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# A note on the IBVP for wave equations with dynamic boundary conditions

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**Abstract**

In this paper, we investigate the controllability on the IBVP for a class of wave equations with dynamic boundary conditions by the HUM method as well as the wellposedness for the related back-ward problems. After proving a new observability inequality, we establish new wellposedness and controllability theorems for the IBVP.

**Keywords:** Wentzell boundary condition; wave equation; wellposedness; controllability

## 1 Introduction

In this paper, we consider the exact boundary controllability on the IBVP for wave equation with dynamic boundary condition as follows:

$$\begin{cases} \phi'' - \Delta\phi + f(\phi) = 0, & (x,t) \in Q = \Omega \times (0,T), \\ -\Delta_T\phi + \frac{\partial\phi}{\partial\nu} = v_1, & \text{on } \Gamma_1 \times (0,T), \\ \phi = 0, & \text{on } \Gamma_0 \times (0,T), \\ \phi(0,x) = \phi_0, \qquad \phi_t(0,x) = \phi_1, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, and $\Delta_T$ is tangential Laplace operator. The boundary condition on $\Gamma_1$ is called the static Wentzell boundary condition and the dynamic Wentzell boundary condition is

$$\phi'' - \Delta_T\phi + \frac{\partial\phi}{\partial\nu} = v_1, \quad \text{on } \Gamma_1 \times (0,T). \tag{1.2}$$

The system models an elastic body's transverse vibration. For details, please see the paper of Lemrabet [1]. In [1–7] and the references therein, one can find more details as regards dynamic boundary conditions. Moreover, Heminna [3] gives the controllability for elasticity system with two controls: both tangential and normal, under the assumption of the wellposedness for the backward system, which is a key assumption for getting controllability. In this paper, we establish first of all the wellposedness theorem for back-ward systems based on the transposition method (*cf.* [8]) and then obtain the controllability on the IBVP for the wave equation above by using the method of HUM.

## 2 Boundary controllability for Wentzell systems

For simplicity, we write

$$V = H^1_{\Gamma_0}(\Omega) := \bigl\{v \in H^1(\Omega) : v|_{\Gamma_1} \in H^1(\Gamma_1), v|_{\Gamma_0} = 0\bigr\}, \qquad \mathcal{H} = V \times L^2(\Omega),$$

with the norm

$$\begin{aligned}
&\|u\|_V^2 = \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla_T u\|^2_{L^2(\Gamma_1)}, \\
&\bigl\|(u,v)\bigr\|^2_{\mathcal{H}} = \|u\|_V^2 + \|v\|^2_{L^2(\Omega)}.
\end{aligned}$$

We study the controllability under the geometric condition:

$$\exists x_0 \in \mathbb{R}^n, \quad (x - x_0)\cdot \nu \le 0, \quad \text{on } \Gamma_0.$$

Take a look at the linear homogeneous system first,

$$\begin{cases}
u'' - \Delta u = 0, & (x,t) \in Q = \Omega \times (0,T), \\
-\Delta_T u + \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_1 \times (0,T), \\
u = 0, & \text{on } \Gamma_0 \times (0,T), \\
u(0,x) = u_0, \qquad u_t(0,x) = u_1, & x \in \Omega.
\end{cases} \tag{2.1}$$

The wellposedness for the problem (2.1) is not hard to see. Define an operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ by

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \end{pmatrix},$$

with

$$\begin{aligned}
&D(\mathcal{A}) := \bigl\{(u,v) \in \mathcal{H} : \Delta u \in L^2(\Omega), v \in V, \partial_\nu u - \Delta_T u = 0\bigr\}, \\
&D\bigl(\mathcal{A}^2\bigr) = \bigl\{(u,v)^T \in D(\mathcal{A}) : \mathcal{A}(u,v)^T \in \mathcal{H}\bigr\}.
\end{aligned}$$

Write

$$E(t) := \frac{1}{2}\int_\Omega \bigl(|\nabla u|^2 + \bigl|u'\bigr|^2\bigr)\,dx + \frac{1}{2}\int_{\Gamma_1} |\nabla_T u|^2\,ds.$$

Then it is clear that $E(t) = E(0)$.

