Li and Gu *Boundary Value Problems* (2016) 2016:39
DOI 10.1186/s13661-016-0552-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence of periodic solutions of Boussinesq system

Hengyan Li* and Liuxin Gu

*Correspondence:
hengyanli989@126.com
School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou, 450011, China

Abstract

This paper is devoted to the study of the dynamical behavior of a Boussinesq system, which is a basic model in describing the flame propagation in a gravitationally stratified medium. This system consists of an incompressible Navier-Stokes equation coupled with a reaction-advection-diffusion equation under the Boussinesq approximation. We prove that this system possesses time dependent periodic solutions, bifurcating from a steady solution.

Keywords: Boussinesq system; periodic solution; Hopf bifurcation

1 Introduction and main results

The Boussinesq-type equation of reactive flows is a basic model in describing the flame propagation in a gravitationally stratified medium, and its non-dimensional form is given by

$$\begin{aligned} &U_t + (U\cdot\nabla)U - \nu\triangle U + \nabla P = T\vec{\rho}, \\ &\nabla U = 0, \\ &T_t + (U\cdot\nabla)T - \triangle T = g(T), \end{aligned} \tag{1.1}$$

where $(t,x)\in\mathbf{R}^+\times\mathbf{R}^3$, $U\in\mathbf{R}^3$ is the velocity field, T is the temperature function, $\nu>0$ denotes the Prandtl number, which is the ratio of the kinematic and thermal diffusivities (inverse proportional to the Reynolds number); $P(x,t)\in\mathbf{R}$ denotes the pressure; the vector $\vec{\rho}=\rho\vec{g}$ corresponds to the non-dimensional gravity $\vec{g}$ scaled by the Rayleigh number $\rho>0$. The reaction term of Kolmogorov-Petrovskii-Piskunov (KPP) type is of the form

$$g(T) = \frac{\alpha T(1-T)}{4}.$$

Here, α is the reaction rate. See [1] for the derivation of this model and the related parameters.

When the initial temperature T_0 is identically zero (or constant), the above system reduces to the classical incompressible Navier-Stokes equation:

$$\begin{aligned} &U_t + (U\cdot\nabla)U - \nu\triangle U + \nabla P = 0, \\ &\nabla U = 0. \end{aligned}$$

Since the work of Sattinger [2], Iudovich [3], and Iooss [4] in 1971, the bifurcation of stationary solutions into time-periodic solutions (*i.e.* Hopf bifurcation) of incompressible Navier-Stokes equation has attracted much attention, see [5–10], *etc.* When the linearized operator possesses a continuous spectrum up to the imaginary axis and that a pair of imaginary eigenvalues crosses the imaginary axis, Melcher, *et al.* [11] proved Hopf bifurcation for the vorticity formulation of the incompressible Navier-Stokes equations in $\mathbf{R}^3$. Their work is mainly motivated by the work of Brand *et al.* [12] who studied the Hopf-bifurcation problem and its exchange of stability for a coupled reaction diffusion model in $\mathbf{R}^a$. Inspired by the work of [11, 12], this paper is to establish the corresponding Hopf-bifurcation result for the three-dimensional Boussinesq system.

The Boussinesq system is a very important model in fluid mechanics, which exhibits extremely rich phenomena, for example, Rayleigh-Bénard convection [13–15], geophysical fluid dynamics [16, 17] *etc.* A key problem in the study of the dynamic behavior of Boussinesq system is how to understand the time-periodic solutions, quasi-periodic solutions and traveling waves, *etc.* There were several papers on the existence of time-periodic solutions [18] and traveling waves [19–24] for the Boussinesq system (1.1). To our knowledge, there is no theoretical result on bifurcation analysis for the Boussinesq system on $\mathbf{R}^3$.

In the present paper, we consider the reactive Boussinesq system with external time-independent force in $\mathbf{R}^3$

$$U_t + (U\cdot\nabla)U - \nu\triangle U + \nabla P = T\vec{\rho} + f(x,\epsilon), \tag{1.2}$$

$$\nabla U = 0, \tag{1.3}$$

$$T_t + (U\cdot\nabla)T - \triangle T = g(T) + h(x,\epsilon), \tag{1.4}$$

with initial conditions

$$U(x,0) = U_0(x), \qquad T(x,0) = T_0(x), \tag{1.5}$$

where $f(x,\epsilon)$ and $h(x,\epsilon) \in \mathbf{R}^3 \times \mathbf{R}$ are external time independent forces, which depend smoothly on some parameter ϵ, $g(T) = T(1-T)$. Meanwhile, external forces $f(x,\epsilon)$ and $h(x,\epsilon)$ can be chosen suitably so that $(U_\epsilon(x), T_\epsilon(x), P_\epsilon(x))$ is the solution of the steady Boussinesq system

$$\begin{aligned}
&-\nu\triangle U + (U\cdot\nabla)U + \nabla P = T\vec{\rho} + f(x,\epsilon),\\
&\nabla U = 0,\\
&-\triangle T + (U\cdot\nabla)T = g(T) + h(x,\epsilon),
\end{aligned}$$

with the condition

$$\lim_{|x|\to\infty} U_\epsilon(x) = 0, \qquad \lim_{|x|\to\infty} T_\epsilon(x) = 0.$$

Furthermore, assume that the steady solution $(U_\epsilon(x), T_\epsilon(x), P_\epsilon(x))$ satisfies the following certain decay properties:

(A0) For $p \in (3,4)$ and $s > 2$,

$$\|U_\epsilon(x)\|_{\mathbf{L}_s^p}, \|T_\epsilon(x)\|_{\mathbf{L}_s^p}, \|P_\epsilon(x)\|_{\mathbf{L}_s^p} \le C,$$

where C and $\mathbf{L}_s^p$ denote a positive constant and the weighted Lebesgue space to be specified blow.

We also assume that the solution of system (1.2)-(1.4) has the form

$$U(x,t) = u(x,t) + U_\epsilon(x), \qquad T(x,t) = v(x,t) + T_\epsilon(x), \qquad P_\epsilon(x,t) = p(x,t) + P_\epsilon(x),$$

where

$$\big(U_\epsilon(x), T_\epsilon(x), P_\epsilon(x)\big) = \big(u_\epsilon(x) + \mathbf{c}, \tilde{T}_\epsilon(x) + \mathbf{1}, P_\epsilon(x)\big),$$

and $(u_\epsilon(x), \tilde{T}_\epsilon(x), P_\epsilon(x))$ is the solution of the following steady problem:

$$-\nu\triangle U + (U\cdot\nabla)U + \nabla P = T\overrightarrow{\rho} + f_\epsilon(x),$$
$$\nabla U = 0,$$
$$-\triangle T + (U\cdot\nabla)T = g(T) + h_\epsilon(x),$$

with the condition

$$\lim_{|x|\to\infty} U_\epsilon(x) = 0, \qquad \lim_{|x|\to\infty} T_\epsilon(x) = 0.$$

Then the deviation $(u(x,t), v(x,t), p(x,t))$ from the stationary $(U_\epsilon(x), T_\epsilon(x), P_\epsilon(x))$ satisfies

$$u_t - \nu\triangle u + c\partial_{x_1}u + (u_\epsilon\cdot\nabla)u + (u\cdot\nabla)u_\epsilon + (u\cdot\nabla)u + \nabla p = v\overrightarrow{\rho}, \tag{1.6}$$
$$\nabla u = 0, \tag{1.7}$$
$$v_t - \triangle v - v + \partial_{x_3}v + (u_\epsilon\cdot\nabla)v + (u\cdot\nabla)\tilde{T}_\epsilon + (u\cdot\nabla)v + 2v\tilde{T}_\epsilon + v^2 = 0. \tag{1.8}$$

