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Some properties and applications related to the $(2,p)$-Laplacian operator

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Abstract
In this paper, we give some properties about the $(2,p)$-Laplacian operator ($p>1$, $p\neq 2$), and consider the existence of solutions to two kinds of partial differential equations related to the $(2,p)$-Laplacian operator by those properties. Specifically, we establish an existence result of positive solutions using fixed point index theory and an existence result of nodal solutions via the quantitative deformation lemma.

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Keywords: properties; fixed point index; quantitative deformation lemma

1 Introduction and main results

Recently, much attention has been paid to the existence of solutions to the following quasilinear elliptic problems of (q,p)-Laplacian type:

$$\begin{cases} -\Delta_q u - \Delta_p u = h(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, $p,q>1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, and the function $h:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. By a solution u of (1.1), we mean that u, belonging to some Sobolev space, solves (1.1) in the weak sense, *i.e.*, u satisfies

$$\int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla v + \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v = \int_\Omega h(x,u)v, \quad v \in C_0^\infty(\Omega).$$

Moreover, by a non-negative nontrivial solution u of (1.1), we mean that u is a solution of (1.1), $u\neq 0$ and $u(x)\geq 0$ for $x\in\Omega$; if u is a solution of (1.1) with $u^\pm\neq 0$, where $u^+ = \max\{u,0\}$ and $u^- = \max\{u,0\}$, then we say that u is a sign-changing solution of (1.1). We know that solutions to (1.1) are the steady state solutions of the general reaction-diffusion equation

$$u_t = \operatorname{div}\bigl(H(u)\nabla u\bigr) + h(x,u), \tag{1.2}$$

where $H(u) = |\nabla u|^{q-2} + |\nabla u|^{p-2}$. Equation (1.2) has a wide range of applications in physics and related sciences such as biophysics [1], plasma physics [2], and chemical reaction design [3]. The stationary solutions to (1.2) have been studied by many authors; see [4–9].

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When $q = 2 \neq p$ and $h(x,u) = a(x)|u|^{r-2}u + b(x)|u|^{s-2}u$, Sidiropoulos in [7] considered the existence of non-negative solutions to (1.1) with the exponents r, s being subcritical and a, b being essentially bounded functions. Their proofs are variational in character and rely either on the fibering method or on the mountain pass theorem of Ambrosetti-Rabinowitz. In [9], with $h(x,u) = \theta|u|^{r-2} + |u|^{p^*-2}u$, $p < N$, $1 < q < p < p^* = Np/(N-p)$, the authors established the existence of multiple positive solutions of (1.1) by some standard variational methods.

The purpose of this article is to give some properties and applications about the operator $-\Delta - \Delta_p$ ($p > 1$, $p \neq 2$). Let $H_0^1(\Omega)$ and $W_0^{1,p}(\Omega)$ be the usual Sobolev spaces defined as the completion of $C_0^\infty(\Omega)$ with respect to the norms $\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2)^{1/2}$ and $\|u\|_{W_0^{1,p}(\Omega)} = (\int_\Omega |\nabla u|^p)^{1/p}$, respectively. When $1 < p < 2$, let $X = H_0^1(\Omega)$ and $\|u\|_X = \|u\|_{H_0^1(\Omega)}$, or when $p > 2$, let $X = W_0^{1,p}(\Omega)$ and $\|u\|_X = \|u\|_{W_0^{1,p}(\Omega)}$. Denote by X^* the dual space of X. In addition, by $|\cdot|_q$, we denote the usual norm in $L^q(\Omega)$, $q \geq 1$.

It is well known that the operator $-\Delta$ or $-\Delta_p$ is a homeomorphism in the Sobolev space X. However, we do not know whether the operator $-\Delta - \Delta_p$ is a homeomorphism in the single Sobolev space X and we do not have the related literature in our hands. Similarly, it is obvious that the functional $\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$ and $\psi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p$ belong to $C^1(H_0^1(\Omega),\mathbb{R})$ and $C^1(W_0^{1,p}(\Omega),\mathbb{R})$, respectively. Whether the functional $J = \varphi + \psi$ belongs to $C^1(X,\mathbb{R})$ is an interesting problem. For the reader's convenience and completeness of the paper, in Section 2, we will answer those questions and obtain the properties (a) and (b):

(a) the operator $-\Delta - \Delta_p$ is a homeomorphism from X to X^* (Proposition 2.3);
(b) the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p, \quad u \in X,$$

belongs to $C^1(X,\mathbb{R})$ (Proposition 2.4).

As an application of the property (a), we will consider in Section 3 the following quasilinear elliptic equation:

$$\begin{cases} -\Delta u - \Delta_p u = f(x,u) + h(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $h \in L^\infty(\Omega)$ and f satisfies the following conditions:

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,t) \geq 0$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}_+$ and $f(x,t) = 0$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}_-$, where $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{R}_- := (-\infty, 0]$;

(f_2)

$$\lim_{t\to\infty} \frac{f(x,t)}{t + t^{p-1}} = f_\infty < \infty, \quad \text{uniformly for } x \in \overline{\Omega}.$$

The asymptotic behavior of f leads us to define the following two constants:

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^2 : u \in H_0^1(\Omega), \int_\Omega |u|^2 = 1\right\}$$

and

$$\mu_1 = \inf\left\{ \int_\Omega |\nabla u|^p : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p = 1 \right\}.$$

Our main result as regards equation (1.3) is the following theorem.

Theorem 1.1 *Suppose that f satisfies* (f_1) *and* (f_2) *with $f_\infty < \lambda_1$ when $1 < p < 2$ or $f_\infty < \mu_1$ when $p > 2$. Then equation* (1.3) *has a non-negative solution. Moreover, equation* (1.3) *has a non-negative nontrivial solution when $h \not\equiv 0$.*

Remark 1.2 When $h = 0$ the conditions in Theorem 1.1 cannot guarantee the existence of nontrivial solution to (1.3). In fact, we can get the following.

Proposition 1.3 *Assume that $f(x,u) = \lambda u^+ + \mu(u^+)^{p-1}$ with $0 < \lambda, \mu < \min\{\lambda_1, \mu_1\}$ and $h = 0$. Then* (1.3) *has only zero solution.*

Proof Suppose that u is a nontrivial solution to (1.3). Then

$$\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u|^p = \lambda \int_\Omega u^2 + \mu \int_\Omega |u|^p.$$

Hence,

$$\begin{aligned} 0 < (\lambda_1 - \lambda) \int_\Omega u^2 &\le \int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 \\ &= \mu \int_\Omega |u|^p - \int_\Omega |\nabla u|^2 \le (\mu - \mu_1) \int_\Omega |u|^p < 0, \end{aligned}$$

which is a contradiction. The proof is completed. □

As an application of the property (b), in Section 4, we will investigate the existence of least energy sign-changing solution of the following equation:

$$\begin{cases} -\Delta u - \Delta_p u = g(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

By the symmetry, we only consider the case $p > 2$. Here, $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following hypotheses:

(g_1) $\lim_{t\to 0} g(t)/t = 0$;
(g_2) for some constant $q \in (p, p^*)$, $\lim_{|t|\to\infty} g(t)/|t|^{q-1} = 0$, where $p^* = \infty$ for $N \le p$, and $p^* = Np/(N-p)$ for $N > p$;
(g_3) there exists $\mu > p$ such that $\lim_{|t|\to\infty} G(t)/|t|^\mu = \infty$, where $G(t) = \int_0^t g(s)\,\mathrm{d}s$ for all $t \in \mathbb{R}$;
(g_4) $g(t)/|t|^{\mu-1}$ is increasing on $(-\infty, 0)$ and $(0, \infty)$, respectively.