**Lemma 2.1** (Observability inequality) *For $T > 2R$,*

$$E(0) \le C\int_{\Sigma_1} \bigl(u'^2 + u^2 + |\nabla_T u|^2 + |\Delta_T u|^2\bigr)\,ds\,dt, \tag{2.2}$$

*where $R = \max_{x\in\bar{\Omega}} |x - x_0|$, $\Sigma_1 = (0,T) \times \Gamma_1$.*

*Proof* Multiply the equation with the radial multiplier $(x-x_0)\cdot\nabla u+\frac{n-1}{2}u$ and integrate by parts in $Q$. Then we obtain

$$\begin{aligned}
&\frac{1}{2}\int_Q\left(\left|u'\right|^2+|\nabla u|^2\right)dx\,dt+\frac{1}{2}\int_{\Sigma_1}|\nabla_T u|^2\,ds\,dt+\left|\left\langle u',(x-x_0)\cdot\nabla u+\frac{n-1}{2}u\right\rangle\right|_0^T\\
&\quad=\frac{1}{2}\int_{\Sigma_1}(x-x_0)\cdot\nu\left|u'\right|^2ds\,dt+\int_{\Sigma_1}\frac{\partial u}{\partial\nu}(x-x_0)\cdot\nabla u\,ds\,dt\\
&\qquad+\frac{n-1}{2}\int_{\Sigma_1}u\frac{\partial u}{\partial\nu}\,ds\,dt+\frac{1}{2}\int_0^T\int_{\Gamma_0}(x-x_0)\cdot\nu\left|\frac{\partial u}{\partial\nu}\right|^2ds\,dt\\
&\qquad+\frac{1}{2}\int_{\Sigma_1}\left(|\nabla_T u|^2-(x-x_0)\cdot\nu|\nabla u|^2\right)ds\,dt.
\end{aligned}\tag{2.3}$$

It is easy to see that

$$\left|\left\langle u',(x-x_0)\cdot\nabla u+\frac{n-1}{2}u\right\rangle\right|_0^T\le 2RE(0)+c(T)\int_{\Sigma_1}\left(u^2+u'^2\right)ds\,dt.$$

Combining with the geometric condition $(x-x_0)\cdot\nu\le 0$ on $\Gamma_0$, we deduce from (2.3) and (2.1) that

$$\begin{aligned}
(T-2R)E_0&\le c_1\int_{\Sigma_1}\left|u'\right|^2ds\,dt+\int_{\Sigma_1}\frac{\partial u}{\partial\nu}(x-x_0)\cdot\nabla u\,ds\,dt\\
&\quad+c(T)\int_{\Sigma_1}u^2\,ds\,dt+\frac{n-1}{2}\int_{\Sigma_1}u\frac{\partial u}{\partial\nu}\,ds\,dt+\frac{1}{2}\int_{\Sigma_1}|\nabla_T u|^2\,ds\,dt\\
&\le c\int_{\Sigma_1}\left(\left|u'\right|^2+|\Delta_T u|^2+u^2+|\nabla_T u|^2\right)ds\,dt.
\end{aligned}$$

So, the observability inequality (2.2) holds. □

The observability inequality (2.2) enables us to define the following norm:

$$\left\|(u_0,u_1)\right\|_F^2:=\int_{\Sigma_1}\left(\left|u'\right|^2+|\Delta_T u|^2+u^2+|\nabla_T u|^2\right)ds\,dt,$$

and the corresponding inner product

$$\left\langle(u_0,u_1),(v_0,v_1)\right\rangle_F:=\int_{\Sigma_1}\left(u'v'+\Delta_T u\Delta_T v+uv+\nabla_T u\nabla_T v\right)ds\,dt,$$

where $u$ (or $v$) is the solution of (2.1) with initial data $(u_0,u_1)$ (or $(v_0,v_1)$). Let

$$F:=\overline{\left\{(u_0,u_1)\in C^\infty(\bar\Omega)\times C^\infty(\bar\Omega):\partial_\nu u_0-\Delta_T u_0=0\right\}}^{\|\cdot\|_F}.\tag{2.4}$$

Then $(F,\langle\cdot,\cdot\rangle_F)$ is a Hilbert space.