Here, for general matrices $u = (u_{ij})_{i,j=1,2,3}$,

$$\nabla\cdot u = \left(\sum_{j=1}^{3}\partial_{x_1}u_{1j}, \sum_{j=1}^{3}\partial_{x_1}u_{2j}, \sum_{j=1}^{3}\partial_{x_1}u_{3j}\right)^T.$$

In fact, by the incompressible condition (1.7), it follows that

$$\nabla\cdot\big(uv^T\big) = u\cdot\nabla u + u\nabla\cdot u = u\cdot\nabla u.$$

So, system (1.6)-(1.8) can be rewritten as

$$u_t - \nu\triangle u + c\partial_{x_1}u + \nabla\cdot\big(u_\epsilon u^T\big) + \nabla\cdot\big(uu_\epsilon^T\big) + \nabla\cdot\big(uu^T\big) + \nabla p = v\overrightarrow{\rho}, \tag{1.9}$$

$$\begin{aligned} &v_t - \triangle v - v + \partial_{x_3}v + \nabla\cdot\big(u_\epsilon v^T\big) + \nabla\cdot\big(u\tilde{T}_\epsilon^T\big) + \nabla\cdot\big(uv^T\big) \\ &\quad - u_\epsilon\nabla\cdot v - u\nabla\cdot\tilde{T}_\epsilon - u\nabla\cdot v + 2v\tilde{T}_\epsilon + v^2 = 0, \end{aligned} \tag{1.10}$$

with the incompressible condition

$$\nabla \cdot u = 0.$$

It is convenient to rewrite system (1.9)-(1.10) in the stream function and vorticity formulation in dimensionless form. The vorticity associated with the velocity field u of the fluid is defined by $\omega = \nabla \times u$. Then, using

$$\nabla \times \nabla \cdot (uu^T) = \nabla \cdot (\omega u^T - u\omega^T),$$

we can rewrite system (1.9)-(1.10) in the stream function and the vorticity formulation in dimensionless form

$$\omega_t - \triangle\omega + c\partial_{x_1}\omega - 2\nabla \cdot M(\omega_\epsilon, \omega) - \nabla \cdot M(\omega, \omega) = \vec{\rho}\nabla \nu, \tag{1.11}$$

$$\nu_t - \triangle\nu - \nu + \partial_{x_3}\nu + \nabla \cdot N(u, u_\epsilon, \nu, \tilde{T}_\epsilon) - B(u, \nu, \tilde{T}_\epsilon) = 0, \tag{1.12}$$

where

$$2M(\omega_1, \omega_2) = \omega_2 u_1^T + \omega_1 u_2^T - u_2\omega_1^T - u_1\omega_2^T,$$
$$N(u, u_\epsilon, \nu, \tilde{T}_\epsilon) = u_\epsilon \nu^T + u\tilde{T}_\epsilon^T + u\nu^T,$$
$$B(u, \nu, \tilde{T}_\epsilon) = u_\epsilon \nabla \cdot \nu + u\nabla \cdot \tilde{T}_\epsilon + u\nabla \cdot \nu - 2\nu\tilde{T}_\epsilon - \nu^2.$$

Note that we can assume that $\nabla \cdot \omega = 0$. This is because the space of divergence free vector fields is invariant under the evolution of (1.11).

Denote $\varphi = (\omega, \nu)^T$. Then we can write system (1.11)-(1.12) as the evolution equation of the form

$$\frac{d\varphi}{dt} + \mathcal{N}\varphi = F(\varphi), \tag{1.13}$$

where

$$\mathcal{N} = \begin{pmatrix} -\triangle + c\partial_{x_1} & \vec{\rho} \cdot \nabla \nu \\ 0 & -\triangle - 1 + \partial_{x_3} \end{pmatrix}$$

and

$$F(\varphi) = \begin{pmatrix} 2\nabla \cdot M(\omega_\epsilon, \omega) + \nabla \cdot M(\omega, \omega) \\ B(u, \nu, \tilde{T}_\epsilon) - \nabla \cdot N(u, u_\epsilon, \nu, \tilde{T}_\epsilon) \end{pmatrix}.$$

For $y \in \mathbf{R}^3$, the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are given by

$$\mathcal{F}(u)(y) = \hat{u}(y) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} u(x)e^{-ix\cdot y}\,dx,$$
$$\mathcal{F}^{-1}(\hat{u})(x) = u(x) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} \hat{u}(y)e^{ix\cdot y}\,dy.$$

For $s \geq 0$, $1 \leq p \leq 2$, and $\frac{1}{p} + \frac{1}{q} = 1$, the Fourier transform is a continuous mapping from $\mathbf{L}_s^p$ into $\mathbf{W}_\kappa^q$. Especially, when $p = 2$, the Fourier transform is an isomorphism between $\mathbf{L}_s^p$ and $\mathbf{W}_\kappa^q$.

Since we deal with the problem in the whole space $\mathbf{R}^3$, it is advantageous to apply the Fourier transform to the evolution equation (1.13). Denote $\hat{\varphi} = (\hat{\omega}, \hat{v})^T$. Then

$$\frac{d\hat{\varphi}}{dt} + \hat{\mathcal{N}}\hat{\varphi} = \hat{F}(\hat{\varphi}), \tag{1.14}$$

where

$$\hat{\mathcal{N}} = \begin{pmatrix} |y|^2 + icy_1 & i\vec{\rho} \cdot y \\ 0 & |y|^2 + 1 + iy_3 \end{pmatrix},$$
$$\hat{F}(\hat{\varphi}) = \begin{pmatrix} 2iy \cdot \hat{M}(\hat{\omega}_\epsilon, \hat{\omega}) + iy \cdot \hat{M}(\hat{\omega}, \hat{\omega}) \\ \hat{P}(\hat{u}_\epsilon, \hat{v}, \hat{\hat{T}}_\epsilon) + \hat{B}(\hat{u}, \hat{v}, \hat{\hat{T}}_\epsilon) \end{pmatrix},$$

and

$$\hat{M}(\hat{\omega}_1, \hat{\omega}_2) = \hat{\omega}_2 * \hat{u}_1^T + \hat{\omega}_1 * \hat{u}_2^T - \hat{u}_2 * \hat{\omega}_1^T - \hat{u}_1 * \hat{\omega}_2^T,$$
$$\hat{P}(\hat{u}_\epsilon, \hat{v}, \hat{\hat{T}}_\epsilon) = -iy \cdot (\hat{u}_\epsilon * \hat{v}^T) + i\hat{u}_\epsilon * (y \cdot \hat{v}) - 2\hat{v} * \hat{\hat{T}}_\epsilon,$$
$$\hat{B}(\hat{u}, \hat{v}, \hat{\hat{T}}_\epsilon) = iy \cdot (\hat{u} * \hat{\hat{T}}_\epsilon^T + \hat{u} * \hat{v}^T) + i\hat{u} * (y \cdot \hat{\hat{T}}_\epsilon) + i\hat{u} * (y \cdot \hat{v}) - \hat{v} * \hat{v}.$$

Here, $*$ denotes the convolution. That is,

$$\hat{u} * \hat{v}(y) = \int_{\mathbf{R}^3} \hat{u}(y - x)\hat{v}(x)\, dx,$$

and for general matrices $u = (u_{kj})_{k,j=1,2,3}$,

$$iy \cdot u = i\left(\sum_{j=1}^3 \partial_{x_1} y_j u_{1j}, \sum_{j=1}^3 \partial_{x_1} y_j u_{2j}, \sum_{j=1}^3 \partial_{x_1} y_j u_{3j}\right)^T.$$

To overcome the essential spectrum of operator $\hat{J}$ (defined in (2.3)) up to the imaginary axis, for $3 < p < 4$ and $s > 3(1 - \frac{1}{p})$, we need the following assumption:

(A1) For any $\epsilon \in [\epsilon_c - \epsilon_0, \epsilon_c + \epsilon_0]$, 0 is not an eigenvalue of $\hat{J}$.