Our main result as regards equation (1.4) is the following theorem.

Theorem 1.4 *If the assumptions* (g_1)-(g_4) *hold, then the problem* (1.4) *has one least energy sign-changing solution.*

Remark 1.5 In general, in order to obtain sign-changing solutions, it is common to assume that the nonlinearity satisfies $g(t)t \geq 0$, $t \in \mathbb{R}$. However, there are functions satisfying (g_1)-(g_4) but not having the property $g(t)t \geq 0$, $t \in \mathbb{R}$. For example, we consider the following function:

$$g(t) = \begin{cases} t^3 \log t, & t > 0, \\ -t^4, & t \leq 0. \end{cases}$$

It is obvious that g satisfies (g_1)-(g_4). But $g(t)t < 0$ when $0 < t < 1$. The proof of Theorem 1.4 is based on the ideas from [10, 11] and we put some new ingredients in the proof process.

The paper is organized as follows. First, in Section 2, we prove the two properties related to the operator $-\Delta - \Delta_p$. Next, the applications of the properties (a) and (b) are given in Sections 3 and 4, respectively. In this paper, c_i $(i = 1, 2, \ldots)$ and the C_ε denote various positive constant whose exact values are not essential to the analysis of the relevant problems.

2 Properties of the operator $-\Delta - \Delta_p$

In this section, we show the properties (a) and (b) for the operator $-\Delta - \Delta_p$. Before completing the proof of the property (a), we introduce some notations and lemmas first. Let $P = \{u \in X : u(x) \geq 0, \text{a.e. } x \in \Omega\}$ be the positive cone in X and let $P^* = \{h \in X^* : \langle h, u\rangle \geq 0, u \in P\}$ be its dual cone. Define a nonlinear operator $A : X \to X^*$ by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v, \quad u, v \in X.$$

Lemma 2.1 [12] *There exist constants c_i $(i = 1, 2, 3, 4)$ such that, for all $x_1, x_2 \in \mathbb{R}^N$, when $1 < p < 2$,*

$$(x_2 - x_1) \cdot \left(|x_2|^{p-2}x_2 - |x_1|^{p-2}x_1\right) \geq c_1\left(|x_2| + |x_1|\right)^{p-2} |x_2 - x_1|^2, \tag{2.1}$$

$$\left||x_2|^{p-2}x_2 - |x_1|^{p-2}x_1\right| \leq c_2|x_2 - x_1|^{p-1}; \tag{2.2}$$

when $p > 2$,

$$(x_2 - x_1) \cdot \left(|x_2|^{p-2}x_2 - |x_1|^{p-2}x_1\right) \geq c_3|x_2 - x_1|^p, \tag{2.3}$$

$$\left||x_2|^{p-2}x_2 - |x_1|^{p-2}x_1\right| \leq c_4\left(|x_2| + |x_1|\right)^{p-2} |x_2 - x_1|. \tag{2.4}$$

Remark 2.2 In (2.1), when $x_1 = x_2 = 0$, we define $(|x_2| + |x_1|)^{p-2}|x_2 - x_1|^2 = 0$.

Proposition 2.3 *The operator A is a homeomorphism from X to X^* and $A^{-1}(P^*) \subset P$.*

Proof First of all, we show that A is a homeomorphism. When $1 < p < 2$, for any $u, v \in X$, by (2.1), we have

$$\begin{aligned} \langle Au - Av, u - v\rangle &= \int_\Omega \left|\nabla(u - v)\right|^2 + \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot \nabla(u - v) \\ &\geq \int_\Omega \left|\nabla(u - v)\right|^2 + c_1 \int_\Omega \left(|\nabla u| + |\nabla v|\right)^{p-2} \left|\nabla(u - v)\right|^2 \\ &\geq \int_\Omega \left|\nabla(u - v)\right|^2. \end{aligned}$$

When $p > 2$, for any $u, v \in X$, by (2.3), we get

$$\begin{aligned}\langle Au - Av, u - v\rangle &= \int_\Omega |\nabla(u-v)|^2 + \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot \nabla(u-v) \\ &\geq \int_\Omega |\nabla(u-v)|^2 + c_3 \int_\Omega |\nabla(u-v)|^p \\ &\geq c_3 \int_\Omega |\nabla(u-v)|^p.\end{aligned}$$

Hence, A is a strongly monotone operator.

Then we claim A is continuous from X to X^*. Assume $u_n \to u$ in X. For all $w \in X$, when $1 < p < 2$, by the Hölder inequality, the Sobolev embedding theorem, and (2.2), we obtain

$$\begin{aligned}&\left|\langle Au_n - Au, w\rangle\right| \\ &\quad= \left|\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot \nabla w + \int_\Omega (\nabla u_n - \nabla u)\cdot \nabla w\right| \\ &\quad\leq \int_\Omega \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right||\nabla w| + \int_\Omega |\nabla(u_n - u)||\nabla w| \\ &\quad\leq c_2 \int_\Omega |\nabla(u_n-u)|^{p-1}|\nabla w| + \int_\Omega |\nabla(u_n-u)||\nabla w| \\ &\quad\leq c_2 \left(\int_\Omega |\nabla(u_n-u)|^2\right)^{(p-1)/2} \left(\int_\Omega |\nabla w|^2\right)^{1/2} |\Omega|^{(2-p)/2} \\ &\qquad+ \left(\int_\Omega |\nabla(u_n-u)|^2\right)^{1/2} \left(\int_\Omega w^2\right)^{1/2} \\ &\quad\leq c_5 \|u_n - u\|_X^{p-1}\|w\|_X + \|u_n - u\|_X\|w\|_X;\end{aligned} \tag{2.5}$$

similarly, when $p > 2$, by the Hölder inequality, the Sobolev embedding theorem, and (2.4), we can get

$$\begin{aligned}&\left|\langle Au_n - Au, w\rangle\right| \\ &\quad\leq \int_\Omega \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right||\nabla w| + \int_\Omega |\nabla(u_n - u)||\nabla w| \\ &\quad\leq c_4 \int_\Omega \left(|\nabla u_n| + |\nabla u|\right)^{p-2}|\nabla(u_n-u)||\nabla w| + \int_\Omega |\nabla(u_n-u)||\nabla w| \\ &\quad\leq c_4 \left(\int_\Omega \left(|\nabla u_n| + |\nabla u|\right)^p\right)^{(p-2)/p} \left(\int_\Omega |\nabla(u_n-u)|^p\right)^{1/p} \left(\int_\Omega |\nabla w|^p\right)^{1/p} \\ &\qquad+ c_6 \|u_n - u\|_X\|w\|_X \\ &\quad\leq c_7 \left(\|u_n\|_X + \|u\|_X\right)^{p-2}\|u_n - u\|_X\|w\|_X + c_6\|u_n - u\|_X\|w\|_X.\end{aligned} \tag{2.6}$$

Thus, $\|Au_n - Au\| \to 0$ as $n \to \infty$. By the strong monotone operator theorem ([13], Theorem 26.A, p.557), A is a homeomorphism.