Now we consider the wellposedness for the linear backward problem

$$\begin{cases}\phi''-\Delta\phi=0, & \text{in } Q,\\ \frac{\partial\phi}{\partial\nu}-\Delta_T\phi=v, & \text{on } \Gamma_1\times(0,T),\\ \phi=0, & \text{on } \Gamma_0\times(0,T),\end{cases}\tag{2.5}$$

with terminal data

$$\phi(T) = \phi_0, \qquad \phi'(T) = \phi_1, \quad \text{in } \Omega, \tag{2.6}$$

where

$$\nu(x,t) = -\partial_t u' + \Delta_T(\Delta_T u) - \Delta_T u + u$$

and $\partial_t$ is taken in the following sense:

$$\langle -\partial_t u', \psi\rangle = \langle u', \psi'\rangle, \quad \forall \psi \in H^1\bigl(0, T; L^2(\Omega)\bigr).$$

For every

$$\bigl(\theta, \theta'\bigr) \in C\bigl((0, T+\varepsilon); D\bigl(\mathcal{A}^2\bigr)\bigr) \cap C^1\bigl((0, T+\varepsilon); D(\mathcal{A})\bigr) \cap C^2\bigl((0, T+\varepsilon); \mathcal{H}\bigr)$$

with $\theta(0) = \theta'(0) = 0$, we say $\phi \in L^\infty(0, T; V')$ is the solution of (2.5)-(2.6) if it satisfies the following equality:

$$\begin{aligned}
&\int_Q \phi f\, dQ + \bigl\langle \phi'(T), \theta(T)\bigr\rangle_{F',F} - \bigl\langle \phi(T), \theta'(T)\bigr\rangle_{F',F} \\
&\quad = -\int_{\Sigma_1} \bigl(\nabla_T u \nabla_T \theta + \Delta_T u \Delta_T \theta + u'\theta' + u\theta\bigr)\, ds\, dt,
\end{aligned} \tag{2.7}$$

where

$$f = \theta'' - \Delta\theta \in L^1(0, T; V).$$

It is clear that $\theta$ satisfies

$$\begin{cases}
\theta'' - \Delta\theta = f, & \text{in } Q, \\
\frac{\partial\theta}{\partial\nu} - \Delta_T\theta = 0, & \text{on } \Gamma_1, \\
\theta = 0, & \text{on } \Gamma_0, \\
\theta(0) = 0, \quad \theta'(0) = 0, & \text{in } \Omega.
\end{cases} \tag{2.8}$$

**Theorem 2.2** *In the sense of* (2.7), *the problem* (2.5)-(2.6) *has a unique solution* $\phi$ *satisfying*

$$\phi \in L^\infty\bigl(0, T; V'\bigr).$$

*Proof* First of all, we give the energy estimate for the nonhomogeneous system (2.8).

For the general energy (the low-order energy), since

$$\frac{1}{2}\frac{d}{dt}\biggl(\int_\Omega \theta'^2 + |\nabla\theta|^2\, dx + \int_{\Gamma_1} |\nabla_T\theta|^2\, ds\biggr) = \int_\Omega f\theta_t\, dx$$

and

$$E(T) = E(t) + \int_t^T \int_\Omega f\theta'\, dx\, dt,$$

we have

$$E(t) \le C_T\big(E(T) + \|f\|^2_{L^2(0,T;L^2(\Omega))}\big), \quad \forall t \in (0,T).$$