(A2) For $\epsilon = \epsilon_c$, the operator $\hat{J}$ has two pair eigenvalues (λ_0^+, μ_0^+) and (λ_0^-, μ_0^-) satisfying

$$\lambda_0^\pm(\epsilon_c) = \mu_0^\pm(\epsilon_c) = \pm i\varpi_c \neq 0, \quad \text{for } \varpi_c > 0, \tag{1.15}$$
$$\frac{d}{d\epsilon}\mathbf{Re}\big(\lambda_0^\pm(\epsilon)\big)\Big|_{\epsilon=\epsilon_c} > 0, \qquad \frac{d}{d\epsilon}\mathbf{Re}\big(\mu_0^\pm(\epsilon)\big)\Big|_{\epsilon=\epsilon_c} > 0. \tag{1.16}$$

(A3) The remaining eigenvalue of $\hat{J}$ is strictly bounded away from the imaginary axis in the left half plane for all $\epsilon \in [\epsilon_c - \epsilon_0, \epsilon_c + \epsilon_0]$.

Here is our main result in this paper.

Theorem 1.1 *Assume that* (A0)-(A3) *hold. Then for* $p \in (3,4)$ *and* $s > 3(1-\frac{1}{p})$, *system* (1.14) *has a time-periodic solution*

$$\hat{\varphi}(y,t) = \bigl(\hat{u}(y,t), \hat{v}(y,t)\bigr) = \left(\sum_{n\in\mathbf{Z}} \hat{u}_n(y)e^{in\varpi t}, \sum_{n\in\mathbf{Z}} \hat{v}_n(y)e^{in\varpi t}\right),$$

with $\epsilon = \epsilon_c + \epsilon_0$, $\epsilon_0 \in (0,\beta)$, $\|(\hat{u},\hat{v})\|_{\mathcal{L}_s^p} = O(\epsilon_0)$, $\varpi - \varpi_c = O(\epsilon_0^2)$.

This paper is organized as follows. In Section 2, we give the basic setting of the problem and derive some priori estimates needed in the proof in next section. The proof of the main result occupies the Section 3.

2 Preliminary and some estimates

We start this section by introducing some notations. Consider the following standard Sobolev space, a spatially weighted Lebesgue space:

$$\mathbf{W}_\kappa^q := \left\{u : \|u\|_\kappa^q := \sum_{|\alpha|\le\kappa} \|D^\alpha u\|_{\mathbf{L}^q}^q < \infty\right\},$$

$$\mathbf{L}_s^p := \left\{u : \|u\|_s^p := \int_{\mathbf{R}^3} \rho^{sp}(x)u^p(x)\,dx < \infty\right\},$$

where the weighted function $\rho(x) = \sqrt{1+|x|^2}$.

To investigate periodic solutions of system (1.2)-(1.4), we also introduce the space

$$\mathbf{X}_s^p := \left\{\hat{u} = (\hat{u}_n)_{n\in\mathbf{Z}} : \|\hat{u}\|_{\mathbf{X}_s^p} := \sum_{n\in\mathbf{Z}} \|\hat{u}_n\|_{\mathbf{L}_s^p} < \infty\right\}$$

and the weighted space

$$\mathcal{L}_s^p = \mathbf{L}_s^p \times \mathbf{L}_{s+1}^p, \qquad \mathcal{X}_s^p = \mathbf{X}_s^p \times \mathbf{X}_{s+1}^p,$$

with norms

$$\|\varphi\|_{\mathcal{L}_s^p} := \|u\|_{\mathbf{L}_s^p} + \|v\|_{\mathbf{L}_{s+1}^p}, \qquad \|\varphi\|_{\mathcal{X}_s^p} := \|u\|_{\mathbf{X}_s^p} + \|v\|_{\mathbf{X}_{s+1}^p},$$

for $\varphi = (u,v)^T \in \mathcal{L}_s^p$ or $\mathcal{X}_s^p$, respectively.

As we known, the vorticity $\omega = \nabla \times u$, where u is the velocity field. By the Biot-Savart law, u is recovered from ω as

$$u(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{(x-y)\times\omega(y)}{|x-y|^3}\,dy.$$

The following estimates are taken from [11], which show the norm relationship $\hat{u}$ with $\hat{\omega}$.

Lemma 2.1 *Let* $p \in [1,+\infty]$. *For* $k = 1,2,3$ *and* $\hat{\omega} \in (\mathbf{L}^p(\mathbf{R}^3))^3$, *there exists a constant* C *such that*

$$\|iy_k\hat{u}\|_{\mathbf{L}^p} \le C\|\hat{\omega}\|_{\mathbf{L}^p}.$$

Furthermore, for every $p \in [1,3)$, $p_1, p_2 \in [1,\infty]$ *with* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$,

$$\|\hat{u}\|_{\mathbf{L}^p} \leq C\bigl(\|\hat{\omega}\|_{\mathbf{L}^{p_1}} + \|\hat{\omega}\|_{\mathbf{L}^p}\bigr).$$

Meanwhile, if $\hat{\omega} \in \mathbf{L}^p(\mathbf{R}^3) \cap \mathbf{L}^{p_1}(\mathbf{R}^3)$, *then* $\hat{u} \in \mathbf{L}^p(\mathbf{R}^3)$ *and the above estimate also holds.*

Then, for the weighted space $\mathcal{L}_s^p$, the following Sobolev embedding holds.

Lemma 2.2 *For* $p_1 \geq p_2$ *and* $s > \frac{a}{p_2} - \frac{a}{p_1}$, *the continuous embedding* $\mathcal{L}_s^{p_1}(\mathbf{R}^a) \subset \mathcal{L}^{p_2}(\mathbf{R}^a)$ *holds.*

Proof Note that $\rho(y) = (1 + |y|^2)^{\frac{1}{2}}$, $y \in \mathbf{R}^a$. By direct computation, we have

$$\begin{aligned}
\bigl\|\rho^{-s}(y)\bigr\|_{\mathbf{L}^{p_3}}^{p_3} &= \int_{\mathbf{R}^a} \frac{dy}{(1+|y|^2)^{\frac{sp_3}{2}}} \\
&= \int_{|y|\leq 1} \frac{dy}{(1+|y|^2)^{\frac{sp_3}{2}}} + \int_{|y|>1} \frac{dy}{(1+|y|^2)^{\frac{sp_3}{2}}} \\
&\leq \int_{|y|\leq 1} \frac{dy}{(1+|y|^2)^{\frac{sp_3}{2}}} + C\int_1^{\infty} \frac{dx}{x^{sp_3-a+1}}, \quad \forall p_3 > 0.
\end{aligned}$$

Hence, for $sp_3 > a$, the above inequality implies that $\|\rho^{-s}(y)\|_{\mathbf{L}^{p_3}}^{p_3}$ is bounded.