To show the second part of this proof, we assume that $w \in P^*$. By the first part of the lemma, there exists $u \in X$ such that

$$\int_\Omega \left[|\nabla u|^{p-2}\nabla u \cdot \nabla v + \nabla u \cdot \nabla v\right] = \langle w, v\rangle, \quad v \in X. \tag{2.7}$$

Taking $v = u^-$ in (2.7), we have $\int_\Omega |\nabla u^-|^p + \int_\Omega |\nabla u^-|^2 \le 0$. Hence $u(x) \ge 0$ almost everywhere for $x \in \Omega$, that is, $u \in P$. Then the proof is completed. □

Next, we show the property (b) for the operator $-\Delta - \Delta_p$.

Proposition 2.4 *The functional*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p, \quad u \in X,$$

belongs to $C^1(X, \mathbb{R})$.

Proof For any $u \in X$, we define the functional $J_1(u) = \int_\Omega |\nabla u|^2$ and $J_2(u) = \int_\Omega |\nabla u|^p$ for convenience. The proof will be completed by considering the following two cases.

(i) When $1 < p < 2$, since $J_1(u) = \int_\Omega |\nabla u|^2 = \|u\|_X^2$, J_1 is of $C^1(X, \mathbb{R})$. Hence we only need to show that J_2 is of $C^1(X, \mathbb{R})$.

We first show J_2 is Gâteaux differentiable. In fact, let $u, v \in X$. For fixed $x \in \Omega$ and $0 < |t| < 1$, there exists a constant $\lambda \in [0, 1]$ such that

$$\frac{1}{|t|}\left|\left|\nabla(u + tv)\right|^p - |\nabla u|^p\right| \le p\left|\nabla(u + \lambda tv)\right|^{p-1}|\nabla v| \le p\left(|\nabla u| + |\nabla v|\right)^{p-1}|\nabla v|.$$

By the Hölder inequality and the Sobolev embedding theorem, we find that

$$\begin{aligned}\int_\Omega \left|\left(|\nabla u| + |\nabla v|\right)^{p-1}|\nabla v|\right| &\le \left(\int_\Omega \left(|\nabla u| + |\nabla v|\right)^2\right)^{(p-1)/2}\left(\int_\Omega |\nabla v|^2\right)^{1/2}|\Omega|^{(2-p)/2} \\ &\le c_8\left(\|u\|_X + \|v\|_X\right)^{p-1}\|v\|_X < \infty.\end{aligned}$$

Hence, $(|\nabla u| + |\nabla v|)^{p-1}|\nabla v| \in L^1$. By the Lebesgue theorem, we obtain

$$\begin{aligned}\left\langle J_2'(u), v\right\rangle &= \lim_{t\to 0}\int_\Omega \frac{|\nabla(u + tv)|^p - |\nabla u|^p}{t} \\ &= p\lim_{t\to 0}\int_\Omega |\nabla u + \lambda t\nabla v|^{p-2}\nabla(u + \lambda tv)\cdot\nabla v \\ &= p\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v.\end{aligned}$$

Next, we show the continuity of the Gâteaux differentiability. Assume $u_n \to u$ as $n \to \infty$ in X. Therefore, similar to (2.5), we can deduce

$$\begin{aligned}\left|\left\langle J_2'(u_n) - J_2'(u), v\right\rangle\right| &\le p\int_\Omega \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right||\nabla v| \\ &\le c_2\int_\Omega \left|\nabla(u_n - u)\right|^{p-1}|\nabla v| \\ &\le c_5\|u_n - u\|_X^{p-1}\|v\|_X.\end{aligned}$$

Obviously,

$$\left\|J_2'(u_n) - J_2'(u)\right\| \le c_5\|u_n - u\|_X^{p-1} \to 0, \quad n \to \infty.$$

Hence $J_2 \in C^1(X, \mathbb{R})$.

(ii) When $p > 2$, $J_2 = \int_\Omega |\nabla u|^p$ is of $C^1(X,\mathbb{R})$ (see [14], Proposition 1.11, p.9). Hence we only show that $J_1(u)$ is of $C^1(X,\mathbb{R})$. Letting $u, v \in X$, for fixed $0 < |t| < 1$, we have

$$\frac{1}{|t|}\left|\left|\nabla(u+tv)\right|^2 - |\nabla u|^2\right| = \left|t|\nabla v|^2 + 2\nabla u \cdot \nabla v\right| \le |\nabla v|^2 + 2|\nabla u||\nabla v|.$$

By the Hölder inequality and the Sobolev embedding theorem, we get

$$\int_\Omega \left(|\nabla v|^2 + 2|\nabla u||\nabla v|\right) \le c_9\|v\|_X^2 + 2c_{10}\|u\|_X\|v\|_X < \infty.$$

Hence, $|\nabla v|^2 + 2|\nabla u||\nabla v| \in L^1$. Then

$$\begin{aligned}\langle J_1'(u), v\rangle &= \lim_{t\to 0}\int_\Omega \frac{|\nabla(u+tv)|^2 - |\nabla u|^2}{t} \\ &= \lim_{t\to 0}\int_\Omega \left(t|\nabla v|^2 + 2\nabla u \cdot \nabla v\right) \\ &= 2\int_\Omega \nabla u \cdot \nabla v.\end{aligned}$$

Assume $u_n \to u$ as $n \to \infty$ in X. Therefore, similar to (2.6), we can find that

$$\left|\langle J_1'(u_n) - J_1'(u), v\rangle\right| \le 2\int_\Omega \left|\nabla(u_n - u)\right||\nabla v| \le 2c_{11}\|u_n - u\|_X\|v\|_X.$$

Then

$$\left\|J_2'(u_n) - J_2'(u)\right\| \le 2c_{11}\|u_n - u\|_X \to 0, \quad n \to \infty.$$

Hence, $J_1 \in C^1(X,\mathbb{R})$. The proof is completed. □

3 An application of the property (a)

In this section, we mainly show Theorem 1.1 by the fixed point index theory. Hence, we first introduce a proposition about the fixed point index.

