Togun *Boundary Value Problems* (2016) 2016:57
DOI 10.1186/s13661-016-0561-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Nonlocal beam theory for nonlinear vibrations of a nanobeam resting on elastic foundation

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Abstract

In the present study, nonlinear vibrations of an Euler-Bernoulli nanobeam resting on an elastic foundation is studied using nonlocal elasticity theory. Hamilton's principle is employed to derive the governing equations and boundary conditions. The nonlinear equation of motion is obtained by including stretching of the neutral axis that introduces cubic nonlinearity into the equations. Forcing and damping effects are included in the equations of the motion. The multiple scale method, a perturbation technique for deriving the approximate solutions of the equations, is applied to the nonlinear systems. Natural frequencies and mode shapes for the linear problem are found and also nonlinear frequencies are found for a nonlocal Euler-Bernoulli nanobeam resting on an elastic foundation. In the numerical calculation, frequency-response curves are drawn for various parameters like nonlocal parameters, elastic foundation, and boundary conditions. The effects of the different nonlocal parameters (γ) and elastic foundation parameters (κ) as well as the effects of different boundary conditions on the vibrations are discussed.

Keywords: vibratio; nanobeam; perturbation metho; nonlocal elasticity; elastic foundation

1 Introduction

Due to the rapid improvement in the nano-mechanics, nanobeams have become one of the most important structures used extensively in nanotechnology, such as those in sensors and actuators. The nonlocal elasticity theory which was formally initiated by the papers of Eringen [1] on nonlocal elasticity can be used for nanotechnology applications due to the small length scale in nanoapplications of the beam. The main differences between continuum (local) elasticity theory and nonlocal elasticity theory come from stress definition. Continuum elasticity theory assumes that stress at a point is a function of strain at that point, whereas in the nonlocal elasticity theory stress at a point is a function of strains at all points in the continuum.

Numerical simulation and analysis of nanostructures have been presented extensively by the researchers. Researchers have searched these structures due to the difficulties in experimental specification at the nanoscale and due to their being time-consuming. Then researchers became interested in the theory applied to different mechanical analyses. The nonlocal elasticity theory has been used to examine the vibration, bending, and buckling of the beam depending on the beam model. Peddieson *et al.* [2] can be considered to be a pioneering work which first applied the nonlocal elasticity theory of Eringen [1] to the nanotechnology.

Vibration analyses are of first priority in the design of nanoelectromechanical systems (NEMS) and new nanodevices. Finding natural frequencies and mode shapes is of primary importance. The effect of the surrounding medium on the vibration response of nanobeams also has practical value. Beams resting on an elastic foundation are usually included in the design of aircraft structures. Niknam and Aghdam [3] studied natural frequency and buckling load of nonlocal functionally graded beams resting on a nonlinear elastic foundation using the Eringen's nonlocal elasticity theory. Kiani [4] researched a single walled carbon nanotube (SWCNT) structure embedded in an elastic matrix by the nonlocal Euler-Bernoulli, Timoshenko, and higher order beams. Nonlocal elasticity and Timoshenko beam theory were implemented to investigate the stability response of SWCNT embedded in an elastic medium [5, 6]. Nonlocal Euler-Bernoulli theory was applied to investigate the influence of viscoelastic foundation on the nonconservative instability of cantilever CNTs under the action of a concentrated follower [7]. The critical buckling temperature of SWCNT, which is embedded in one parameter elastic medium (Winkler foundation) was estimated by using continuum mechanics theory [8]. Electro-thermal loadings on the transverse free vibration of double walled boron nitride nanotubes (DWBNNTs) embedded in an elastic medium were considered by Arani *et al.* [9], who investigated the influence of spring modulus, shear modulus, electric field, and temperature change on the natural frequency. Also this author and coauthors [10] carried out an electro-thermal vibration of the DWBNNTs which are coupled by a visco-Pasternak medium based on strain gradient theory. The thermal-mechanical vibration and buckling instability of a SWCNT conveying fluid and resting on an elastic medium were carried out to obtain the effects of temperature change, nonlocal parameter, and elastic medium on the vibration frequency [11]. The thermo-mechanical vibration of a SWCNT embedded in an elastic medium was studied by Murmu and Pradhan [12], who presented the effect of nonlocal small-scale effects, temperature change, Winkler constant, and vibration modes on the frequency. Also Murmu and Pradhan [13] applied a nonlocal beam model to the buckling analysis of SWCNT with the effect of temperature change and the surrounding elastic medium. An electro-thermo-mechanical vibration analysis of nonuniform and nonhomogeneous boron nitride nanorod embedded in elastic medium was presented [14]. Localized modes of free vibrations of SWCNTs embedded in a nonhomogeneous elastic medium were studied on the base of the nonlocal continuum shell theory [15]. The vibration of an axially loaded nonprismatic SWCNT embedded in a two parameter elastic medium [16], a viscous fluid conveying SWCNT embedded in an elastic medium [17], an elastically supported DWCNT embedded in an elastic foundation subject to axial load [18], SWCNT for delivering nanoparticles [19], a carbon nanotube resting on a linear viscoelastic Winkler foundation [20], SWCNT resting on elastic foundation [21], nonuniform SWCNT conveying fluid embedded in viscoelastic medium [22], carbon nanotubes embedded in an elastic medium [23], nanotubes embedded in an elastic matrix [24], and curved SWCNT on a Pasternak elastic foundation [25] were investigated based on the Euler-Bernoulli beam model and the Timoshenko beam model. Aydogdu and Arda [26] researched the torsional vibration behavior of DWCNTs based on nonlocal elasticity theory. Ahangar *et al.* [27] studied the size dependent vibration of a microbeam. The work of

Marin is based on the thermoelasticity of initial stress bodies [28] and of dipolar bodies [29].

Differential equation solutions are much more complicated and time consuming. Further, in many cases, it is impossible to solve nonlinear equations exactly, and therefore there is a need for some approximate solution method. There are several numerical and analytical methods used for solution of nonlinear equations. Some of them are the homotopy perturbation method [30], the multiple scale method [31–34], He's variational method [3, 35–38], the direct iterative method [39], the finite element method [11, 20, 22], and the differential quadrature method [5, 10, 21, 40, 41].

The above investigations clearly show that most of the studies presented in the literature are related to the linear vibration analysis of nanostructures, but studies on the nonlinear vibration are rather limited. Studies related to a nonlinear vibration analysis of nanotubes [32, 39, 40, 42–47], functionally graded beams [37, 38, 48, 49], microbeams [36], nanobeams [35], and boron nitride nanotubes [41] have been reported. When searching the literature, most of the work is related to the frequency amplitude response of nanotubes/nanobeams. However, a nonlinear vibration of nanosystems with damping effect is very rare. The nonlinear free vibration of the nanotube with damping effect was studied by using nonlocal elasticity theory [32]. To the best of the knowledge of the author, there is no published work on a nonlinear free vibration of nanobeam resting on elastic foundation with the effect of damping and forcing terms. The nonlinearity of the problem is obtained by including stretching of the neutral axis that introduces a cubic nonlinearity into the equations. Nonlinear frequency-response curves are drawn for nanobeams with different end conditions.