Let $\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_3}$. By Hölder's inequality and the above inequality, it follows that for $\forall \varphi \in \mathcal{L}^{p_2}(\mathbf{R}^a)$,

$$\begin{aligned}
\|\varphi\|_{\mathcal{L}^{p_2}(\mathbf{R}^a)} &= \bigl\|\rho^s\varphi\rho^{-s}\bigr\|_{\mathcal{L}^{p_2}(\mathbf{R}^a)} \leq \bigl\|\rho^s\varphi\bigr\|_{\mathcal{L}^{p_1}(\mathbf{R}^a)}\bigl\|\rho^{-s}\bigr\|_{\mathbf{L}^{p_3}(\mathbf{R}^a)} \\
&= \|\varphi\|_{\mathcal{L}_s^{p_1}(\mathbf{R}^a)}\bigl\|\rho^{-s}\bigr\|_{\mathbf{L}^{p_3}(\mathbf{R}^a)} \leq C\|\varphi\|_{\mathcal{L}_s^{p_1}(\mathbf{R}^a)}.
\end{aligned}$$

This completes the proof. □

From Corollary 2.6 in [11], the following result holds.

Lemma 2.3 *Let* $p \in (\frac{3}{2}, +\infty]$. *For any* $\hat{\omega}_1, \hat{\omega}_2 \in \mathbf{L}_s^p$ *and* $s > 3(1 - \frac{1}{p})$, *there exists a positive constant C such that*

$$\bigl\|\hat{M}(\hat{\omega}_1, \hat{\omega}_2)\bigr\|_{\mathbf{L}_s^p} \leq C\|\hat{\omega}_1\|_{\mathbf{L}_s^p}\|\hat{\omega}_2\|_{\mathbf{L}_s^p}.$$

Furthermore, let $p \in (3,4)$. *Then, for* $s > 0$,

$$\bigl\|\hat{M}(\hat{\omega}_1, \hat{\omega}_2)\bigr\|_{\mathbf{L}^\infty} \leq C\|\hat{\omega}_1\|_{\mathbf{L}_s^p}\|\hat{\omega}_2\|_{\mathbf{L}_s^p}.$$

Lemma 2.4 *Let* $p \in (\frac{3}{2}, +\infty]$ *and* $s > 3(1 - \frac{1}{p})$. *Then there exists a constant* $C > 0$ *such that*

$$\bigl\|i\hat{u} * (y \cdot \hat{v})\bigr\|_{\mathbf{L}_s^p} \leq C\|\hat{\omega}\|_{\mathbf{L}_s^p}\|\hat{v}\|_{\mathbf{L}_{s+1}^p}, \quad \textit{for } (\hat{u}, \hat{v}) \in \mathcal{L}_s^p.$$

Proof By Young's inequality and Lemma 2.1, it follows that

$$\begin{aligned}
&\left\| i\hat{u} * (y \cdot \hat{v}) \right\|_{\mathbf{L}^p_s} \\
&\quad \le C\left(\|\hat{u}\|_{\mathbf{L}^p} + \|\hat{u}\|_{\mathbf{L}^p_s} \right) \|y\hat{v}\|_{\mathbf{L}^1} + \|\hat{u}\|_{\mathbf{L}^1} \|\hat{v}\|_{\mathbf{L}^p_{s+1}} \\
&\quad \le C\left(\|\hat{\omega}\|_{\mathbf{L}^{p_1}} + \|\hat{\omega}\|_{\mathbf{L}^p} + \|\hat{\omega}\|_{\mathbf{L}^p_{s-1}} \right) \|y\hat{v}\|_{\mathbf{L}^1} + \|\hat{v}\|_{\mathbf{L}^p_{s+1}} \left(\|\hat{\omega}\|_{\mathbf{L}^1} + \|\hat{\omega}\|_{\mathbf{L}^{p_1}} \right),
\end{aligned} \tag{2.1}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and $p_2 \in [1,3)$.

For $s > 3(1 - \frac{1}{p})$ and $p > p_1$, applying the Sobolev embedding $\mathbf{L}^p_s \subset \mathbf{L}^1 \cap \mathbf{L}^{p_1}$ to (2.1) yields

$$\left\| i\hat{u} * (y \cdot \hat{v}) \right\|_{\mathbf{L}^p_s} \le C\|\hat{\omega}\|_{\mathbf{L}^p_s} \|\hat{v}\|_{\mathbf{L}^p_{s+1}}.$$

This completes the proof. □

Lemma 2.5 *Let $p \in (3,4)$ and $s > 1$. Then there exists a constant $C > 0$ such that*

$$\left\| i\hat{u} * (y \cdot \hat{v}) \right\|_{\mathbf{L}^\infty} \le C\|\hat{\omega}\|_{\mathbf{L}^p_s} \|\hat{v}\|_{\mathbf{L}^p_{s+1}}, \quad \textit{for } (\hat{u}, \hat{v}) \in \mathcal{L}^p_s.$$

Proof By Young's inequality and Lemma 2.1, we have

$$\left\| i\hat{u} * (y \cdot \hat{v}) \right\|_{\mathbf{L}^\infty} \le \|\hat{u}\|_{\mathbf{L}^p} \|y\hat{v}\|_{\mathbf{L}^{p_1}} \le C\left(\|\hat{\omega}\|_{\mathbf{L}^{p_1'}} + \|\hat{\omega}\|_{\mathbf{L}^p} \right) \|y\hat{v}\|_{\mathbf{L}^{p_1}}, \tag{2.2}$$

where $\frac{1}{p} + \frac{1}{p_1} = 1$, $\frac{1}{p_1'} + \frac{1}{p_2} = \frac{1}{p_1}$, and $p_2 \in [1,3)$.

For $p \ge p_1$ and $s > 3(\frac{1}{p_1} - \frac{1}{p})$, $p \ge p_1'$ and $s > 3(\frac{1}{p_1'} - \frac{1}{p})$, applying the Sobolev embedding $\mathbf{L}^p_s \subset \mathbf{L}^{p_1}$ and $\mathbf{L}^p_s \subset \mathbf{L}^{p_1'}$ to (2.2), we derive

$$\left\| i\hat{u} * (y \cdot \hat{v}) \right\|_{\mathbf{L}^\infty} \le C\|\hat{\omega}\|_{\mathbf{L}^p_s} \|\hat{v}\|_{\mathbf{L}^p_{s+1}}.$$

This completes the proof. □

Consider the linearized operator of (1.14)

$$\mathcal{J}_\epsilon(\hat{\varphi}) = \left(\hat{N} + DF(\varphi_\epsilon) \right) \hat{\varphi}, \tag{2.3}$$

where

$$DF(\varphi_\epsilon)\hat{\varphi} = \begin{pmatrix} 2iy \cdot \hat{M}(\hat{\omega}_\epsilon, \hat{\omega}) \\ \hat{P}(\hat{u}_\epsilon, \hat{v}, \hat{\hat{T}}_\epsilon) \end{pmatrix},$$

and

$$\hat{P}(\hat{u}_\epsilon, \hat{v}, \hat{\hat{T}}_\epsilon) = i\hat{u}_\epsilon * (y \cdot \hat{v}) - 2\hat{v} * \hat{\hat{T}}_\epsilon - iy \cdot \left(\hat{u}_\epsilon * \hat{v}^T \right).$$