Proposition 3.1 (see [15]) *Let E be a real Banach space, $V \subset E$ be a cone, and $U \subset V$ be a bounded open subset of V. If the completely continuous operator $B : \overline{U} \to V$ has no fixed point on ∂U, then there exists an integer $i(B,U,V)$, which is regarded as the fixed point index, and the following statements hold*:

(i) *If $B : \overline{U} \to U$ is a constant mapping, then $i(B,U,V) = 1$.*

(ii) *Assume that U_1 and U_2 are disjoint open subsets of U and B has no fixed point in $\overline{U} \setminus (U_1 \cup U_2)$, then $i(B,U,V) = i(B,U_1,V) + i(B,U_2,V)$, where $i(B,U_i,V) = i(B_{\overline{U_i}},U_i,V)$, $i = 1,2$.*

(iii) *If $H : [0,1] \times \overline{U} \to V$ is a completely continuous homotopy and $H(t,u) \neq u$ for any $(t,u) \in [0,1] \times \partial U$, then $i(H(t,\cdot),U,V)$ is independent of $t \in [0,1]$.*

(iv) *If $i(B,U,V) \neq 0$, then B has a fixed point in U.*

Before proving Theorem 1.1, we need to give some definitions and a lemma, which will be used to prove Theorem 1.1. We define the operators L and K by

$$\langle Lu, v\rangle = \int_\Omega f(x,u)v, \quad u, v \in X,$$

$$\langle Ku, v\rangle = \begin{cases} \int_\Omega uv, & u, v \in X, p \in (1,2), \\ \int_\Omega |u|^{p-2}uv, & u, v \in X, p \in (2,\infty). \end{cases}$$

Lemma 3.2 *Suppose that f satisfies* (f_1) *and* (f_2). *Then when $1 < p < 2$, we have*

$$\lim_{\|u\|_X \to \infty, u \in P} \frac{Lu - f_\infty Ku}{\|u\|_X} = 0;$$

when $p > 2$, we have

$$\lim_{\|u\|_X \to \infty, u \in P} \frac{Lu - f_\infty Ku}{\|u\|_X^{p-1}} = 0.$$

Proof When $1 < p < 2$, by (f_1), (f_2), for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\bigl|f(x,t) - f_\infty t\bigr| \le \varepsilon t + C_\varepsilon, \quad x \in \overline{\Omega}, t \ge 0.$$

For $u \in P \setminus \{0\}$, letting $w = u/\|u\|_X$, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\begin{aligned} \sup_{\|v\|_X \le 1} \left| \left\langle \frac{Lu - f_\infty Ku}{\|u\|_X}, v \right\rangle \right| &\le \sup_{\|v\|_X \le 1} \int_\Omega \frac{|f(x,u) - f_\infty u|}{\|u\|_X} |v| \\ &\le \sup_{\|v\|_X \le 1} \int_\Omega \bigl(\varepsilon w|v| + C_\varepsilon \|u\|_X^{-1}|v|\bigr) \\ &\le \sup_{\|v\|_X \le 1} \bigl(\varepsilon |w|_2|v|_2 + C_\varepsilon \|u\|_X^{-1}|v|_2|\Omega|^{1/2}\bigr) \\ &\le \sup_{\|v\|_X \le 1} \bigl(\varepsilon c_1 \|w\|_X\|v\|_X + C_\varepsilon c_2 \|u\|_X^{-1}\|v\|_X\bigr) \\ &\le \varepsilon c_1 + C_\varepsilon c_2 \|u\|_X^{-1}. \end{aligned}$$

Then, when $\|u\|_X \to \infty$, we have

$$\lim_{\|u\|_X \to \infty, u \in P} \frac{Lu - f_\infty Ku}{\|u\|_X} = 0.$$

Similarly, when $p > 2$, we get

$$\bigl|f(x,t) - f_\infty t^{p-1}\bigr| \le \varepsilon t^{p-1} + C_\varepsilon, \quad x \in \overline{\Omega}, t \ge 0.$$

For $u \in P \setminus \{0\}$, letting $w = u/\|u\|_X$, we can find that

$$\begin{aligned} \sup_{\|v\|_X \le 1} \left| \left\langle \frac{Lu - f_\infty Ku}{\|u\|_X^{p-1}}, v \right\rangle \right| &\le \sup_{\|v\|_X \le 1} \int_\Omega \frac{|f(x,u) - f_\infty u^{p-1}|}{\|u\|_X^{p-1}} |v| \\ &\le \sup_{\|v\|_X \le 1} \int_\Omega \bigl(\varepsilon w|v| + C_\varepsilon \|u\|_X^{1-p}|v|\bigr) \\ &\le C_\varepsilon \varepsilon c_3 + c_4 \|u\|_X^{1-p}. \end{aligned}$$

Then, when $\|u\|_X \to \infty$, we have

$$\lim_{\|u\|_X\to\infty, u\in P} \frac{Lu - f_\infty Ku}{\|u\|_X^{p-1}} = 0.$$

The proof is completed. □

Because of the assumptions (f_1) and (f_2), f satisfies the subcritical condition. By [16], Proposition B.10, p.90, we know $L : X \to X^*$ is compact. Hence, $A^{-1}L$ is a completely continuous operator. Since $A^{-1}h$ is a constant operator, then the operator $T : X \to X$, where T is defined by $T = A^{-1}L + A^{-1}h$, is a completely continuous operator.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 In order to prove Theorem 1.1, we only need to show the fixed point index $i(T, P_r, P) = 1$ for large r. To this end, we define a completely continuous homotopy function $H : [0,1] \times X \to X^*$ by

$$H(t,u) = tLu + th, \quad (t,u) \in [0,1] \times P.$$

We claim that there exists $R_0 > 0$ such that the operator equation

$$Au = H(t,u) \tag{3.1}$$

has no solution on $[0,1] \times \partial P_r$ for $r > R_0$. We prove by contradiction. Suppose that there exists a sequence $\{(t_n, u_n)\} \subset [0,1] \times P$ such that

$$t_n \to t_0, \qquad \|u_n\| \to \infty, \quad n \to \infty,$$

where (t_n, u_n) satisfies (3.1), that is,

$$\int_\Omega \nabla u_n \cdot \nabla v + \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v = t_n \int_\Omega f(x, u_n) v + t_n \int_\Omega hv, \quad v \in X.$$

Let $\omega_n = u_n / \|u_n\|_X$. Then

$$\begin{aligned} &\|u_n\|_X \int_\Omega \nabla \omega_n \cdot \nabla v + \|u_n\|_X^{p-1} \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla v \\ &\quad = t_n \int_\Omega f(x, u_n) v + t_n \int_\Omega hv. \end{aligned} \tag{3.2}$$

Since $\{w_n\}$ is bounded in P, we may assume for some $w_0 \in P$, by passing to a subsequence if necessary, that $w_n \rightharpoonup w_0 \in P$.

