nabla\tilde{u}_t-\nabla u_t\|^2\|\nabla\tilde{u}-\nabla u\|^2 + \|\nabla\tilde{u}_t-\nabla u_t\|^4\right),
\end{aligned}
$$

where $C_1 > 0$, $C_2 > 0$. By the Gronwall inequality and the estimates (4.4), (4.5), (4.6), we obtain

$$\begin{aligned}\|\theta_t\|^2 + \|\nabla\theta\|^2 &\leq \frac{C_2}{C_1}\exp^{C_1 t}\int_0^t \big(\|\nabla\tilde{u}(s) - \nabla u(s)\|^4 \\ &\quad + \|\tilde{u}_t(s) - u_t(s)\|^2\|\nabla\tilde{u}(s) - \nabla u(s)\|^2 + \|\nabla\tilde{u}_t(s) - \nabla u_t(s)\|^4\big)\,ds \\ &\leq C_3\big(|\eta|^2 + \|\xi\|^2\big)^2 \cdot \exp^{C_4 t}, \quad \forall t \geq 0,\end{aligned}$$

where $C_3 > 0$, $C_4 > 0$, that is,

$$\|\tilde{\psi}(t) - \psi(t) - U(t)\|^2_{H_0^1\times L^2} \leq C_3\big(\|(\xi,\eta)^T\|^2_{H_0^1\times L^2}\big)^2 \cdot \exp^{C_4 t} \quad \forall t \geq 0.$$

Therefore,

$$\begin{aligned}&\frac{\|\tilde{\psi}(t) - \psi(t) - U(t)\|^2_{H_0^1\times L^2}}{\|(\xi,\eta)^T\|^2_{H_0^1\times L^2}} \\ &\quad \leq C_4\|(\xi,\eta)^T\|^2_{H_0^1\times L^2} \cdot \exp^{C_4 t} \\ &\quad \to 0 \quad \text{as } (\xi,\eta)^T \to 0 \text{ in } D(\mathbb{L}).\end{aligned} \tag{4.15}$$

Since $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ is dense in $D(\mathbb{L})$, (4.15) is true for solutions $\tilde{\psi}(t), \psi(t), U(t) \in \mathcal{H}$.

Next, to prove the decomposition (4.1), one has the following.

Lemma 4.4 $L \cdot ((\xi,\eta)^T) = (U, U_t) = (U_1, U_{1t}) + (U_2, U_{2t}) = C \cdot ((\xi,\eta)^T) + K \cdot ((\xi,\eta)^T)$ (*where the operator C is contractive and K is compact as in Lemma* 4.1), *separately, satisfying the following equations*:

$$\begin{cases} U_{1tt} - \Delta U_{1t} - \Delta U_1 = 0, \\ U_1(x,t)|_{\partial\Omega} = 0, \\ (U_1(x,0), U_{1t}(x,0))^T = (\xi,\eta)^T; \end{cases} \tag{4.16}$$

$$\begin{cases} U_{2tt} - \Delta U_{2t} - \Delta U_2 + f_1'(u,u_t)U_2 + f_2'(u,u_t)U_{2t} = 0, \\ U_2(x,t)|_{\partial\Omega} = 0, \\ (U_2(x,0), U_{2t}(x,0))^T = (0,0)^T. \end{cases} \tag{4.17}$$

Proof For (U_1, U_{1t}), we set

$$\zeta(t) = U_{1t}(t) + \epsilon U_1(t).$$

Here $\epsilon \in (0, \epsilon_0)$, for some $\epsilon_0 \leq 1$ to be determined later. Testing equation (4.16) with ζ yields

$$\frac{1}{2}\frac{d}{dt}E + \epsilon(1-\epsilon)\|A^{1/2}U_1\|^2 + \|A^{1/2}\zeta\|^2 = \epsilon\|\zeta\|^2 - \epsilon^2\langle U_1, \zeta\rangle, \tag{4.18}$$

where the energy functional E is given as

$$E = (1-\epsilon)\|A^{1/2}U_1(t)\|^2 + \|\zeta(t)\|^2.$$

We have the inequality

$$-\epsilon^2\langle U_1,\zeta\rangle \le \frac{\epsilon^3}{4}\|A^{1/2}U_1\|^2+\epsilon\|\zeta\|^2.$$

Inserting it into (4.18), one gets

$$\frac{d}{dt}E+2\epsilon\left(1-\epsilon-\frac{\epsilon^2}{4}\right)\|A^{1/2}U_1\|^2+(2\lambda_1-4\epsilon)\|\zeta\|^2\le 0, \tag{4.19}$$

so, for ϵ_0 small enough,

$$\frac{d}{dt}E+\epsilon\|A^{1/2}U_1\|^2+(2\lambda_1-4\epsilon)\|\zeta\|^2\le 0, \tag{4.20}$$

which implies that $(U_1,U_{1t})=C\cdot((\xi,\eta)^T)$ is contractive.