Then we can rewrite system (1.14) as

$$\frac{d\hat{\varphi}}{dt} + \mathcal{J}_\epsilon(\hat{\varphi}) = G(\hat{\varphi}),$$

where

$$G(\hat{\varphi}) = \begin{pmatrix} iy \cdot \hat{M}(\hat{\omega}, \hat{\omega}) \\ \hat{B}(\hat{u}, \hat{v}, \hat{\tilde{T}}_\epsilon) \end{pmatrix}$$

and

$$\hat{M}(\hat{\omega}_1, \hat{\omega}_2) = \hat{\omega}_2 * \hat{u}_1^T + \hat{\omega}_1 * \hat{u}_2^T - \hat{u}_2 * \hat{\omega}_1^T - \hat{u}_1 * \hat{\omega}_2^T,$$

$$\hat{B}(\hat{u}, \hat{v}, \hat{\tilde{T}}_\epsilon) = iy \cdot \left(\hat{u} * \hat{\tilde{T}}_\epsilon^T + \hat{u} * \hat{v}^T\right) + i\hat{u} * (y \cdot \hat{\tilde{T}}_\epsilon) + i\hat{u} * (y \cdot \hat{v}) - \hat{v} * \hat{v}.$$

Remark 2.1 By applying the theorem of Riesz, it is easy to see that the operators $\mathcal{J}_\epsilon$ and $\hat{N}$ differ by a relatively compact perturbation in $\mathcal{L}_s^p$, for $p \in (3,4)$ and $s > 3(1 - \frac{1}{p})$. Hence, the essential spectrum of the operator $\mathcal{J}$ equals the essential spectrum of the operator $\hat{N}$ (see [25], p. 136).

Lemma 2.6 *Let $p \in (\frac{3}{2}, +\infty]$ and $s > 3(1 - \frac{1}{p})$. Then, for $(\hat{\omega}, \hat{v}) \in \mathcal{L}_s^p$, there exists a positive constant C such that*

$$\left\|\hat{B}(\hat{u}, \hat{v}, \hat{\tilde{T}}_\epsilon)\right\|_{\mathbf{L}_s^p} \le C\left(\|\hat{\omega}\|_{\mathbf{L}_s^p} \|\hat{\tilde{T}}_\epsilon\|_{\mathbf{L}_{s+1}^p} + \|\hat{\omega}\|_{\mathbf{L}_s^p}^2 + \|\hat{v}\|_{\mathbf{L}_{s+1}^p}^2\right).$$

Moreover, for $p \in (3,4)$ and $s > 0$, there exists a positive constant C such that

$$\left\|\hat{B}(\hat{u}, \hat{v}, \hat{\tilde{T}}_\epsilon)\right\|_{\mathbf{L}^\infty} \le C\left(\|\hat{\omega}\|_{\mathbf{L}_s^p} \|\hat{\tilde{T}}_\epsilon\|_{\mathbf{L}_{s+1}^p} + \|\hat{\omega}\|_{\mathbf{L}_s^p}^2 + \|\hat{v}\|_{\mathbf{L}_{s+1}^p}^2\right).$$

Proof Applying Lemma 2.4 and Young's inequality for convolution, it is easy to derive this result. □

Lemma 2.7 *Let $p, p_1 \ge 1$. Then, for $s > 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\hat{f} \in \mathcal{L}_s^p(\mathbf{R}^3)$, the equation*

$$\hat{N}\hat{\varphi} = \hat{f}$$

has a unique solution $\varphi = \hat{N}^{-1}\hat{f} \in \mathcal{L}_s^p(\mathbf{R}^3)$.

Proof Let $\eta(y) \in \mathbf{C}^\infty(\mathbf{R}, [0,1])$ be a cut-off function satisfying

$$\eta(y) = 1, \quad \text{for } |y| \le 1;$$

$$\eta(y) = 0, \quad \text{for } |y| \ge 2.$$

Consider

$$\hat{\varphi} = \hat{\mathcal{N}}^{-1}\hat{f} = \eta(y)\hat{\mathcal{N}}^{-1}\hat{f} + \left(1 - \eta(y)\right)\hat{\mathcal{N}}^{-1}\hat{f},$$

where

$$\hat{\mathcal{N}}^{-1} = \frac{1}{(|y|^2 + icy_1)(|y|^2 + iy_3 + 1)} \begin{pmatrix} |y|^2 + icy_1 & -i\vec{\rho}y \\ 0 & |y|^2 + 1 + iy_3 \end{pmatrix}.$$

Denote

$$\Upsilon^1_{|y|\le 1} = \frac{1}{|y|^2 + iy_3 + 1}, \qquad \tilde{\Upsilon}^1_{|y|>1} = \frac{1}{|y|^2 + iy_3 + 1},$$
$$\Upsilon^2_{|y|\le 1} = \frac{1}{|y|^2 + icy_1}, \qquad \tilde{\Upsilon}^2_{|y|>1} = \frac{1}{|y|^2 + icy_1},$$
$$\Upsilon^3_{|y|\le 1} = \frac{-i\vec{\rho}y}{(|y|^2 + icy_1)(|y|^2 + iy_3 + 1)}, \qquad \tilde{\Upsilon}^3_{|y|>1} = \frac{-i\vec{\rho}y}{(|y|^2 + icy_1)(|y|^2 + iy_3 + 1)}.$$

By Minkowski's inequality, Hölder's inequality, and Lemma 2.2, for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $p' > p_1$, and $s > 3(\frac{1}{p_1} - \frac{1}{p'})$, we have

$$\begin{aligned}
\|\hat{\varphi}\|_{\mathcal{L}^p_s} &\le \big\|\eta(y)\hat{\mathcal{N}}^{-1}\hat{f}\big\|_{\mathcal{L}^p_s} + \big\|\big(1-\eta(y)\big)\hat{\mathcal{N}}^{-1}\hat{f}\big\|_{\mathcal{L}^{p'}_s} \\
&\le C\big(\big\|\eta(y)\hat{\mathcal{N}}^{-1}\big\|_{\mathcal{L}^{p_1}}\|\hat{f}\|_{\mathcal{L}^{p_2}_s} + \big\|\big(1-\eta(y)\big)\hat{\mathcal{N}}^{-1}\big\|_{\mathcal{L}^\infty}\|\hat{f}\|_{\mathcal{L}^p_s}\big) \\
&\le C\big(\big\|\eta(y)\hat{\mathcal{N}}^{-1}\big\|_{\mathcal{L}^{p'}_s}\|\hat{f}\|_{\mathcal{L}^{p_2}_s} + \big\|\big(1-\eta(y)\big)\hat{\mathcal{N}}^{-1}\big\|_{\mathcal{L}^\infty}\|\hat{f}\|_{\mathcal{L}^p_s}\big).
\end{aligned} \tag{2.4}$$

It is easy to check that $\|\tilde{\Upsilon}^j_{|y|>1}\|_{\mathbf{L}^\infty} < +\infty$ for $j = 1,2,3$. For $\frac{1}{2} + \frac{s}{2} < p' < \frac{3}{2} + \frac{s}{2}$, we have

$$\big\|\Upsilon^1_{|y|\le 1}\big\|^{p'}_{\mathbf{L}^{p'}_s} = \int_{|y|\le 1} \frac{(1+|y|^2)^{\frac{s}{2}}}{||y|^2 + iy_3 + 1|^{p'}}\,dy \le C\int_0^1 \frac{1}{x^{2p'-s-2}}\,dx < +\infty.$$