When $1 < p < 2$, by (3.2), we can calculate that

$$\begin{aligned} &\int_\Omega \nabla \omega_n \cdot \nabla v + \|u_n\|_X^{p-2} \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla v \\ &\quad = t_n \int_\Omega \frac{f(x, u_n) - f_\infty u_n}{\|u_n\|_X} v + t_n f_\infty \int_\Omega \omega_n v + t_n \int_\Omega \frac{hv}{\|u_n\|_X}, \end{aligned} \tag{3.3}$$

taking $v = w_n$ in (3.3) and letting $n \to \infty$, by Lemma 3.2, we have

$$1 = t_0 f_\infty \int_\Omega \omega_0^2.$$

Hence,

$$\lambda_1 = t_0 \lambda_1 f_\infty \int_\Omega w_0^2 \leq f_\infty \lambda_1 \int_\Omega \omega_0^2 \leq f_\infty,$$

which contradicts $f_\infty < \lambda_1$.

When $p > 2$, by (3.2), we can deduce

$$\begin{aligned} &\frac{1}{\|u_n\|_X^{p-2}} \int_\Omega \nabla \omega_n \cdot \nabla v + \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla v \\ &\quad = t_n \int_\Omega \frac{f(x,u_n) - f_\infty u_n^{p-1}}{\|u_n\|_X^{p-1}} v + t_n f_\infty \int_\Omega \omega_n^{p-1} v + t_n \int_\Omega \frac{hv}{\|u_n\|_X^{p-1}}. \end{aligned}$$

Similarly, we get

$$1 = t_0 f_\infty \int_\Omega \omega_0^p.$$

Hence,

$$\mu_1 = t_0 \mu_1 f_\infty \int_\Omega w_0^p \leq f_\infty \mu_1 \int_\Omega \omega_0^p \leq f_\infty,$$

which contradicts $f_\infty < \mu_1$.

Consequently, taking $r > R_0$, we obtain

$$i(AT, P_r, P) = i\big(A^{-1}H(1,\cdot), P_r, P\big) = i\big(A^{-1}H(0,\cdot), P_r, P\big) = 1. \tag{3.4}$$

By (3.4), we know the problem (1.1) has a non-negative solution $u \in X$. Especially, when $h \not\equiv 0$, it is quite evident that $u = 0$ is not the solution of the problem (1.1). Hence, (1.1) has a non-negative nontrivial solution. The proof is completed. □

4 An application of the property (b)

In this section, we first define the energy functional $I : X \to \mathbb{R}$ by

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{p} \int_\Omega |\nabla u|^p - \int_\Omega G(u), \quad v \in X.$$

It is obvious that the functional I is well defined and belongs to $C^1(X,\mathbb{R})$ by Proposition 2.4. Furthermore,

$$\langle I'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v - \int_\Omega g(u) v, \quad u, v \in X.$$

Clearly, critical points of I are the weak solutions to (1.4). Moreover, if $u \in X$ is a solution to (1.4) and $u^\pm \not\equiv 0$, where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$, then u is a sign-changing solution.

In order to get a sign-changing solution to (1.4), we first need to seek a minimizer of the energy functional I over the constraint:

$$M = \{u \in X : u^{\pm} \neq 0, \langle I'(u), u^{+}\rangle = \langle I'(u), u^{-}\rangle = 0\},$$

and then we need to prove that the minimizer is a sign-changing solution to (1.4).

Now we first state the following lemmas.

Lemma 4.1 *Assume that* (g_1)-(g_4) *hold, and* $u \in X$ *with* $u^{\pm} \neq 0$. *Then there is a unique pair* $(s_0, t_0) \in (0,\infty) \times (0,\infty)$ *such that* $s_0 u^+ + t_0 u^- \in M$.

Proof For any $u \in X$ with $u^{\pm} \neq 0$, we define

$$\begin{aligned} P(s) &= \langle I'(su^+ + tu^-), su^+\rangle \\ &= s^2 \int_\Omega |\nabla u^+|^2 + s^p \int_\Omega |\nabla u^+|^p - \int_\Omega g(su^+) su^+ \qquad (4.1)\\ &= s^2 \left[\int_\Omega |\nabla u^+|^2 + s^{p-2} \int_\Omega |\nabla u^+|^p - \int_\Omega \frac{g(su^+)u^+}{s}\right]. \qquad (4.2) \end{aligned}$$

By the conditions (g_1) and (g_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(t)| \leq \varepsilon|t| + C_\varepsilon |t|^{q-1}, \quad t \in \mathbb{R}. \qquad (4.3)$$

According to the condition (g_1), we have, for each $\eta \in \mathbb{R}$,

$$\lim_{t\to 0} \frac{g(t\eta)}{t} = 0. \qquad (4.4)$$

Thus, by (4.3), (4.4), and the Lebesgue theorem, we get

$$\lim_{s\to 0^+} \int_\Omega \frac{g(su^+)u^+}{s} = 0. \qquad (4.5)$$

Since $u^+ \neq 0$, then it follows from (4.2) and (4.5) that $P(s) > 0$ for $s > 0$ small.

By the conditions (g_3) and (g_4), we find that

$$\lim_{|t|\to\infty} \frac{g(t)}{|t|^{\mu-2}t} = \infty. \qquad (4.6)$$

By (4.6) and $g \in C^1(\mathbb{R},\mathbb{R})$, we see, for any $M > 0$, there exists a constant $c_1 > 0$ such that, for any $t \in \mathbb{R}$,

$$g(t)t \geq M|t|^\mu - c_1. \qquad (4.7)$$

According to (4.1) and (4.7), we obtain

$$\begin{aligned} P(s) &\leq s^2 \int_\Omega |\nabla u^+|^2 + s^p \int_\Omega |\nabla u^+|^p - Ms^\mu \int_\Omega |u^+|^\mu + c_1|\Omega| \\ &= s^\mu \left[s^{2-\mu} \int_\Omega |\nabla u^+|^2 + s^{p-\mu} \int_\Omega |\nabla u^+|^p + s^{-\mu} c_1|\Omega| - M\int_\Omega |u^+|^\mu\right]. \end{aligned}$$

Because of the arbitrariness of M, we see that $P(s) < 0$ for s large. Thus, there exists $s_0 > 0$ such that $P(s_0) = 0$.

Similarly, we define

$$Q(t) = \langle I'(su^+ + tu^-), tu^-\rangle = t^2 \int_\Omega |\nabla u^-|^2 + t^p \int_\Omega |\nabla u^-|^p - \int_\Omega g(tu^-)tu^-.$$

By the similar way, we get there exists $t_0 > 0$ such that $Q(t_0) = 0$.