2 Nonlocal elastic models

Recently, the nonlocal continuum mechanics method has been successfully applied to analyze the mechanical behaviors of nano-structures.

2.1 General theory

According to the Eringen's nonlocal elasticity theory [1], the stress at a reference point x in an elastic continuum not only depends on the strain field at the same point but also on the strains at all other points of the body. So, the nonlocal stress tensor σ at a point x is given by

$$\sigma_{ij}\bigl(x^*\bigr) = \int_V K\bigl(\bigl|x^* - x'^*\bigr|, \gamma\bigr) C_{ijkl}\varepsilon_{kl}\bigl(x'^*\bigr)\, dV\bigl(x'^*\bigr), \tag{1}$$

where σ_{ij} and ε_{ij} are the stress and strain tensors, respectively; C_{ijkl} is the elastic modulus tensor of classical isotropic elasticity, and $K(|x^* - x'^*|, \gamma)$ is the kernel function. $|x^* - x'^*|$ is the Euclidean distance, and $\gamma = e_0 a/L$, where e_0 is a constant appropriate to each material, a is an internal characteristic length (*e.g.*, lattice parameter, granular distance), and L is an external characteristic length (*e.g.*, the crack length, the wavelength). $e_0 a$ used in this study is usually taken as a small-scale parameter. It is very hard to solve the elasticity problems by using the integral constitutive relation in equation (1). Therefore, a simplified constitutive relation in a differential form is given by Eringen [1] as follows:

$$\bigl(1 - (e_0 a)^2 \nabla^2\bigr)\sigma = T. \tag{2}$$

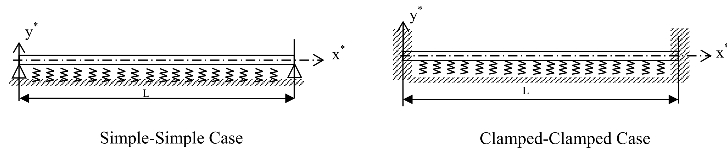

Figure 1 Boundary conditions for different beam supports.

Here ∇^2 is the Laplacian operator. The nonlocal constitutive relation for the present nanobeam can be written as

$$\sigma(x^*) - (e_0 a)^2 \frac{\partial^2 \sigma(x^*)}{\partial x^{*2}} = E\varepsilon(x^*), \tag{3}$$

where E is the elasticity modulus.

2.2 Governing equations of the nanobeam resting on elastic foundation

The considered nanobeam resting on elastic foundation as shown schematically in Figure 1. Simply supported and clamped-clamped supported boundary conditions are considered. The nonlocal Euler-Bernoulli beam model is used to model the nanobeam.

Here y^* the beam's transverse displacement, t^* the time variable, ρA the mass per unit length, k the elastic foundation stiffness, L the length of the beam, A the area of the cross-section of the beam, I the area moment of inertia, $e_0 a$ the small-scale parameter, and N the axial force. The equations of motion are obtained by using Hamilton's principle.

$$\begin{aligned} £ = {} & \frac{1}{2}\int_0^L \rho A\left(\frac{\partial y^*}{\partial t^*}\right)^2 dx^* \\ & + \frac{1}{2}\int_0^L \left((e_0 a)^2 \rho A \frac{\partial^2 y^*}{\partial t^{*2}} - (e_0 a)^2 N \frac{\partial^2 y^*}{\partial x^{*2}} - (e_0 a)^2 k - EI\frac{\partial^2 y^*}{\partial x^{*2}}\right)\frac{\partial^2 y^*}{\partial x^{*2}}\, dx^* \\ & - \frac{1}{2}\int_0^L N\left(\frac{\partial y^*}{\partial x^*}\right)^2 dx^* - \frac{1}{2}\int_0^L k y^{*2}\, dx^*. \end{aligned} \tag{4}$$

In equation (4), kinetic energy of the beam is shown in the first integral, the elastic energy induced by the bending is shown in the second integral, the elastic energy in extension due to stretching of the neutral axis is shown in the third integral, and the elastic energy due to elastic foundation is shown in the last integral. The dimensional form of nonlocal governing equation and the boundary conditions can be obtained by applying equation (4) as follows:

$$\begin{aligned} & EI\frac{\partial^4 y^*}{\partial x^{*4}} + \rho A \frac{\partial^2}{\partial t^{*2}}\left(y^* - (e_0 a)^2 \frac{\partial^2 y^*}{\partial x^{*2}}\right) + k\left(y^* - (e_0 a)^2\frac{\partial^2 y^*}{\partial x^{*2}}\right) \\ & \quad = \frac{EA}{2L}\left[\int_0^L \left(\frac{\partial y^*}{\partial x^*}\right)^2 dx^*\right]\left[\frac{\partial^2 y^*}{\partial x^{*2}} - (e_0 a)^2 \frac{\partial^4 y^*}{\partial x^{*4}}\right]. \end{aligned} \tag{5}$$

Furthermore, the following possible boundary conditions at the beam ends (at $x^* = 0$ *and* $x^* = L$) are obtained:

Simple-Simple Case		Clamped-Clamped Case	
$y^*(0) = 0,$	$y^*(L) = 0,$	$y^*(0) = 0,$	$y^*(L) = 0,$
$y''^*(0) = 0,$	$y''^*(L) = 0,$	$y'^*(0) = 0,$	$y^{*\prime}(L) = 0.$

The following nondimensional parameters can be defined in order to obtain general results:

$$x = \frac{x^*}{L}, \qquad w = \frac{w^*}{L}, \qquad t = \frac{t^*}{L}\sqrt{\frac{EI}{\rho A}}, \qquad \gamma = \frac{e_0 a}{L}, \qquad \kappa = \frac{kL^4}{EI}. \tag{6}$$

The equation of motion can be written in the dimensionless form by using equation (6) as follows:

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} - \gamma^2 \frac{\partial^4 y}{\partial x^2\,\partial t^2} + \kappa y - \kappa\gamma^2 \frac{\partial^2 y}{\partial x^2} = \frac{1}{2}\left[\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx\right]\left[\frac{\partial^2 y}{\partial x^2} - \gamma^2 \frac{\partial^4 y}{\partial x^4}\right]. \tag{7}$$