In the same way, we get

$$\begin{aligned}
&\big\|\Upsilon^2_{|y|\le 1}\big\|^{p'}_{\mathbf{L}^{p_1}} < +\infty, \quad \text{for } \frac{1}{2} + \frac{s}{2} < p' < \frac{3}{2} + \frac{s}{2}, \\
&\big\|\Upsilon^3_{|y|\le 1}\big\|^{p'}_{\mathbf{L}^{p_1}} < +\infty, \quad \text{for } \frac{1}{3} + \frac{s}{3} < p' < 1 + \frac{s}{3}.
\end{aligned}$$

Therefore, by (2.4) and the above estimates, for $\frac{1}{2} + \frac{s}{2} < p' < 1 + \frac{s}{3}$ and $\frac{1}{p_1} - \frac{1}{p'} \ge \frac{1}{3}$, we obtain

$$\big\|\eta(y)\hat{\mathcal{N}}^{-1}\big\|_{\mathcal{L}^{p'}_s} < +\infty. \tag{2.5}$$

This completes the proof. □

Lemma 2.8 *Let $p, p_2 \in (\frac{2}{3}, +\infty]$. For $s \ge 2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then the operator $\hat{N}^{-1} \cdot DF(\hat{\varphi})$ is a compact operator on $\mathcal{L}^p_s$. Furthermore, the operator*

$$\Gamma := I + \hat{N}^{-1} \cdot DF(\hat{\varphi}) : \mathcal{L}^p_s \to \mathcal{L}^p_s$$

is a Fredholm operator with index 0.

Proof Denote the set

$$\mathcal{S} = \big\{\hat{\varphi} \in \mathcal{L}^p_s : \|\hat{\varphi}\|_{\mathcal{L}^p_s} \le 1\big\}$$

and $\chi_n = (\hat{N}^{-1} \cdot DF(\hat{\varphi}))\hat{\varphi}_n$ for any sequence $\hat{\varphi}_n = (\hat{\omega}_n, \hat{v}_n) \in \mathcal{S}$.

By Minkowski's inequality, Hölder's inequality, and Lemma 2.2, for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $p' > p_1$, and $s > 3(\frac{1}{p_1} - \frac{1}{p'})$, we have

$$\begin{aligned} \|\chi_n\|_{\mathcal{L}_s^p} &= \big\|\big(\hat{N}^{-1} \cdot DF(\hat{\varphi})\big)\hat{\varphi}_n\big\|_{\mathcal{L}_s^p} \\ &\le \big\|\eta(y)\big(\hat{N}^{-1} \cdot DF(\hat{\varphi})\big)\hat{\varphi}_n\big\|_{\mathcal{L}_s^p} + \big\|\big(1-\eta(y)\big)\big(\hat{N}^{-1} \cdot DF(\hat{\varphi})\big)\hat{\varphi}_n\big\|_{\mathcal{L}_s^p} \\ &\le C\big(\big\|\eta(y)\hat{N}^{-1}\big\|_{\mathcal{L}^{p_1}}\big\|DF(\hat{\varphi})\hat{\varphi}_n\big\|_{\mathcal{L}_s^{p_2}} + \big\|\big(1-\eta(y)\big)\hat{N}^{-1}\big\|_{\mathcal{L}^\infty}\big\|DF(\hat{\varphi})\hat{\varphi}_n\big\|_{\mathcal{L}_s^p}\big) \\ &\le C\big(\big\|\eta(y)\hat{N}^{-1}\big\|_{\mathcal{L}_s^{p'}}\big\|DF(\hat{\varphi})\hat{\varphi}_n\big\|_{\mathcal{L}_s^{p_2}} + \big\|\big(1-\eta(y)\big)\hat{N}^{-1}\big\|_{\mathcal{L}^\infty}\big\|DF(\hat{\varphi})\hat{\varphi}_n\big\|_{\mathcal{L}_s^p}\big). \end{aligned} \tag{2.6}$$

As proved in Lemma 2.4, by Young's inequality for convolutions, for $p, p_2 \in (\frac{3}{2}, +\infty]$ and $s > 3(1 - \frac{1}{p_2})$, we can get

$$\begin{aligned} &\big\|DF(\hat{\varphi})\hat{\varphi}_n\big\|_{\mathcal{L}_s^{p_2}} \\ &\quad\le C\big(\|\hat{\omega}_n\|^2_{\mathbf{L}_s^{p_2}} + \|\hat{v}_n\|^2_{\mathbf{L}_{s+1}^{p_2}} + \|\hat{u}_\epsilon\|^2_{\mathbf{L}_s^{p_2}} + \|\hat{\omega}_\epsilon\|^2_{\mathbf{L}_s^{p_2}} + \|\hat{\hat{T}}_\epsilon\|^2_{\mathbf{L}_{s+1}^{p_2}}\big) < +\infty, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\big\|DF(\hat{\varphi})\hat{\varphi}_n\big\|_{\mathcal{L}_s^{p'}} \\ &\quad\le C\big(\|\hat{\omega}_n\|^2_{\mathbf{L}_s^{p'}} + \|\hat{v}_n\|^2_{\mathbf{L}_{s+1}^{p'}} + \|\hat{u}_\epsilon\|^2_{\mathbf{L}_s^{p'}} + \|\hat{\omega}_\epsilon\|^2_{\mathbf{L}_s^{p'}} + \|\hat{\hat{T}}_\epsilon\|^2_{\mathbf{L}_{s+1}^{p'}}\big) < +\infty. \end{aligned} \tag{2.8}$$

From (2.5), for $\frac{1}{2} + \frac{s}{2} < p' < 1 + \frac{s}{3}$ and $\frac{1}{p_1} - \frac{1}{p'} \ge \frac{2}{3}$, we derive

$$\big\|\eta(y)\hat{\mathcal{N}}^{-1}\big\|_{\mathcal{L}_s^{p'}} < +\infty. \tag{2.9}$$

By (2.7)-(2.9), it follows that

$$\|\chi_n\|_{\mathcal{L}_s^p} = \big\|\big(\hat{N}^{-1} \cdot DF(\hat{\varphi})\big)\hat{\varphi}_n\big\|_{\mathcal{L}_s^p} < +\infty.$$

Therefore, $\hat{N}^{-1} \cdot DF(\hat{\varphi})\mathcal{S}$ is a precompact set in $\mathcal{L}_s^p$. This completes the proof. □

Lemma 2.9 *Let $p, p_2 \in (\frac{2}{3}, +\infty]$. For $s \ge 2$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\hat{f} \in \mathcal{L}_s^p(\mathbf{R}^3)$, the equation*

$$\mathcal{J}_\epsilon \hat{\varphi} = \hat{f}$$

has a unique solution $\hat{\varphi} = \mathcal{J}_\epsilon^{-1}\hat{f} = \Gamma^{-1}\hat{N}^{-1}\hat{f} \in \mathcal{L}_s^p(\mathbf{R}^3)$, where the operator

$$\Gamma^{-1} = \big(I + \hat{N}^{-1} \cdot DF(\hat{\varphi})\big) : \mathcal{L}_s^p \to \mathcal{L}_s^p.$$