Next, we prove the uniqueness. Suppose that there exist s_1, s_2 such that $0 < s_1 < s_2$ and $P(s_1) = P(s_2) = 0$, that is,

$$\frac{1}{s_1^{\mu-2}} \int_\Omega |\nabla u^+|^2 + \frac{1}{s_1^{\mu-p}} \int_\Omega |\nabla u^+|^p = \int_\Omega \frac{g(s_1u^+)}{|s_1u^+|^{\mu-1}} |u^+|^\mu.$$

It also holds if s_1 is replaced by s_2. Therefore,

$$\begin{aligned}
&\left(\frac{1}{s_1^{\mu-2}} - \frac{1}{s_2^{\mu-2}}\right) \int_\Omega |\nabla u^+|^2 + \left(\frac{1}{s_1^{\mu-p}} - \frac{1}{s_2^{\mu-p}}\right) \int_\Omega |\nabla u^+|^p \\
&\quad = \int_\Omega \left[\frac{g(s_1u^+)}{|s_1u^+|^{\mu-1}} - \frac{g(s_2u^+)}{|s_2u^+|^{\mu-1}}\right] |u^+|^\mu,
\end{aligned}$$

which is absurd in view of (g_4) and $0 < s_1 < s_2$. Then there exists a unique s_0 such that $P(s_0) = 0$. Similarly, the uniqueness of t_0 can be proved. The proof is completed. □

Lemma 4.2 *For fixed $u \in X$ with $u^\pm \neq 0$, the vector (s_0, t_0) obtained in Lemma* 4.1 *is the unique maximum point of the function $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined as $\phi(s,t) = I(su^+ + tu^-)$.*

Proof From the proof of Lemma 4.1, (s_0, t_0) is the unique critical point of ϕ in $(0,\infty) \times (0,\infty)$. By the assumption ($g_3$), we deduce that $\phi(s,t) \to -\infty$ uniformly as $|(s,t)| \to \infty$. So it is sufficient to check that a maximum point cannot be achieved on the boundary of $[0,\infty) \times [0,\infty)$. Without loss of generality, we may assume that $(0,\bar{t})$ is a maximum point of ϕ. By (4.5) and

$$\begin{aligned}
\phi'_s(s,\bar{t}) &= \langle I'(su^+ + \bar{t}u^-), u^+\rangle \\
&= s\int_\Omega |\nabla u^+|^2 + s^{p-1} \int_\Omega |\nabla u^+|^p - \int_\Omega f(su^+)u^+ \\
&= s\left[\int_\Omega |\nabla u^+|^2 + s^{p-2} \int_\Omega |\nabla u^+|^p - \int_\Omega \frac{f(su^+)u^+}{s}\right],
\end{aligned}$$

we have there exists $\tilde{s} > 0$ small enough such that for any $s \in (0,\tilde{s})$ we have $\phi'_s(s,\bar{t}) > 0$. It implies that $\phi(s,t)$ is an increasing function with respect to s when $s \in (0,\tilde{s})$, that is, the pair $(0,\bar{t})$ is not a maximum point of ϕ in $\mathbb{R}_+ \times \mathbb{R}_+$. Hence (s_0, t_0) is the unique maximum point. The proof is completed. □

Lemma 4.3 *Assume that* (g_1)-(g_4) *hold. Then $m := \inf\{I(u) : u \in M\} > 0$ can be achieved.*

Proof For every $u \in M$, we have $\langle I'(u), u^\pm\rangle = 0$. Thus by (4.3), we get

$$\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla u|^p = \int_\Omega g(u)u \le \varepsilon \int_\Omega |u|^2 + C_\varepsilon \int_\Omega |u|^q. \tag{4.8}$$

According to the Sobolev embedding theorem, we have

$$\|u\|^2_{H^1_0(\Omega)} + \|u\|^p_X \le \varepsilon c_1 \|u\|^2_{H^1_0(\Omega)} + C_\varepsilon c_2 \|u\|^q_X. \tag{4.9}$$

It suggests there exists a constant $\alpha > 0$ such that $\|u\|_X \ge \alpha$.

Since the conditions (g_3) and (g_4) imply that

$$H(t) = tg(t) - \mu G(t) \ge 0, \quad t \in \mathbb{R}, \tag{4.10}$$

and $H(t)$ is increasing when $t > 0$ and decreasing when $t < 0$, then we can find that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{\mu}\langle I'(u), u\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|^2_{H^1_0(\Omega)} + \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u\|^p_X + \frac{1}{\mu}\big[g(u)u - \mu G(u)\big] \\ &\ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u\|^p_X \\ &\ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\alpha^p, \end{aligned} \tag{4.11}$$

which implies $m \ge (1/p - 1/\mu)\alpha^p > 0$.

Let $\{u_n\} \subset M$ satisfy $I(u_n) \to m$. Then $\{u_n\}$ is bounded in X by (4.11). We may assume there exists $u_0 \in X$, by passing to a subsequence if necessary, such that $u_n^\pm \rightharpoonup u_0^\pm$ weakly in X. Next, we show $u_0^\pm \neq 0$. In fact, since $u_n \in M$, then similar to (4.8), we have

$$\int_\Omega |\nabla u_n^\pm|^2 + \int_\Omega |\nabla u_n^\pm|^p = \int_\Omega g(u_n^\pm)u_n^\pm \le \varepsilon \int_\Omega |u_n^\pm|^2 + C_\varepsilon \int_\Omega |u_n^\pm|^q. \tag{4.12}$$

Similar to the discussion below (4.9) there exists a constant $\alpha_1 > 0$ such that $\|u_n^\pm\|_X \ge \alpha_1$, and then it follows from (4.12) that

$$\alpha_1^p \le \varepsilon \int_\Omega |u_n^\pm|^2 + C_\varepsilon \int_\Omega |u_n^\pm|^q, \quad n = 1, 2, \ldots. \tag{4.13}$$

Since $\{u_n\}$ is bounded in X and the embedding $X \hookrightarrow L^2(\Omega)$ holds, there is $c_3 > 0$ such that

$$\alpha_1^p \le \varepsilon c_3 + C_\varepsilon \int_\Omega |u_n^\pm|^q.$$

Choosing $\varepsilon = \alpha^p/(2c_3)$, we obtain

$$\int_\Omega |u_n^\pm|^q \ge \frac{\alpha_1^p}{2C_\varepsilon}.$$

By $u_n^\pm \rightharpoonup u_0^\pm$ weakly in X and the compactness of the embedding $X \hookrightarrow L^q(\Omega)$, we get

$$\int_\Omega |u_0^\pm|^q \ge \frac{\alpha_1^p}{2C_\varepsilon}, \tag{4.14}$$

then $u_0^\pm \neq 0$.