3 Approximate solution

In order to obtain the approximate solution for the problem, the multiple scale method will be employed to the partial differential equation system and boundary conditions directly [50, 51]. Forcing and damping terms are included in equation (7):

$$\begin{aligned}\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} - \gamma^2 \frac{\partial^4 y}{\partial x^2\,\partial t^2} + \kappa y - \kappa\gamma^2 \frac{\partial^2 y}{\partial x^2} &= \frac{1}{2}\left[\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx\right]\left[\frac{\partial^2 y}{\partial x^2} - \gamma^2 \frac{\partial^4 y}{\partial x^4}\right] \\ &\quad + \overline{F}\cos\Omega t - 2\overline{\mu}\frac{\partial y}{\partial t}.\end{aligned} \tag{8}$$

Because of the absence of quadratic nonlinearities, a straightforward asymptotic expansion can be written in order to include stretching and damping effects at order ε^3. Thus we assume that

$$y(x,t;\varepsilon) = \varepsilon y_1(x, T_0; T_2) + \varepsilon^3 y_3(x, T_0; T_2), \tag{9}$$

where ε is a small bookkeeping parameter to denote the deflections. Hence, a weakly nonlinear system can be investigated by this procedure. $T_0 = t$ and $T_2 = \varepsilon^2 t$ are the fast and slow time scales, respectively. These are used to characterize the modulation of the amplitude and phase due to damping, nonlinearity, and a possible resonance case. In this analysis, only the primary resonance case is considered. Hence, the damping and forcing terms are ordered as defined below so that they are a counter effect of nonlinearity:

$$\overline{\mu} = \varepsilon^2\mu, \qquad \overline{F} = \varepsilon^3 F.$$

Using the chain rule, the derivatives with respect to time are transformed according to

$$\frac{\partial}{\partial t} = D_0 + \varepsilon^2 D_2, \qquad \frac{\partial^2}{\partial t^2} = D_0^2 + 2\varepsilon^2 D_0 D_2, \tag{10}$$

where $D_n = \partial/\partial T_n$ ($n = 0, 2$). In order to obtain the equations of motion and boundary conditions at different orders, we apply expansions as follows:

Order (ε):

$$y_1^{\mathrm{iv}} + D_0^2 y_1 - \gamma^2 D_0^2 y_1'' + \kappa y_1 - \kappa\gamma^2 y_1'' = 0. \tag{11}$$

Order (ε^3):

$$\begin{aligned} &y_3^{\mathrm{iv}} + D_0^2 y_3 - \gamma^2 D_0^2 y_3'' + \kappa y_3 - \kappa\gamma^2 y_3'' \\ &\quad = -2D_0 D_2 y_1 + 2\gamma^2 D_0 D_2 y_1'' \\ &\qquad + \frac{1}{2}\left[\int_0^1 y_1'^2\,dx\right] y_1'' - \frac{1}{2}\gamma^2\left[\int_0^1 y_1'^2\,dx\right] y_1^{\mathrm{iv}} + F\cos\Omega t - 2\mu D_0 y_1. \end{aligned} \tag{12}$$

By using the first order of expansion, we obtain linear natural frequency values and by using the second order of expansion, we obtain a solvability condition.

3.1 Linear problem

The first terms in the expansions lead to the linear problem. The solution of the form is assumed as follows:

$$y_1(x, T_0, T_2) = \left[A(T_2)e^{i\omega T_0} + cc\right] Y(x), \tag{13}$$

where cc represents the complex conjugate of the preceding terms. Equation (13) is substituted into equation (11) as follows:

$$Y^{\mathrm{iv}}(x) + \left(\omega^2 + \kappa\right)\gamma^2 Y''(x) - \left(\omega^2 - \kappa\right) Y(x) = 0, \tag{14}$$

$$\begin{aligned} &\text{S-F Case:} \quad Y(0) = 0, \quad Y''(0) = 0, \quad Y(1) = 0, \quad Y''(1) = 0, \\ &\text{F-F Case:} \quad Y(0) = 0, \quad Y'(0) = 0, \quad Y(1) = 0, \quad Y'(1) = 0. \end{aligned} \tag{15}$$

The solutions to equation (15) are given by

$$Y(x) = c_1 e^{i\beta_1 x} + c_2 e^{i\beta_2 x} + c_3 e^{i\beta_3 x} + c_4 e^{i\beta_4 x}. \tag{16}$$

The boundary conditions are applied; the frequency equations can be obtained. Mode shapes of the linear first three frequency are plotted in Figure 2 with the values $\kappa = 1$ and $\gamma = 1$ and simply supported and clamped-clamped supported boundary conditions, respectively.

3.2 Non-linear problem

The solution of the nonlinear equation (12) gives corrections to the problem. They will have a solution only if a solvability condition is satisfied as explained in [51, 52]. The solvability condition is reached by separating the secular and nonsecular terms. Then the solution can be written by assuming the following expansions:

$$y_3(x, T_0, T_2) = \varphi(x, T_2)e^{i\omega T_0} + cc + W(x, T_0, T_2). \tag{17}$$

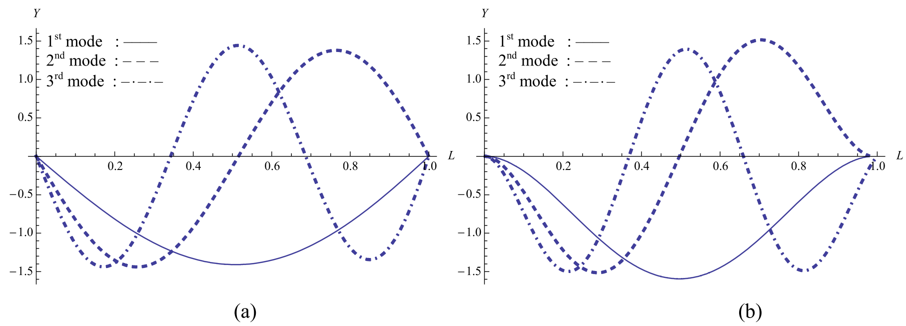

Figure 2 First three vibration modes shape. (a) Simply supported for $\kappa = 1$ and $\gamma = 1$; **(b)** clamped-clamped supported for $\kappa = 1$ and $\gamma = 0.1$.