Proof This is a direct result from Lemma 2.8. □

3 Proof of the main result

This section is devoted to proving the main result. Since the linear operator which we get in solving equation (1.13) is not invertible for $\epsilon = \epsilon_0$, the implicit function theorem cannot be applied directly. The Lyapunov-Schmidt reduction is a powerful method to deal with this case. Assume that $|\epsilon - \epsilon_0|$ and $|\varpi - \varpi_0|$ are suitable small. We find the $2\pi/\varpi$-time-periodic solution $\hat{\varphi} = (\hat{\omega}, \hat{v}) \in \mathcal{X}_s^p$, which can be made the ansatz as

$$\hat{\varphi}(y,t) = \left(\sum_{n\in\mathbf{Z}} \hat{\omega}_n(y)e^{in\varpi t}, \sum_{n\in\mathbf{Z}} \hat{v}_n(y)e^{in\varpi t}\right),$$

and satisfying

$$\frac{d\hat{\varphi}}{dt} + \mathcal{J}_\epsilon\hat{\varphi} = G(\hat{\varphi}). \tag{3.1}$$

Introduce the projection S_n onto the nth Fourier mode, *i.e.*,

$$(S_n\hat{\varphi})(y) = \left(\frac{\varpi}{2\pi}\int_0^{\frac{2\pi}{\varpi}} \hat{\omega}(y,t)e^{in\varpi t}\,dt, \frac{\varpi}{2\pi}\int_0^{\frac{2\pi}{\varpi}} \hat{v}(y,t)e^{in\varpi t}\,dt\right)$$

and the $\mathcal{J}$-invariant orthogonal projection $\mathcal{P}_{n,c}$ onto the subspace spanned by the eigenvector associated with the eigenvalue $(in\varpi, in\varpi)$. We denote $\mathcal{P}_{n,s} = 1 - \mathcal{P}_{n,c}$.

Applying the projection S_n to equation (3.1) we get lattice systems for the Fourier modes $S_n\hat{\varphi}$

$$in\varpi\hat{\varphi}_n + \mathcal{J}_\epsilon\hat{\varphi}_n = G_n(\hat{\varphi}), \quad n \in \mathbf{Z}, \tag{3.2}$$

where

$$G_n(\hat{\varphi}) = \sum_{n\in\mathbf{Z}} G(\hat{\varphi}_{n-m}, \hat{\varphi}_m).$$

Denote

$$in\varpi^* = \begin{pmatrix} in\varpi & 0 \\ 0 & in\varpi \end{pmatrix}.$$

Then we rewrite lattice system (3.2) as

$$in\varpi^*\hat{\varphi}_n + \mathcal{J}_\epsilon\hat{\varphi}_n = G_n(\hat{\varphi}), \quad \text{for } n = \pm 2, \pm 3, \ldots, \tag{3.3}$$

$$\pm i\varpi^*\hat{\varphi}_{n,s} + \mathcal{J}_\epsilon\hat{\varphi}_{n,s} = \mathcal{P}_{n,s}G_{\pm 1}(\hat{\varphi}), \quad \text{for } n = \pm 1, \tag{3.4}$$

$$\mathcal{J}_\epsilon\hat{\varphi}_0 = G_0(\hat{\varphi}), \quad \text{for } n = 0, \tag{3.5}$$

$$\pm i\varpi^*\hat{\varphi}_{n,c} + \mathcal{J}_\epsilon\hat{\varphi}_{n,c} = \mathcal{P}_{n,c}G_{\pm 1}(\hat{\varphi}), \quad \text{for } n = \pm 1. \tag{3.6}$$

In the following, we want to prove that if

$$\hat{\varphi}_{\pm 1,c} = \mathcal{P}_{\pm 1,c}\hat{\varphi}_{\pm 1} = (\mathcal{P}_{\pm 1,c}\hat{\omega}_{\pm 1}, \mathcal{P}_{\pm 1,c}\hat{v}_{\pm 1}) \in \mathcal{L}_s^p$$

is given, then the above lattice systems are solvable.

By assumptions (A1)-(A3), (3.3)-(3.5), and Lemma 2.9, we obtain

$$\hat{\varphi}_n = \left(in\varpi^* + \mathcal{J}_\epsilon\right)^{-1} G_n(\hat{\varphi}), \quad \text{for } n = \pm 2, \pm 3, \ldots, \tag{3.7}$$

$$\hat{\varphi}_{n,s} = \left(\pm i\varpi^* + \mathcal{J}_\epsilon\right)^{-1} \mathcal{P}_{n,s} G_{\pm 1}(\hat{\varphi}), \quad \text{for } n = \pm 1, \tag{3.8}$$

$$\hat{\varphi}_0 = \mathcal{J}_\epsilon^{-1} G_0(\hat{\varphi}), \quad \text{for } n = 0. \tag{3.9}$$

So we rewrite (3.7)-(3.9) as

$$\mathcal{F}(\hat{\varphi}_c, \hat{\varphi}_s) = 0, \tag{3.10}$$

where

$$\hat{\varphi}_c = (\ldots, 0, \hat{\varphi}_{-1,c}, 0, \hat{\varphi}_{1,c}, 0, \ldots),$$
$$\hat{\varphi}_s = (\ldots, \hat{\varphi}_{-2}, \hat{\varphi}_{-1,s}, \hat{\varphi}_0, \hat{\varphi}_{1,s}, \hat{\varphi}_{-2}, \ldots).$$

Lemma 3.1 *Define* $\Xi = (\Xi_{1n}, \Xi_{2n})_{n\in\mathbf{Z}} : \mathcal{L}_s^p \to \mathcal{L}_s^p$ *and* $(\Xi\hat{\varphi})_{n\in\mathbf{Z}} = (\Xi_{1n}\hat{\omega}_n, \Xi_{2n}\hat{v}_n)$. *Then*

$$\|\Xi\hat{\varphi}\|_{\mathcal{X}_s^p} \le \sup_{n\in\mathbf{Z}} \|\Xi\|_{\mathcal{L}_s^p \mapsto \mathcal{L}_s^p} \|\hat{\varphi}\|_{\mathcal{X}_s^p}.$$

Proof

$$\begin{aligned} \|\Xi\hat{\varphi}\|_{\mathcal{X}_s^p} &\le \sum_{n\in\mathbf{Z}} \left(\|\Xi_{1n}\hat{\omega}_n\|_{\mathbf{L}_s^p} + \|\Xi_{2n}\hat{v}_n\|_{\mathbf{L}_{s+1}^p}\right) \\ &\le \sup_{n\in\mathbf{Z}} \left(\|\Xi_{1n}\|_{\mathbf{L}_s^p \mapsto \mathbf{L}_s^p} + \|\Xi_{2n}\|_{\mathbf{L}_{s+1}^p \mapsto \mathbf{L}_{s+1}^p}\right) \sum_{n\in\mathbf{Z}} \left(\|\hat{\omega}_n\|_{\mathbf{L}_s^p} + \|\hat{v}_n\|_{\mathbf{L}_{s+1}^p}\right) \\ &\le \sup_{n\in\mathbf{Z}} \|\Xi\|_{\mathcal{L}_s^p \mapsto \mathcal{L}_s^p} \|\hat{\varphi}\|_{\mathcal{X}_s^p}. \end{aligned}$$

□

Lemma 3.2 *Let* $p > \frac{3}{2}$ *and* $s > 3(1 - \frac{1}{p})$. *Then, for any* $\hat{\varphi} \in \mathcal{X}_s^p$,

$$\left\|\left(\hat{B}_n(\hat{\varphi})\right)_{n\in\mathbf{Z}}\right\|_{\mathcal{X}_s^p} \le C\|\hat{\varphi}\|_{\mathcal{X}_s^p} \left(\|\hat{\tilde{T}}_\epsilon\|_{\mathbf{X}_{s+1}^p} + \|\hat{\varphi}\|_{\mathcal{X}_s^p}\right).$$