For the high-order energy, we have

$$E_1(t) = \frac{1}{2}\int_\Omega \left|\nabla\theta'\right|^2 + |\Delta\theta|^2\,dx + \frac{1}{2}\int_{\Gamma_1}\left|\nabla_T\theta'\right|^2\,ds$$

and

$$E_1(T) = E_1(t) + \int_t^T\int_\Omega f\Delta\theta'\,dx\,dt.$$

Hence,

$$\begin{aligned}
E_1(t) &= E_1(T) + \int_t^T\int_{\partial\Omega} f\frac{\partial\theta'}{\partial\nu}\,ds\,dt - \int_t^T\int_\Omega \nabla f\nabla\theta'\,dx\,dt \\
&= E_1(T) + \int_t^T\int_{\Gamma_1} f\Delta_T\theta'\,ds\,dt - \int_t^T\int_\Omega \nabla f\nabla\theta'\,dx\,dt \\
&\le E_1(T) + \int_t^T\left(\int_{\Gamma_1}\|\nabla_T f\|^2\,ds\right)^{\frac{1}{2}}\left(\int_{\Gamma_1}\left|\nabla_T\theta'\right|^2\,ds\right)^{\frac{1}{2}}dt \\
&\quad + \int_t^T\left(\int_\Omega|\nabla f|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega\left|\nabla\theta'\right|^2\,dx\right)^{\frac{1}{2}}dt \\
&\le E_1(T) + \left\|E_1(t)\right\|^{\frac{1}{2}}_{L^\infty(0,T)}\|f\|_{L^1(0,T;V)},
\end{aligned}$$

which implies that

$$E_1(t) \le C\big(E_1(T) + \|f\|^2_{L^1(0,T;V)}\big), \quad 0 \le t \le T.$$

Let $\theta = \theta_1 + \theta_2$, where $\theta_1$ satisfies

$$\begin{cases}
\theta_1'' - \Delta\theta_1 = 0, & \text{in } Q, \\
\frac{\partial\theta_1}{\partial\nu} - \Delta_T\theta_1 = 0, & \text{on } \Sigma_1, \\
\theta_1 = 0, & \text{on } \Sigma_0, \\
\theta_1(T) = \theta(T), \qquad \theta_1'(T) = \theta'(T), & \text{in } \Omega,
\end{cases}$$

and $\theta_2$ satisfies

$$\begin{cases}
\theta_2'' - \Delta\theta_2 = f, & \text{in } Q, \\
\frac{\partial\theta_2}{\partial\nu} - \Delta_T\theta_2 = 0, & \text{on } \Sigma_1, \\
\theta_2 = 0, & \text{on } \Sigma_0, \\
\theta_2(T) = 0, \qquad \theta_2'(T) = 0, & \text{in } \Omega.
\end{cases}$$

Let

$$L\big(\theta(T),\theta'(T),f\big) = \int_{\Sigma_1}(\nabla_T u\nabla_T\theta + \Delta_T u\Delta_T\theta + u_t\theta_t d + u\theta)\,ds\,dt.$$

Then we obtain

$$\begin{aligned}
&L\bigl(\theta(T),\theta'(T),f\bigr)\\
&\quad=\int_{\Sigma_1}(\nabla_T u\nabla_T\theta+\Delta_T u\Delta_T\theta+u_t\theta_t+u\theta)\,ds\,dt\\
&\quad\le\int_{\Sigma_1}\bigl(\nabla_T u\nabla_T\theta_1+\Delta_T u\Delta_T\theta_1+u_t\theta_{1t}+u\theta_1\\
&\qquad+\nabla_T u\nabla_T\theta_2+\Delta_T u\Delta_T\theta_2+u'\theta_2'+u\theta_2\bigr)\,ds\,dt\\
&\quad\le C\bigl(\bigl\|\bigl\{\theta(T),\theta(T)\bigr\}\bigr\|_F^2+\|f\|_{L^1(0,T;V)}^2\bigr)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, $L:F\times L^1(0,T;V)\to L^\infty(0,T;V')\times F'$ is a bounded operator. So $\exists\phi\in L^\infty(0,T;V')$, $(\rho_1,-\rho_0)\in F'$ such that