Substituting equation (17) into equation (12), the terms that produce secular terms will be eliminated. Here, the solution related with nonsecular terms is represented by $W(x, T_0, T_2)$ and leads to

$$\begin{aligned}&\varphi^{\mathrm{iv}} - \omega^2\varphi - \gamma^2\omega^2\varphi'' + \kappa\varphi - \kappa\gamma^2\varphi'' \\ &\quad = -2i\omega A'Y(x) + 2i\omega\gamma^2 A'Y''(x) \\ &\qquad + \frac{3}{2}A^2\overline{A}\left(\int_0^1 Y'^2\,dx\right)Y''(x) - \frac{3}{2}\gamma^2 A^2\overline{A}\left(\int_0^1 Y'^2(x)\,dx\right)Y^{\mathrm{iv}}(x) \\ &\qquad + \frac{1}{2}Fe^{i\sigma T_2} - 4i\mu\omega AY(x) + cc + NST,\end{aligned} \tag{18}$$

where NST stands for nonsecular terms. The nearness of the external excitation is represented by a detuning parameter of order 1σ defined by

$$\Omega = \omega + \varepsilon^2\sigma. \tag{19}$$

Substituting equation (19) into equation (18) and after some algebraic manipulations, the solvability conditions are given by

$$2i\omega(D_2A + 2\mu A) + 2i\omega\gamma^2 D_2Ab + \frac{3}{2}A^2\overline{A}\left(b^2 + \gamma^2 b\Lambda\right) - \frac{1}{2}e^{i\sigma T_2}f = 0. \tag{20}$$

Here the coefficients are defined as $\int_0^1 Y^2\,dx = 1$, $\int_0^1 Y'^2\,dx = b$, $\int_0^1 Y''^2\,dx = \Lambda$, $\int_0^1 FY\,dx = f$. A is the complex amplitude in equation (20), which can be expressed as a real amplitude a and a phase θ,

$$A = \frac{1}{2}a(T_2)e^{i\theta(T_2)}, \tag{21}$$

where $a(T_2)$ and $\theta(T_2)$ represent the amplitude and phase angle of the response, respectively. Substituting equation (21) into equation (20), the modulation equation can be written as

$$\begin{aligned}&\omega a D_2\Phi = \omega a\sigma + \omega\gamma^2 ab\sigma - \omega\gamma^2 abD_2\Phi - \frac{3}{16}a^3\left(b^2 + \gamma^2 b\Lambda\right) + \frac{1}{2}f\cos\Phi, \\ &\omega D_2 a\left(1 + \gamma^2\right) + 2\mu\omega a = \frac{1}{2}f\sin\Phi,\end{aligned} \tag{22}$$

where $\Phi = \sigma T_2 - \theta$. The nonlinear frequencies can be calculated from equation (22) by considering free undamped vibrations.

4 Numerical results

In this section, numerical results can be achieved using the multiple scale method explained in Section 3. Only γ (the nonlocal parameter) and κ (the dimensionless elastic foundation stiffness) values are required in the calculations because of the dimensionless equations obtained. In this study, two different boundary conditions are applied, *i.e.*, simply supported and clamped-clamped supported. Nanobeams boundary conditions are represented by a two-letter symbol in order to simplify the notations. For example, the symbols S-S and C-C indicate that the nanobeam is simply supported and clamped-clamped supported, respectively. The linear natural frequencies for S-S and C-C boundary conditions are calculated for various γ and κ values. The nonlinear frequencies for free undamped vibrations were calculated. We took $\mu = f = \sigma = 0$, and obtained

$$D_2 a = 0 \quad \text{and} \quad a = a_0 \quad \text{(constant)} \tag{23}$$

from equation (17). a_0 is the steady-state real amplitude and hence the nonlinear frequency is

$$\begin{aligned} \omega_{n1} &= \omega + D_2\theta = \omega + \frac{3}{16}\frac{a_0^2(b^2 + \gamma^2 b\Lambda)}{\omega(1+\gamma^2 b)}, \\ \omega_{n1} &= \omega + a_0^2\lambda, \end{aligned} \tag{24}$$

where $\lambda = \frac{3}{16}\frac{(b^2+\gamma^2 b\Lambda)}{\omega(1+\gamma^2 b)}$ is the nonlinear correction coefficient, a measure of the effect of stretching.

Table 1 and Table 2 show the linear frequencies and nonlinear correction terms for the first three frequencies of the S-S and C-C supported nanobeams and various γ and κ values, respectively. The effects of the support conditions and various nonlocal parameters γ and dimensionless elastic foundation parameters κ are given. The elastic foundation stiffness has an important influence on the vibration properties. Nondimensional elastic foundation parameters with $\kappa = 1, 10, 50, 100, 200$ and 500 are given in Table 1 and Table 2. An increase in the dimensionless elastic foundation stiffness κ increases the system stiffness and it becomes more stable than previously. It can be found in Table 1 that increasing the κ values generally increases the linear frequencies. It should be noted that the influence of elastic stiffness is more obvious for the smallest natural frequencies. This phenomenon is in agreement with that described by Ghavanloo *et al.* [20] and Rafiei *et al.* [22]. The same situation can be observed for the clamped-clamped boundary condition. The small-scale effects play an important role in the vibration analysis of the nanobeam. In this paper, the nonlocal parameter ($\gamma = \frac{e_0 a}{L}$) is taken as $\gamma = 0, 0.1, 0.2, 0.3, 0.4$ and 0.5. As seen in Table 1 and Table 2, the linear frequencies obtained from the nonlocal theory for both boundary conditions are smaller than those obtained from the local (classical) theory. With the nonlocal parameter taken as $\gamma = 0$, the nonlocal elasticity reduces to the classical beam theory. Furthermore, an increase in the nonlocal parameter (γ) decreases the linear frequencies for both boundary conditions, as expected, as shown in Table 1 and Table 2. It can be found in the same tables that the γ values increase, and the correction

Table 1 The first five frequencies and nonlinear correction term for various γ and κ values for simple-simple support condition