Proof By Lemma 2.6, we obtain

$$\begin{aligned} \left\|\left(\hat{B}_n(\hat{\varphi})\right)_{n\in\mathbf{Z}}\right\|_{\mathcal{X}_s^p} &= \sum_{n\in\mathbf{Z}} \left\|\left(\hat{B}(\hat{u}, \hat{v}, \hat{\tilde{T}}_\epsilon)\right)_n\right\|_{\mathbf{L}_s^p} \\ &\le C \sum_{n\in\mathbf{Z}} \left(\|\hat{\omega}_n\|_{\mathbf{L}_s^p} \|\hat{\tilde{T}}_{\epsilon n}\|_{\mathbf{L}_{s+1}^p} + \|\hat{\omega}_n\|_{\mathbf{L}_s^p}^2 + \|\hat{v}_n\|_{\mathbf{L}_{s+1}^p}^2\right) \\ &\le C\left(\|\hat{\varphi}\|_{\mathcal{X}_s^p} \|\hat{\tilde{T}}_\epsilon\|_{\mathbf{X}_{s+1}^p} + \|\hat{\varphi}\|_{\mathcal{X}_s^p}^2\right). \end{aligned}$$

□

Lemma 3.3 *There exists a constant* $C > 0$ *such that*

$$\left\|\left(in\varpi^* - \mathcal{J}_\epsilon\right)^{-1}(iy, 1)\cdot\right\|_{\mathcal{L}_s^p \mapsto \mathcal{L}_s^p} \le C, \quad n \in \mathbf{Z} \setminus \{\pm 1, 0\},$$
$$\left\|\left(\pm i\varpi^* - \mathcal{J}_\epsilon\right)^{-1} \mathcal{P}_{n,s}(iy, 1)\cdot\right\|_{\mathcal{L}_s^p \mapsto \mathcal{L}_s^p} \le C, \quad n = \pm 1.$$

Proof This result is directly derived from assumption (A3), the property of the sectorial operators $\mathcal{J}_\epsilon$ and $\hat{N}$, Lemma 2.3, and Lemma 2.8. □

Now, we return to equation (3.10). From Lemmas 3.1-3.3, $\mathcal{F}:\mathcal{X}_s^p \to \mathcal{X}_s^p$ is well defined and smooth for $p\in(3,4)$ and $s\geq 2$. It is obvious that $\mathcal{F}(0,0)=0$, $D_{\hat{\varphi}}\mathcal{F}:\mathcal{X}_s^p\to\mathcal{X}_s^p$ is invertible and $D_{\hat{\varphi}}\mathcal{F}(0,0)=I$. Therefore, by the implicit function theorem, there exists a unique smooth solution $\hat{\varphi}_s=\hat{\varphi}_s(\hat{\varphi}_c)$ satisfying $\|\hat{\varphi}_s(\hat{\varphi}_c)\|_{\mathcal{X}_s^p}\leq C\|\hat{\varphi}_c\|_{\mathcal{X}_s^p}$.

Finally, we give the proof of our main result. This proof is based on the classical Hopf bifurcation (see [26]) applied to solve the equation (3.6) by the implicit function theorem. Let $\psi_n^+\in\mathcal{X}_s^p$ denote the eigenfunctions associated with the eigenvalues $(\pm i\varpi_0,\pm i\varpi_0)$. $\chi_0^+(\epsilon)$ is the eigenvalues of operator $\mathcal{J}_\epsilon$ under the basis (ψ_n^+,ψ_n^+). Introduce $p_{n,c}$ by $\mathcal{P}_{n,c}\varphi=p_{n,c}(\varphi)\psi_n^+$. Then, for $\xi\in\mathbf{C}\setminus 0$, it follows (3.6) that

$$-i\varpi^*\xi\psi_n^+ + \chi_0^+(\epsilon)\psi_n^+ - p_{n,c}\big(G_{+1}\big(\hat{\varphi}_s\big(\xi\psi_n^+\big)\big)\big)\psi_n^+ = 0,$$

which implies that

$$-i\varpi^*\xi + \chi_0^+(\epsilon) - p_{n,c}\big(G_{+1}\big(\hat{\varphi}_s\big(\xi\psi_n^+\big)\big)\big) = 0.$$

Define the complex-valued smooth function

$$\Upsilon(\alpha;\varrho,\beta) := \begin{cases} -i(\varpi_c^*+\varrho)+\chi_0^+(\epsilon_c+\beta)-\alpha^{-1}\Lambda(\epsilon_c+\beta,\alpha), & \alpha\neq 0,\\ -i(\varpi_c^*+\varrho)+\chi_0^+(\epsilon_c+\beta), & \alpha=0,\end{cases}$$

where $\Lambda(\epsilon_c+\beta,\alpha):=p_{n,c}(G_{+1}(\hat{\varphi}_s(\xi\psi_n^+)))$.

Denote

$$i\varpi_c^* = \begin{pmatrix} i\varpi_c & 0\\ 0 & i\varpi_c\end{pmatrix}$$

and

$$\chi_0^+(\epsilon) = \begin{pmatrix} \lambda_0^+ & 0\\ 0 & \mu_0^+\end{pmatrix}.$$

By (1.15)-(1.16) and Lemma 3.2, we know that $\Upsilon(0;0,0)=0$ and the determinant of the Jacobi matrix

$$\begin{aligned}\det D_{\varrho,\beta}\Upsilon(\alpha;\varrho,\beta)|_{\alpha=\varrho=\beta=0} &= \det\begin{pmatrix}\mathbf{0} & \frac{d}{d\epsilon}\operatorname{Re}\chi_0^+(\epsilon)|_{\epsilon=\epsilon_c}\\ \mathbf{-1} & \frac{d}{d\epsilon}\operatorname{Im}\chi_0^+(\epsilon)|_{\epsilon=\epsilon_c}\end{pmatrix}_{4\times 4}\\ &= \frac{d}{d\epsilon}\operatorname{Re}\lambda_0^+(\epsilon)\bigg|_{\epsilon=\epsilon_c} + \frac{d}{d\epsilon}\operatorname{Re}\mu_0^+(\epsilon)\bigg|_{\epsilon=\epsilon_c} > 0.\end{aligned}$$

Therefore, there exists a function $\alpha\mapsto(\varrho(\alpha),\beta(\alpha))$ with $\varrho(0)=\beta(0)=0$ satisfying

$$-i\alpha\big(\varpi_c^*+\varrho(\alpha)\big)+\alpha\chi_0^+\big(\epsilon_c+\beta(\alpha)\big)-\Lambda\big(\epsilon_c+\beta(\alpha),\alpha\big)=0, \tag{3.11}$$

for $|\alpha|$ sufficient small.

Due to the degree of nonlinearity term in (3.11), it is easy to see that there exists a function $\alpha(\beta)$ such that $\hat{\varphi}_{n,c} = \alpha(\beta)\psi_n^+$ is the solution of (3.6) for $\varpi = \varpi_0 + \varrho(\alpha(\beta))$ and $\epsilon = \epsilon_0 + \beta$. This completes the proof.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
This work is supported by the project of Education department of Henan Province 14A110013, and the project of Tianyuan 11426106.

Received: 19 August 2015 Accepted: 2 February 2016 Published online: 09 February 2016

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