The conditions (g_1) and (g_2) combined with $u_n^\pm \to u_0^\pm$ in $L^q(\Omega)$ yield

$$\lim_{n\to\infty}\int_\Omega g(u_n^\pm)u_n^\pm = \int_\Omega g(u_0^\pm)u_0^\pm, \qquad \lim_{n\to\infty}\int_\Omega G(u_n^\pm) = \int_\Omega G(u_0^\pm). \tag{4.15}$$

By (4.12), (4.15), and the weak lower semicontinuity of the norm, we can deduce

$$\begin{aligned}\int_\Omega|\nabla u_0^\pm|^2 + \int_\Omega|\nabla u_0^\pm|^p &\le \liminf_{n\to\infty}\left[\int_\Omega|\nabla u_n^\pm|^2 + \int_\Omega|\nabla u_n^\pm|^p\right]\\ &= \liminf_{n\to\infty}\int_\Omega g(u_n^\pm)u_n^\pm = \int_\Omega g(u_0^\pm)u_0^\pm.\end{aligned} \tag{4.16}$$

According to Lemma 4.1 and (4.16), there exists $(s_0,t_0)\in(0,1]\times(0,1]$ such that $\tilde{u} = s_0u_0^+ + t_0u_0^- \in M$. Since (4.11) and (4.15), we then have

$$\begin{aligned} m &\le I(\tilde{u}) - \frac{1}{\mu}\langle I'(\tilde{u}),\tilde{u}\rangle\\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega|\nabla\tilde{u}|^2 + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_\Omega|\nabla\tilde{u}|^p + \frac{1}{\mu}\int_\Omega\bigl[g(\tilde{u})\tilde{u}-\mu G(\tilde{u})\bigr]\\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega|\nabla s_0u_0^+|^2 + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega|\nabla t_0u_0^-|^2\\ &\quad + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_\Omega|\nabla s_0u_0^+|^p + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_\Omega|\nabla t_0u_0^-|^p\\ &\quad + \frac{1}{\mu}\int_\Omega\bigl[g(s_0u_0^+)s_0u_0^+ - \mu G(s_0u_0^+)\bigr] + \frac{1}{\mu}\int_\Omega\bigl[g(t_0u_0^-)t_0u_0^- - \mu G(t_0u_0^-)\bigr]\\ &\le \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega|\nabla u_0^+|^2 + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega|\nabla u_0^-|^2\\ &\quad + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_\Omega|\nabla u_0^+|^p + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_\Omega|\nabla u_0^-|^p\\ &\quad + \frac{1}{\mu}\int_\Omega\bigl[g(u_0^+)u_0^+ - \mu G(u_0^+)\bigr] + \frac{1}{\mu}\int_\Omega\bigl[g(u_0^-)u_0^- - \mu G(u_0^-)\bigr]\\ &= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega|\nabla u_0|^2 + \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_\Omega|\nabla u_0|^p + \frac{1}{\mu}\int_\Omega\bigl[g(u_0)u_0-\mu G(u_0)\bigr]\\ &\le \liminf_{n\to\infty}\left[I(u_n) - \frac{1}{\mu}\langle I'(u_n),u_n\rangle\right] = m.\end{aligned}$$

Thus we deduce that $s_0 = t_0 = 1$, that is, $\tilde{u} = u_0$ and $I(u_0) = m$. Then the proof is completed. □

Proof of Theorem 1.4 In order to prove Theorem 1.4, we need to show $I'(u_0) = 0$ by the quantitative deformation lemma.

It is clear that $\langle I'(u_0),u_0^+\rangle = \langle I'(u_0),u_0^-\rangle = 0$. It follows from Lemma 4.3 that, for $(s,t)\in\mathbb{R}_+\times\mathbb{R}_+$ and $(s,t)\neq(1,1)$, we have $I(su_0^+ + tu_0^-) < I(u_0^+ + u_0^-) = m$. It follows from (4.14) that $\int_\Omega|u_0^\pm|^q \ge \alpha_1^p/(2C_\varepsilon) := \beta^q$. Then $|u_0^\pm|_q \ge \beta$. We denote

$$S_q = \inf_{u\in X,|u|_q=1}\int_\Omega|\nabla u|^q. \tag{4.17}$$

We assume that $I'(u_0) \neq 0$. Then there exist $r_0, \rho > 0$ such that $\|I'(v)\| \geq \rho$ for all $\|v - u_0\|_X \leq r_0$. Let $\delta \in (0, \min\{(\beta S_q)/2, r_0/3\})$, $D = (1-\sigma, 1+\sigma) \times (1-\sigma, 1+\sigma)$, and $\varphi(s,t) = su_0^+ + tu_0^-$, where $0 < \sigma < \min\{1/2, \delta/\|u_0\|_X\}$. By Lemma 4.3 again, we get

$$\bar{m} = \max_{\partial D} I \circ \varphi < m. \tag{4.18}$$

For $\varepsilon = \min\{(m - \bar{m})/2, \rho\delta/8\}$ and $S = B(u_0, \delta)$, [14], Lemma 2.3, p.38 yields a deformation η such that

(i) $\eta(1,u) = u$, if $u \notin I^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, I^{m+\varepsilon} \cap S) \subset I^{m-\varepsilon}$;

(iii) $\|\eta(1,u) - u\| \leq \delta$ for all $u \in X$.

By Lemmas 4.2 and 4.3, for $(s,t) \in \overline{D}$, we know $I(\varphi(s,t)) \leq m < m+\varepsilon$, that is, $\varphi(s,t) \in I^{m+\varepsilon}$. Since

$$\begin{aligned}
\|\varphi(s,t) - u_0\|_X^p &= \|su_0^+ + tu_0^- - u_0^+ - u_0^-\|_X^p \\
&= |s-1|^p \|u_0^+\|_X^p + |t-1|^p \|u_0^-\|_X^p \\
&\leq \sigma^p \|u_0\|_X^p \\
&< \delta^p,
\end{aligned}$$

we know that $\varphi(s,t) \in S$. By (ii), we have $I(\eta(1, \varphi(s,t))) < m - \varepsilon$. It is clear that

$$\max_{(s,t)\in\overline{D}} I\bigl(\eta\bigl(1, \varphi(s,t)\bigr)\bigr) < m. \tag{4.19}$$

We claim that $\eta(1, \varphi(D)) \cap M \neq \emptyset$. In fact, define $\psi(s,t) = \eta(1, \varphi(s,t))$ on $\overline{D}$,

$$\begin{aligned}
\Phi(s,t) &= \bigl(\bigl\langle I'\bigl(\varphi(s,t)\bigr), su_0^+\bigr\rangle, \bigl\langle I'\bigl(\varphi(s,t)\bigr), tu_0^-\bigr\rangle\bigr) \\
&= \bigl(\bigl\langle I'\bigl(su_0^+\bigr), su_0^+\bigr\rangle, \bigl\langle I'\bigl(tu_0^-\bigr), tu_0^-\bigr\rangle\bigr), \\
\Psi(s,t) &= \bigl(\bigl\langle I'\bigl(\psi(s,t)\bigr), \bigl(\psi(s,t)\bigr)^+\bigr\rangle, \bigl\langle I'\bigl(\psi(s,t)\bigr), \bigl(\psi(s,t)\bigr)^-\bigr\rangle\bigr),
\end{aligned}$$

and define

$$\begin{aligned}
P(s) &= \bigl\langle I'\bigl(su_0^+\bigr), su_0^+\bigr\rangle = s^2 \int_\Omega \bigl|\nabla u_0^+\bigr|^2 + s^p \int_\Omega \bigl|\nabla u_0^+\bigr|^p - \int_\Omega g\bigl(su_0^+\bigr)su_0^+, \\
Q(t) &= \bigl\langle I'\bigl(tu_0^-\bigr), tu_0^-\bigr\rangle = t^2 \int_\Omega \bigl|\nabla u_0^-\bigr|^2 + t^p \int_\Omega \bigl|\nabla u_0^-\bigr|^p - \int_\Omega g\bigl(su_0^-\bigr)su_0^-,
\end{aligned}$$