	κ	γ	ω_1	ω_2	ω_3	ω_4	ω_5	λ
Simple-Simple	1	0	9.9201	39.4911	88.8321	157.917	246.742	1.84113
		0.1	9.46883	33.4426	64.6491	98.3343	132.51	1.93244
		0.2	8.41654	24.6026	41.6405	58.3889	74.8465	2.23004
		0.3	7.25166	18.5286	29.6349	40.5001	51.2291	2.77290
		0.4	6.2264	14.6293	22.7963	30.8283	38.7947	3.52508
		0.5	5.39378	12.0161	18.4661	24.8405	31.1804	4.40404
	10	0	10.3638	39.6049	88.8827	157.945	246.76	1.76231
		0.1	9.93271	33.5769	64.7187	98.38	132.544	1.84219
		0.2	8.93522	24.7849	41.7484	58.4659	74.9066	2.10059
		0.3	7.84771	18.7699	29.7864	40.611	51.3169	2.56230
		0.4	6.91145	14.9337	22.9928	30.9739	38.9105	3.17568
		0.5	6.17194	12.3849	18.7082	25.021	31.3244	3.84878
	50	0	12.1412	40.1067	89.1074	158.072	246.841	1.50432
		0.1	11.7753	34.1674	65.027	98.5831	132.695	1.55392
		0.2	10.9471	25.5791	42.2248	58.807	75.1732	1.71454
		0.3	10.079	19.8068	30.4504	41.1005	51.7051	1.99506
		0.4	9.36846	16.2178	23.8468	31.6131	39.4212	2.34281
		0.5	8.83701	13.9063	19.7483	25.8079	31.9565	2.68806
	100	0	14.0502	40.7252	89.3876	158.23	246.943	1.29992
		0.1	13.7353	34.8914	65.4103	98.8364	132.883	1.33218
		0.2	13.0322	26.5385	42.8127	59.2306	75.505	1.44022
		0.3	12.312	21.0311	31.2607	41.7044	52.1864	1.63322
		0.4	11.7375	17.6923	24.8731	32.3942	40.0503	1.86995
		0.5	11.3178	15.6008	20.9761	26.7591	32.7294	2.09886
	200	0	17.2456	41.935	89.9452	158.546	247.145	1.05906
		0.1	16.99	36.2961	66.1703	99.341	133.259	1.07698
		0.2	16.4267	28.36	43.9651	60.0688	76.1643	1.14261
		0.3	15.8615	23.2875	32.8212	42.8865	53.1359	1.26773
		0.4	15.4197	20.3228	26.808	33.9026	41.2799	1.42342
		0.5	15.1027	18.5307	23.2378	28.5666	34.223	1.57287
	500	0	24.442	45.3712	91.5977	159.489	247.751	0.74725
		0.1	24.2623	40.217	68.3997	100.84	134.38	0.75417
		0.2	23.8713	33.2309	47.2539	62.5161	78.1089	0.78627
		0.3	23.4859	29.0225	37.111	46.2521	55.8876	0.85618
		0.4	23.1898	26.7024	31.9166	38.0708	51.7907	0.94648
		0.5	22.9803	25.365	28.9827	33.4073	38.3564	1.03368

terms (λ) increase. The stretching effect is measured by λ. The C-C supported nanobeam has both linear and nonlinear frequencies higher than the S-S supported one, which can be seen in Table 1 and Table 2. The C-C supported nanobeam has the strongest end condition, while the S-S supported one has the weakest end condition.

We took $D_2a = 0$, $D_2\Phi = 0$ at the steady state. The detuning parameter frequency was defined by

$$\sigma = \frac{3}{16}\frac{a^2(b^2 + \gamma^2 b\Lambda)}{\omega(1 + \gamma^2 b)} \mp \sqrt{\frac{f^2}{4\omega^2 a^2(1 + \gamma^2 b)^2} - \mu^2}, \tag{25}$$

$$\sigma = a^2\lambda \mp \sqrt{\frac{f^2}{4\omega^2 a^2(1 + \gamma^2 b)^2} - \mu^2}. \tag{26}$$

Vibrating system frequencies are sensitive to the selection of the nonlocal parameters (γ). Hence, the selection of the appropriate nonlocal parameter is very important for a particular system. Also, the elastic stiffness has a significant effect on the system frequen-

Table 2 The first five frequencies and nonlinear correction term for various γ and κ values for clamped-clamped support condition

	κ	γ	ω_1	ω_2	ω_3	ω_4	ω_5	λ
Clamped-Clamped	1	0	22.3956	61.6809	120.908	199.862	298.557	1.26716
		0.1	21.1327	50.993	85.7223	121.352	156.744	1.45493
		0.2	18.3167	36.4377	54.5331	71.6196	88.4926	1.73402
		0.3	15.3861	27.0202	38.8468	49.687	60.7786	2.30212
		0.4	12.9434	21.1635	29.9791	37.841	46.1611	3.33512
		0.5	11.0368	17.3019	24.3529	30.501	37.1779	4.68740
	10	0	22.5957	61.7538	120.945	199.884	298.572	1.25595
		0.1	21.3446	51.0811	85.7747	121.389	156.772	1.44049
		0.2	18.5608	36.5609	54.6156	71.6824	88.5434	1.71122
		0.3	15.6759	27.1862	38.9625	49.7775	60.8526	2.25956
		0.4	13.2865	21.375	30.1288	37.9597	46.2584	3.24900
		0.5	11.4372	17.5601	24.537	30.6482	37.2987	4.52332
	50	0	23.4641	62.0769	121.11	199.984	298.639	1.20946
		0.1	22.2619	51.4712	86.0076	121.554	156.9	1.38114
		0.2	19.6087	37.1039	54.9806	71.9608	88.769	1.61976
		0.3	16.9036	27.9122	39.4725	50.1777	61.1804	2.09545
		0.4	14.715	22.2911	30.7855	38.483	46.6888	2.93359
		0.5	13.0694	18.6643	25.339	31.2939	37.8312	3.95841
	100	0	24.5064	62.4783	121.316	200.109	298.723	1.15802
		0.1	23.3579	51.9546	86.2978	121.759	157.059	1.31633
		0.2	20.8447	37.7717	55.4334	72.3074	89.0502	1.52372
		0.3	18.323	28.7939	40.1008	50.6734	61.5876	1.93312
		0.4	16.3258	23.3857	31.5871	39.1272	47.2212	2.64414
		0.5	14.8597	19.9588	26.3071	32.0829	38.4863	3.48149
	200	0	26.4682	63.2735	121.728	200.359	298.89	1.07219
		0.1	25.4085	52.9083	86.8752	122.169	157.377	1.21010
		0.2	23.1193	39.073	56.3282	72.9956	89.6099	1.37381
		0.3	20.8742	30.481	41.3289	51.6507	62.3942	1.69686
		0.4	19.145	25.4341	33.1323	40.3849	48.2685	2.25478
		0.5	17.9112	22.3239	28.1437	33.6052	39.7643	2.88835
	500	0	31.6317	65.6013	122.954	201.106	299.392	0.89717
		0.1	30.7505	55.6712	88.585	123.391	158.327	0.99988
		0.2	28.8878	42.7399	58.931	75.0224	91.2685	1.09948
		0.3	27.1244	35.0584	44.8116	54.4775	64.7537	1.30586
		0.4	25.8173	30.7716	37.3865	43.9424	51.282	1.67204
		0.5	24.9161	28.2552	33.0464	37.8062	43.3728	2.07632

cies. Figure 3 shows the nonlocal parameter effect on the nonlinear natural frequencies of the nanobeam with $\kappa = 1$, the first mode and S-S and C-C boundary conditions. It can be observed that with the increase in nonlocal parameter ($\gamma = 0, 0.1, 0.2, 0.3, 0.4$ and 0.5), the nonlinear frequencies decrease. The dimensionless linear elastic foundation parameter effect on the nonlinear natural frequencies of the nanobeam with $\gamma = 0.3$ and both boundary conditions are investigated in Figure 4 that plot the nonlinear natural frequency variation versus amplitude with $\kappa = 1, 10, 50, 100$ and 200. It can be seen in the same figures that an increase in dimensionless linear elastic stiffness and a fixed in nonlocal parameter increase in nonlinear frequency value occur regardless of the type of boundary condition.