$$\begin{aligned}
&\int_Q\phi f\,dx\,dt-\bigl\langle\rho_1,\theta(T)\bigr\rangle+\bigl\langle\rho_0,\theta'(T)\bigr\rangle\\
&\quad=\int_{\Sigma_1}\nabla_T u\nabla_T\theta+\Delta_T u\Delta_T\theta+u'\theta'+u\theta\,ds\,dt,
\end{aligned}$$

where $\int_Q\phi f\,dx\,dt$ means $\langle\cdot,\cdot\rangle_{L^\infty(0,T;V'),L^1(0,T;H^1(\Omega))}$. Next, we prove that

$$\phi(T)=\rho_0,\qquad\phi'(T)=\rho_1.$$

Let $\lambda$ be the eigenvalue for the $\Delta$ operator with mixed Wentzell, Dirichlet boundary conditions and $m$ be the corresponding eigenvector. The existence of eigenvalue for the $\Delta$ operator with mixed Wentzell, Dirichlet boundary condition is based on the fact that $\Delta^{-1}:L^2(\Omega)\to V$ is a compact operator. That is,

$$\begin{cases}-\Delta m=\lambda m, & \text{in }\Omega,\\ \frac{\partial m}{\partial\nu}-\Delta_T m=0, & \text{on }\Gamma_1,\\ m=0, & \text{on }\Gamma_0.\end{cases}$$

Set $f:=g(t)m$, where $g$ is a smooth function in $[0,T+\varepsilon]$, and let $\theta:=h(t)m$. Then

$$\begin{cases}h''+\lambda h=g,\\ h(0)=0,\qquad h'(0)=0.\end{cases}\tag{2.9}$$

**Claim** *$\exists g=g_0$ such that*

$$h(T)=h'(T)=0,\qquad h''(T)\neq0.$$

*If this is true, then*

$$\begin{aligned}
&\int_Q\phi g_0(t)m\,dx\,dt-\bigl\langle\rho_1,h(T)m\bigr\rangle+\bigl\langle\rho_0,h'(T)m\bigr\rangle\\
&\quad=\int_{\Sigma_1}\bigl(\Delta_T u\Delta_T m-u''m+\nabla_T u\nabla_T m+mu\bigr)h(t)\,ds\,dt.
\end{aligned}$$

*Since $h'' + \lambda h = g_0$, we have*

$$\begin{aligned}&\int_0^T \langle \phi'' + \lambda\phi, m\rangle h(t)\,dt + \langle \phi(T), m\rangle h'(T) - \langle \phi'(T), m\rangle h(T) + \langle \rho_1, m\rangle h(T) - \langle \rho_0, m\rangle h'(T)\\ &\quad = \int_{\Sigma_1} \Delta_T u \Delta_T m h(t) - u'' m h(t) + \nabla_T u \nabla_T m h(t) + u m h(t)\,ds\,dt. \end{aligned} \tag{2.10}$$

*Differentiate* (2.10) *with respect to $T$, we get*

$$\begin{aligned}&\langle \phi'' + \lambda\phi, m\rangle h(T) + \langle \phi(T), m\rangle h''(T) + \langle \phi'(T), m\rangle h'(T) - \langle \phi''(T), m\rangle h(T)\\ &\qquad - \langle \phi'(T), m\rangle h'(T) + \langle \rho_1, m\rangle h'(T) - \langle \rho_0, m\rangle h''(T)\\ &\quad = \int_{\Gamma_1} \bigl(\Delta_T u \Delta_T m - u'' m + \nabla_T u \nabla_T m + u m\bigr)\,ds\,h(T).\end{aligned}$$

*Therefore*

$$\langle \phi(T), m\rangle h''(T) - \langle \rho_0, m\rangle h''(T) = 0,$$

*which implies that $\phi(T) = \rho_0$. Similarly, we obtain $\phi'(T) = \rho_1$.*