and the matrix

$$B = \begin{bmatrix} P'(1) & 0 \\ 0 & Q'(1) \end{bmatrix}.$$

By (g_4) and $g \in C^1(\mathbb{R}, \mathbb{R})$, we obtain

$$g'(t)t^2 \geq (\mu - 1)g(t)t, \quad t \in \mathbb{R}. \tag{4.20}$$

According to (4.20) and $\langle I'(u_0), u_0^+\rangle = 0$, we get

$$\begin{aligned}
P'(1) &= 2\int_\Omega \left|\nabla u_0^+\right|^2 + p\int_\Omega \left|\nabla u_0^+\right|^p - \int_\Omega g'\left(u_0^+\right)\left|u_0^+\right|^2 - \int_\Omega g\left(u_0^+\right)u_0^+ \\
&\le 2\int_\Omega \left|\nabla u_0^+\right|^2 + p\int_\Omega \left|\nabla u_0^+\right|^p - (\mu-1)\int_\Omega g\left(u_0^+\right)u_0^+ - \int_\Omega g\left(u_0^+\right)u_0^+ \\
&= 2\int_\Omega \left|\nabla u_0^+\right|^2 + p\int_\Omega \left|\nabla u_0^+\right|^p - \mu\int_\Omega g\left(u_0^+\right)u_0^+ \\
&= 2\int_\Omega \left|\nabla u_0^+\right|^2 + p\int_\Omega \left|\nabla u_0^+\right|^p - \mu\int_\Omega \left|\nabla u_0^+\right|^2 - \mu\int_\Omega \left|\nabla u_0^+\right|^p \\
&= (2-\mu)\int_\Omega \left|\nabla u_0^+\right|^2 + (p-\mu)\int_\Omega \left|\nabla u_0^+\right|^p < 0.
\end{aligned}$$

Similarly, we have

$$Q'(1) \le (2-\mu)\int_\Omega \left|\nabla u_0^-\right|^2 + (p-\mu)\int_\Omega \left|\nabla u_0^-\right|^p < 0.$$

Thus, we can deduce

$$J_\Phi(1,1) = \det B > 0.$$

Therefore, by the fact that $(1,1)$ is the unique isolated zero point of the C^1 function, we have

$$\deg(\Phi, D, 0) = \operatorname{ind}\bigl(\Phi, (1,1)\bigr) = \operatorname{sgn} J_\Phi(1,1) = 1.$$

It follows from $\bar{m} < m - 2\varepsilon$, (4.17), and (i) above that $\varphi = \psi$ on ∂D. Thus $\deg(\Psi, D, 0) = \deg(\Phi, D, 0) = 1$. Hence, there exists a pair $(s_0, t_0) \in D$ such that $\Psi(s_0, t_0) = 0$. Next we need to prove $\psi^\pm(s_0, t_0) \neq 0$. We first prove $\psi^+(s_0, t_0) \neq 0$. Since $|u_0^\pm|_q \ge \beta$, for $(s_0, t_0) \in D$, we have $|\varphi^+(s_0, t_0)|_q = s_0|u^+|_q \ge \beta/2$ and $|\varphi^-(s_0, t_0)|_q = t_0|u^-|_q \ge \beta/2$. By (iii) and (4.17), we have

$$\left|\psi(s_0, t_0) - \varphi(s_0, t_0)\right|_q \le S_q^{-1}\left\|\psi(s_0, t_0) - \varphi(s_0, t_0)\right\|_X \le S_q^{-1}\delta.$$

This implies that

$$\left|\psi^\pm(s_0, t_0) - \varphi^\pm(s_0, t_0)\right|_q \le \left|\psi(s_0, t_0) - \varphi(s_0, t_0)\right|_q \le S_q^{-1}\delta.$$

Thus we obtain

$$\left|\psi^\pm(s_0, t_0)\right|_q \ge \left|\varphi^\pm(s_0, t_0)\right|_q - S_q^{-1}\delta \ge \frac{\beta}{2} - S_q^{-1}\delta > 0,$$

which yields $\psi^\pm(s_0, t_0) \neq 0$. Thus $\eta(1, \varphi(s_0, t_0)) = \psi(s_0, t_0) \in \mathcal{M}$, which is a contradiction to (4.19). Then u_0 is a critical point of I, that is, u_0 is a least energy sign-changing solution for equation (1.4). The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors participated in drafting, revising, and commenting the manuscript. All authors read and approved the final manuscript.

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References
1. Fife, PC: Mathematical Aspects of Reacting and Diffusing Systems. Lecture Notes in Biomathematics, vol. 28. Springer, Berlin (1979)
2. Struwe, M: Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 2nd edn. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)), vol. 34. Springer, Berlin (1996)
3. Aris, R: Mathematical Modelling Techniques. Research Notes in Mathematics, vol. 24. Pitman, Boston (1979)
4. Benci, V, Micheletti, AM, Visetti, D: An eigenvalue problem for a quasilinear elliptic field equation. J. Differ. Equ. **184**(2), 299-320 (2002)
5. He, C, Li, G: The existence of a nontrivial solution to the p & q-Laplacian problem with nonlinearity asymptotic to u^{p-1} at infinity in $\mathbb{R}^N$. Nonlinear Anal. **68**(5), 1100-1119 (2008)
6. Li, G, Zhang, G: Multiple solutions for the p & q-Laplacian problem with critical exponent. Acta Math. Sci. Ser. B Engl. Ed. **29**(4), 903-918 (2009)
7. Sidiropoulos, NE: Existence of solutions to indefinite quasilinear elliptic problems of p-q-Laplacian type. Electron. J. Differ. Equ. **2010**, 162 (2010)
8. Yin, H, Yang, Z: A class of p-q-Laplacian type equation with concave-convex nonlinearities in bounded domain. J. Math. Anal. Appl. **382**(2), 843-855 (2011)
9. Yin, H, Yang, Z: Multiplicity of positive solutions to a p-q-Laplacian equation involving critical nonlinearity. Nonlinear Anal. **75**(6), 3021-3035 (2012)
10. Alves, CO, Souto, MAS: Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains. Z. Angew. Math. Phys. **65**(6), 1153-1166 (2014)
11. Shuai, W: Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains. J. Differ. Equ. **259**(4), 1256-1274 (2015)
12. Damascelli, L: Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **15**(4), 493-516 (1998)
13. Zeidler, E: Nonlinear Functional Analysis and Its Applications. II/A: Linear Monotone Operators. Springer, New York (1990). Translated from the German by the author and Leo F Boron
14. Willem, M: Minimax Theorems. Progress in Nonlinear Differential Equations and Their Applications, vol. 24. Birkhäuser Boston, Boston (1996)
15. Amann, H: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev. **18**(4), 620-709 (1976)
16. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Regional Conference Series in Mathematics, vol. 65. Am. Math. Soc., Providence (1986). Published for the Conference Board of the Mathematical Sciences, Washington, DC