The detuning parameter σ determines the instability region about the external excitation frequency when it is close to the natural frequency of system. The values $f = 1$ and damping coefficient $\mu = 0.1$ in equation (26) are assumed in drawing the figures. When $\sigma < 0$, we have an increase in forcing term, an increase in amplitudes and when $\sigma > 0$, an increase in forcing term, and a decrease in amplitudes at different values. The maximum amplitudes is reached when the detuning parameter is $\sigma > 0$. Figures 5, 6, 7 plot

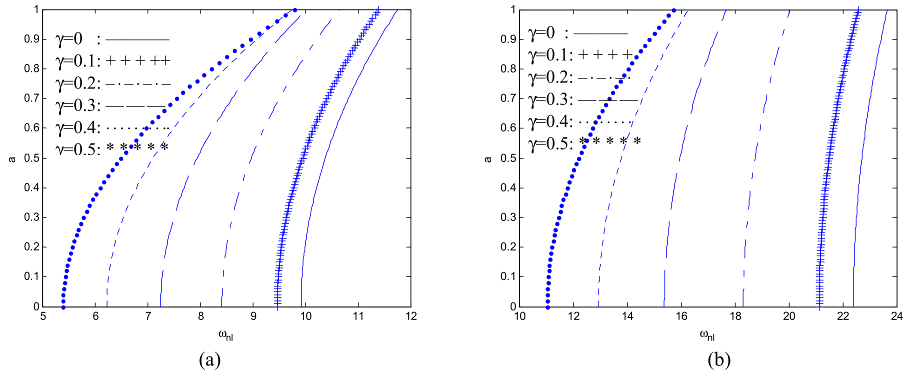

Figure 3 Nonlinear natural frequency versus amplitude for different nonlocal parameters with κ = 1 and first mode. (a) S-S case; **(b)** C-C case.

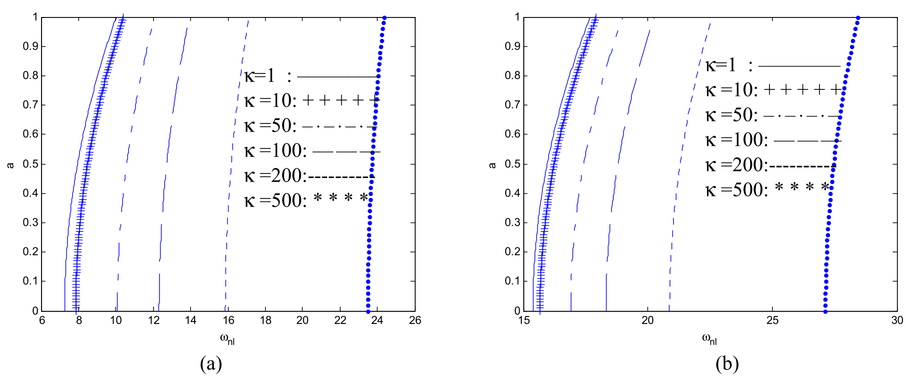

Figure 4 Nonlinear natural frequency versus amplitude for different κ values with γ = 0.3. (a) S-S case; **(b)** C-C case.

the frequency response curves for various nonlocal parameters γ, mode numbers, and dimensionless linear elastic foundation parameter κ with two boundary conditions, *i.e.* S-S and C-C. From Figure 5, it is noted that the increase in γ values leads to a decrease in the amplitude of vibration in both types of boundary conditions. In the same figures we show that the maximum amplitude decreases by increasing the γ values. The same trend can also be seen in Figure 6 with an increase in the modes. Figure 7 shows that an increase in κ values leads to a decrease of the amplitude of the vibration. It can be noted that the stiffness of the elastic foundation has a significant effect on the frequency response curves of the nanobeam. The frequency response curve tends to be a straight line with an increase of the κ values. This indicates that the nonlinear vibration will return to the linear vibration when the stiffness is large enough. This phenomenon is in agreement with that described by Fu *et al.* [44].

5 Conclusions

The nonlinear vibrations of a nanobeam resting on an elastic foundation are investigated for different end conditions. The nanobeam is described by the nonlocal Euler-Bernoulli beam model. The effect of stretching of the neutral axis is included in the nonlinear equations of motion. The multiple scale method, a perturbation technique, is used to obtain

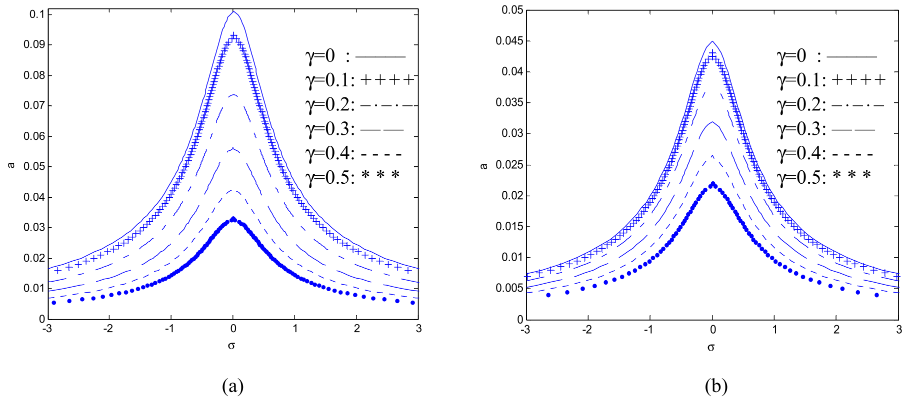

Figure 5 Frequency-response curves for different nonlocal parameters with $\kappa = 1$ and first mode. **(a)** S-S case; **(b)** C-C case.

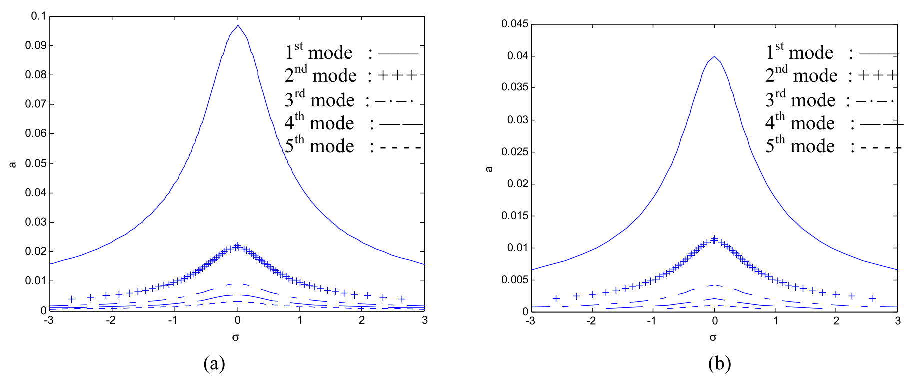

Figure 6 Frequency-response curves for different modes with $\kappa = 1$ and $\gamma = 0.1$. **(a)** S-S case; **(b)** C-C case.