Furthermore, multiplying (4.16) by $A^\sigma U_{1t}+\epsilon A^\sigma U_1$ as in Lemma 3.5, we have

$$\|C\cdot((\xi,\eta)^T)\|^2_{\mathcal{H}^\sigma}=\|(U_1,U_{1t})\|^2_{\mathcal{H}^\sigma}\le J_{B,\sigma}\quad\text{for all } t\ge 0 \text{ and } \xi_u(0)\in B.$$

Similarly, multiplying (4.1) by $A^\sigma U_t+\epsilon A^\sigma U$, we have

$$\|L\cdot(\xi,\eta)^T\|^2_{\mathcal{H}^\sigma}=\|(U,U_t)\|^2_{\mathcal{H}^\sigma}\le J_{B,\sigma}\quad\text{for all } t\ge 0 \text{ and } \xi_u(0)\in B.$$

Thus,

$$\|K\cdot(\xi,\eta)^T\|^2_{\mathcal{H}^\sigma}=\|(U_2,U_{2t})\|^2_{\mathcal{H}^\sigma}=\|(U,U_t)-(U_1,U_{1t})\|^2_{\mathcal{H}^\sigma}\le J_{B,\sigma}$$
$$\text{for all } t\ge 0 \text{ and } \xi_u(0)\in B,$$

which implies that $K\cdot(\xi,\eta)^T=(U_2,U_{2t})$ is compact and the proof of Lemma 4.4 is finished. □

We also need the following Lipschitz continuity of $\{S(t)\}$.

Lemma 4.5 *The mapping* $(t,\xi_u(0))\mapsto\xi_u(t)$ *is Lipschitz continuous on* $[0,t^*]\times\mathcal{B}_\sigma$, *where the absorbing set* $\mathcal{B}_\sigma$ *is given in Theorem* 3.1.

Proof For any $\xi_{u_i}(0)\in\mathcal{B}_\sigma$, $t_i\in[0,t^*]$, $i=1,2$, we have

$$\begin{aligned}&\|S(t_1)\xi_{u_1}(0)-S(t_2)\xi_{u_2}(0)\|_{\mathcal{H}}\\&\quad\le\|S(t_1)\xi_{u_1}(0)-S(t_1)\xi_{u_2}(0)\|_{\mathcal{H}}+\|S(t_1)\xi_{u_2}(0)-S(t_2)\xi_{u_2}(0)\|_{\mathcal{H}}.\end{aligned}$$

The first term has been estimated in (4.4); for the second term, we have

$$\begin{aligned}\|S(t_1)\xi_{u_2}(0)-S(t_2)\xi_{u_2}(0)\|_{\mathcal{H}}&\le\left|\int_{t_1}^{t_2}\left\|\frac{d}{dt}\big(S(t)\xi_{u_2}(0)\big)\right\|_{\mathcal{H}}\right|\\&\le\left\|\frac{d}{dt}\big(S(t)\xi_{u_2}(0)\big)\right\|_{L^\infty(0,t^*;\mathcal{H})}\cdot|t_1-t_2|\end{aligned}$$

and we note that $\|\frac{d}{dt}(S(t)\xi_{u_2}(0))\|_{L^\infty(0,t^*;\mathcal{H})}$ can be estimated as in [6] with the assumptions (2.4)-(2.6). □

Therefore, applying the abstract results devised in [12] to Lemmas 4.4, 4.5, we obtain the exponential attractor for the original semigroup $\{S(t)\}_{t\geq 0}$ in the space $\mathcal{H}$.

Also applying the same argument as in [6] with the assumptions (2.4)-(2.6), we can obtain the same estimates about $\|\nabla u_t(t)\|$ and $u_{tt}(t)$. Thus, similar to Theorem 4.13 in [8], we indeed have the following results (with a stronger attraction for the second component $u_t(t)$ of $(u(t), u_t(t))$).

Theorem 4.1 *Let the assumptions of Theorem* 3.1 *hold, then there exists a set* $\mathcal{E}$, *such that*

(i) $\mathcal{E}$ *is compact in* $H_0^1(\Omega) \times H_0^1(\Omega)$ *and positively invariant, i.e.,* $S(t)\mathcal{E} \subset \mathcal{E}$ *for all* $t \geq 0$;

(ii) $\dim_F(\mathcal{E}, H_0^1(\Omega) \times H_0^1(\Omega)) < \infty$;

(iii) *there exist a constant* $\alpha > 0$ *and an increasing function* $Q : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, for any subset* $B \subset \mathcal{H}$ *with* $\|B\|_{\mathcal{H}} \leq R$,

$$\operatorname{dist}_{H_0^1(\Omega)\times H_0^1(\Omega)}\bigl(S(t)B, \mathcal{E}\bigr) \leq Q(R)\frac{1}{\sqrt{t}}e^{-\alpha t} \quad \textit{for all } t \geq 0;$$

(iv) $\mathcal{E} = (\phi(x), 0) + \mathcal{E}_\sigma$, *with* $\mathcal{E}_\sigma$ *bounded in* $H_0^1(\Omega) \cap H^{1+\sigma}(\Omega) \times H_0^1(\Omega)$ $(\sigma < \frac{1}{2})$, *where* $\phi(x)$ *is the unique solution of* (3.1).

Competing interests

The author declares that they have no competing interests.

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