Now we prove the claim above. Write

$$A := \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, by the Kalman condition [9], we know that (2.9) is controllable. Set $X(t) := (h(t), h'(t))^T$. Then $\exists g_1(s)$, $s \in (0, \frac{T}{2})$, such that $X(\frac{T}{2}) = X_0 \neq 0$. Write

$$g_2\left(s - \frac{T}{2}\right) := B^T e^{A^T(T-s)} w^{-1}\bigl(-e^{A\frac{T}{2}} X_0\bigr),$$

where $w = \int_{\frac{T}{2}}^T e^{A(T-s)} B B^T e^{A^T(T-s)}\,ds$. Then

$$X(t) = e^{A(t-\frac{T}{2})} X_0 + \int_{\frac{T}{2}}^t e^{A(t-s)} B g_2\left(s - \frac{T}{2}\right) ds.$$

Clearly, $X(T) = 0$, $X'(T) \neq 0$. This proof is then complete. □

The following is our exact controllability theorem.

**Theorem 2.3** *Let $T > 2R$ and $F$ be the Hilbert space defined in* (2.4)*. Then for every $(\phi'(0), -\phi(0)) \in F'$, there are $(u_0, u_1) \in F$ and a control function*

$$v(x,t) = -\partial_t u' + \Delta_T(\Delta_T u) - \Delta_T u + u,$$

*where $u$ is the solution to* (2.1)*, such that the solution $\phi(t)$ of system* (2.5) *with initial data $(\phi(0), \phi'(0))$ satisfies*

$$\phi(T) = 0, \qquad \phi'(T) = 0.$$

For the nonlinear case, we assume that $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ satisfies $f(0) = 0$ and the super-linear condition (see [10]):

$$\exists C > 0, p > 1: \quad \left|f'(s)\right| \leq C|s|^{p-1}, \quad \forall s \in R \text{ with } p < \frac{n}{n - \frac{6}{5} + \varepsilon} \text{ if } n \geq 2. \tag{2.11}$$

**Proposition 2.4** *Assume that f satisfies the super-linear condition* (2.11). *Then there exists* $T_0 > 0$ *such that for every* $T > T_0$, *there is a neighborhood* $\omega$ *of* $(0, 0)$ *in* $V \times L^2(\Omega)$ *such that for each* $(\phi_0, \phi_1) \in \omega$, *there exists a control* $v_1 \in H^{-2}(\Gamma)$ *such that the solution to* (1.1) *satisfies*

$$\phi(T) = 0, \qquad \phi'(T) = 0.$$

*Proof* From the results for the nonlinear system of Neumann problems (see [10]), we see that there exists a controllability $v \in L^2(\Gamma_1)$ such that the solution $(\phi, \phi')$ of the following system:

$$\begin{cases} \phi'' - \Delta\phi + f(\phi) = 0, & \text{in } Q, \\ \frac{\partial\phi}{\partial\nu} = v, & \text{on } \Sigma_1, \\ \phi = 0, & \text{on } \Sigma_0, \\ \phi(0) = \phi_0, \qquad \phi'(0) = \phi_1, & \text{in } \Omega, \end{cases}$$

satisfies $(\phi(T), \phi'(T)) = (0, 0)$, and $\phi \in H^\beta(\Omega)$ where $\beta \leq \frac{3}{5} - \varepsilon$. The regularity of $\phi$ for Neumann problems can be found in Theorem 1.4 of [11]. Let $v_1 = v - \Delta_T\phi$. Then

$$\frac{\partial\phi}{\partial\nu} - \Delta_T\phi = v_1,$$

and $v_1 \in H^{-2}(\Gamma_1)$ such that $\phi(T) = 0$, $\phi'(T) = 0$. □

**Remark 2.1** For dynamic Wentzell systems with boundary condition (1.2), we can also prove the results as Theorem 2.3 and Proposition 2.4 by similar arguments.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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