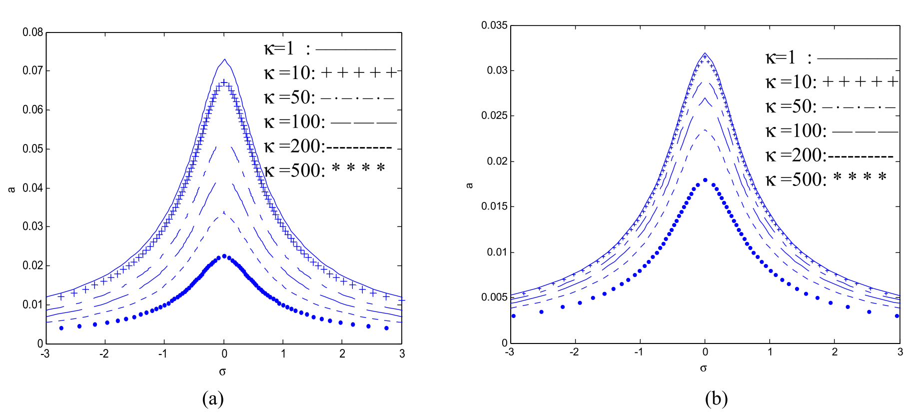

Figure 7 Frequency-response curves for different κ values with first mode and $\gamma = 0.3$. **(a)** S-S case; **(b)** C-C case.

approximate solutions. For the linear problem, exact solutions, and numerical values for natural frequencies are obtained. For the nonlinear problem, nonlinear correction terms are obtained. Nonlinear terms in the perturbation series appear as corrections to the linear problem. The effects of the nonlocal parameter (γ), dimensionless elastic foundation parameter (κ), and boundary conditions are discussed. For each of the end conditions the natural frequencies and mode shapes are tabulated and found. When nonlinear terms are added to the equations, corrections to the linear problem are introduced. The numerical result shows that the nonlinear frequency of the nanobeam decreases with increasing the nonlocal parameters. The present numerical results also reveal that an increase in the dimensionless elastic stiffness (κ) increases the nonlinear frequency value regardless of the type of boundary conditions.

Competing interests
The author declares to have no competing interests.

Received: 7 January 2016 Accepted: 14 February 2016 Published online: 29 February 2016

References

1. Eringen, AC: On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J. Appl. Phys. **54**, 4703-4710 (1983)
2. Peddieson, J, Buchanan, GR, McNitt, RP: Application of nonlocal continuum models to nanotechnology. Int. J. Eng. Sci. **41**, 305-312 (2003)
3. Niknam, H, Aghdam, MM: A semi analytical approach for large amplitude free vibration and buckling of nonlocal FG beams resting on elastic foundation. Compos. Struct. **119**, 452-462 (2015)
4. Kiani, K: A meshless approach for free transverse vibration of embedded single walled nanotubes with arbitrary boundary conditions accounting for nonlocal effect. Int. J. Mech. Sci. **52**, 1343-1356 (2010)
5. Murmu, T, Pradhan, SC: Buckling analysis of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity and Timoshenko beam theory and using DQM. Physica E **41**, 1232-1239 (2009)
6. Pradhan, SC, Reddy, GK: Buckling analysis of single walled carbon nanotube on Winkler foundation using nonlocal elasticity theory and DTM. Comput. Mater. Sci. **50**, 1052-1056 (2011)
7. Kazemi-Lari, MA, Fazelzadeh, SA, Ghavanloo, E: Non-conservative instability of cantilever carbon nanotubes resting on viscoelastic foundation. Physica E **44**, 1623-1630 (2012)
8. Narendar, S, Gopalakrishnan, S: Critical buckling temperature of single-walled carbon nanotubes embedded in a one-parameter elastic medium based on nonlocal continuum mechanics. Physica E **43**, 1185-1191 (2011)
9. Arani, AG, Amir, S, Shajari, AR, Mozdianfard, MR, Maraghi, ZK, Mohammadimehr, M: Electro-thermal non-local vibration analysis of embedded DWBNNTs. Proc. Inst. Mech. Eng., Part C, J. Mech. Eng. Sci. **224**, 745-757 (2011)
10. Arani, AG, Amir, S: Electro-thermal vibration of visco-elastically coupled BNNT systems conveying fluid embedded on elastic foundation via strain gradient theory. Physica B **419**, 1-6 (2013)
11. Chang, TP: Thermal-mechanical vibration and instability of a fluid-conveying single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity theory. Appl. Math. Model. **36**, 1964-1973 (2012)
12. Murmu, T, Pradhan, SC: Thermo-mechanical vibration of a single-walled carbon nanotube embedded in a elastic medium based on nonlocal elasticity theory. Comput. Mater. Sci. **46**, 854-859 (2009)
13. Murmu, T, Pradhan, SC: Thermal effects on the stability of embedded carbon nanotubes. Comput. Mater. Sci. **47**, 721-726 (2010)
14. Rahmati, AH, Mohammadimehr, M: Vibration analysis of non-uniform and non-homogeneous boron nitride nanorods embedded in an elastic medium under combined loadings using DQM. Physica B **440**, 88-98 (2014)
15. Mikhasev, G: On localized modes of free vibrations of single-walled carbon nanotubes embedded in nonhomogeneous elastic medium. Z. Angew. Math. Mech. **94**, 130-141 (2014)
16. Mustapha, KB, Zhong, ZW: Free transverse vibration of an axially loaded non-prismatic single-walled carbon nanotube embedded in a two parameter elastic medium. Comput. Mater. Sci. **50**, 742-751 (2010)
17. Lee, HL, Chang, WJ: Vibration analysis of a viscous-fluid-conveying single-walled carbon nanotube embedded in an elastic medium. Physica E **41**, 529-532 (2009)
18. Kiani, K: Vibration analysis of elastically restrained double-walled carbon nanotubes on elastic foundation subject to axial load using nonlocal shear deformable beam theories. Int. J. Mech. Sci. **68**, 16-34 (2013)
19. Kiani, K: Nonlinear vibrations of a single-walled carbon nanotube for delivering of nanoparticles. Nonlinear Dyn. **76**(4), 1885-1903 (2014)
20. Ghavanloo, E, Daneshmand, F, Rafiei, M: Vibration and instability analysis of carbon nanotubes conveying fluid and resting on a linear viscoelastic Winkler foundation. Physica E **42**, 2218-2224 (2010)
21. Yas, MH, Samadi, N: Free vibrations and buckling analysis of carbon nanotube-reinforced composite Timoshenko beams on elastic foundation. Int. J. Press. Vessels Piping **98**, 119-128 (2012)
22. Rafiei, M, Mohebpour, SR, Daneshmand, F: Small-scale effect on the vibration of non-uniform carbon nanotubes conveying fluid and embedded in viscoelastic medium. Physica E **44**, 1372-1379 (2012)
23. Aydogdu, M: Axial vibration analysis of nanorods (carbon nanotubes) embedded in an elastic medium using nonlocal elasticity. Mech. Res. Commun. **43**, 34-40 (2012)

24. Wang, BL, Wang, KF: Vibration analysis of embedded nanotubes using nonlocal continuum theory. Composites, Part B, Eng. **47**, 96-101 (2013)
25. Mehdipour, I, Barari, A, Kimiaeifar, A, Domairry, G: Vibrational analysis of curved single-walled carbon nanotube on a Pasternak elastic foundation. Adv. Eng. Softw. **48**, 1-5 (2012)
26. Aydogdu, M, Arda, M: Torsional vibration analysis of double walled carbon nanotubes using nonlocal elasticity. Int. J. Mech. Mater. Des. **12**(1), 71-84 (2016). doi:10.1007/s10999-014-9292-8
27. Ahangar, S, Rezazadeh, G, Shabani, R, Ahmadi, G, Toloei, A: On the stability of a microbeam conveying fluid considering modified couple stress theory. Int. J. Mech. Mater. Des. **135**, 327-342 (2011)
28. Marin, M, Marinescu, C: Thermoelasticity of initially stressed bodies, asymptotic equipartition of energies. Int. J. Eng. Sci. **36**(1), 73-86 (1998)
29. Marin, M: An evolutionary equation in thermoelasticity of dipolar bodies. J. Math. Phys. **40**(3), 1391-1399 (1999)
30. Ozturk, B, Coskun, SB: The homotopy perturbation method for free vibration analysis of beams on elastic foundation. Struct. Eng. Mech. **37**(4), 415-425 (2011)
31. Yan, Y, Wang, W, Zhang, L: Applied multiscale method to analysis of nonlinear vibration for double-walled carbon nanotubes. Appl. Math. Model. **35**, 2279-2289 (2011)
32. Wang, YZ, Li, FM: Nonlinear free vibration of nanotube with small scale effects embedded in viscous matrix. Mech. Res. Commun. **60**, 45-51 (2014)
33. Bağdatlı, SM: Nonlinear vibration of nanobeams with various boundary condition based on nonlocal elasticity theory. Composites, Part B, Eng. **80**, 43-52 (2015)
34. Bağdatlı, SM: Non-linear transverse vibrations of tensioned nanobeams using nonlocal beam theory. Struct. Eng. Mech. **55**(2), 281-298 (2015)
35. Şimşek, M: Large amplitude free vibration of nanobeams with various boundary conditions based on the nonlocal elasticity theory. Composites, Part B, Eng. **56**, 621-628 (2014)
36. Şimşek, M: Nonlinear static and free vibration analysis of microbeams based on the nonlinear elastic foundation using modified couple stress theory and He's variational method. Compos. Struct. **112**, 264-272 (2014)
37. Fallah, A, Aghdam, MM: Nonlinear free vibration and post-buckling analysis of functionally graded beams on nonlinear elastic foundation. Eur. J. Mech. A, Solids **30**, 571-583 (2011)
38. Fallah, A, Aghdam, MM: Thermo-mechanical buckling and nonlinear free vibration analysis of functionally graded beams on nonlinear elastic foundation. Composites, Part B, Eng. **43**, 1523-1530 (2012)
39. Ke, LL, Xiang, Y, Yang, J, Kitipornchai, S: Nonlinear free vibration of embedded double-walled carbon nanotubes based on nonlocal Timoshenko beam theory. Comput. Mater. Sci. **47**, 409-417 (2009)
40. Fang, B, Zhen, YX, Zhang, CP, Tang, Y: Nonlinear vibration analysis of double-walled carbon nanotubes based on nonlocal elasticity theory. Appl. Math. Model. **37**, 1096-1107 (2013)
41. Arani, AG, Atabakhshian, V, Loghman, A, Shajari, AR, Amir, S: Nonlinear vibration of embedded SWBNNTs based on nonlocal Timoshenko beam theory using DQ method. Physica B **407**, 2549-2555 (2012)
42. Shen, HS, Zhang, CL: Nonlocal beam model for nonlinear analysis of carbon nanotubes on elastomeric substrates. Comput. Mater. Sci. **50**, 1022-1029 (2011)
43. Şimşek, M: Nonlocal effects in the forced vibration of an elastically connected double-carbon nanotube system under a moving nanoparticle. Comput. Mater. Sci. **50**, 2112-2123 (2011)
44. Fu, YM, Hong, JW, Wang, XQ: Analysis of nonlinear vibration for embedded carbon nanotubes. J. Sound Vib. **296**, 746-756 (2006)
45. Ansari, R, Ramezannezhad, H: Nonlocal Timoshenko beam model for the large-amplitude vibrations of embedded multiwalled carbon nanotubes including thermal effects. Physica E **43**, 1171-1178 (2011)
46. Ansari, R, Ramezannezhad, H, Gholami, R: Nonlocal beam theory for nonlinear vibrations of embedded multiwalled carbon nanotubes in thermal environment. Nonlinear Dyn. **67**, 2241-2254 (2012)
47. Mahdavi, MH, Jiang, LY, Sun, X: Nonlinear vibration of a double-walled carbon nanotube embedded in a polymer matrix. Physica E **43**, 1813-1819 (2011)
48. Komijani, M, Esfahani, SE, Reddy, JN, Liu, YP, Eslami, MR: Nonlinear thermal stability and vibration of pre/post-buckled temperature- and microstructure-dependent functionally graded beams resting on elastic foundation. Compos. Struct. **112**, 292-307 (2014)
49. Sharabiani, PA, Yazdi, MRH: Nonlinear free vibrations of functionally graded nanobeams with surface effects. Composites, Part B, Eng. **45**, 581-586 (2013)
50. Nayfeh, AH, Mook, DT: Nonlinear Oscillations. Wiley, New York (1979)
51. Nayfeh, AH: Introduction to Perturbation Techniques. Wiley, New York (1981)
52. Öz, HR, Pakdemirli, M: Two-to-one internal resonances in a shallow curved beam resting on an elastic foundation. Acta Mech. **185**(3-4), 245-260 (2006)