Zhang and Xuan *Boundary Value Problems* (2016) 2016:60
DOI 10.1186/s13661-016-0568-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Multiple positive solutions for a second-order boundary value problem with integral boundary conditions

Lixin Zhang* and Zuxing Xuan

*Correspondence:
ldtlixin@buu.edu.cn
Department of Basic Courses,
Beijing Union University, Beijing,
100101, P.R. China

Abstract

In view of the Avery-Peterson fixed point theorem, this paper investigates the existence of three positive solutions for the second-order boundary value problem with integral boundary conditions

$$\begin{cases} u''(t) + h(t)f(t,u(t),u'(t)) = 0, \quad 0 < t < 1, \\ u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)\,ds, \\ u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)\,ds. \end{cases}$$

The interesting point is that the nonlinear term involves the first-order derivative explicitly.

MSC: 34B15

Keywords: positive solutions; fixed point theorem; integral boundary conditions

1 Introduction

In this paper, we consider the positive solutions of the following boundary value problem:

$$\begin{cases} u''(t) + h(t)f(t,u(t),u'(t)) = 0, \quad 0 < t < 1, \\ u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)\,ds, \\ u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)\,ds, \end{cases} \tag{1.1}$$

where α and β are nonnegative constants.

Boundary value problems of ordinary differential equations arise in kinds of different areas of applied mathematics and physics. Many authors have studied two-point, three-point, multi-point boundary value problems for second-order differential equations extensively, see [1–4] and the references therein. In recent years, boundary value problems with integral boundary conditions also arise in thermal conduction, chemical engineering, underground water flow, and plasma physics. Some authors have investigated boundary value problems with integral boundary conditions; see [5–13]. Boucherif [6] considered the following problem:

$$\begin{cases} y''(t) = f(t,y(t)), \quad 0 < t < 1, \\ y(0) - ay'(0) = \int_0^1 g_0(s)y(s)\,ds, \\ y(1) - by'(1) = \int_0^1 g_1(s)y(s)\,ds, \end{cases}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $g_0, g_1 \in C([0,1] \to [0,+\infty))$, $a, b \geq 0$. By using Krasnoselskii's fixed point theorem, the existence of positive solutions was obtained.

To the best knowledge of the authors, no work has been done for boundary value problem (1.1) by applying the Avery-Peterson fixed point theorem. In this paper, we will study the existence of three positive solutions of BVP (1.1). Now, we give the following assumptions:

(H_1) $f \in C([0,1] \times [0,\infty) \times (-\infty,\infty), [0,\infty))$, $h \in C([0,1],[0,\infty))$;
(H_2) $g_1, g_2 \in C([0,1],[0,\infty))$, and $0 \leq \sigma_1 + \sigma_2 < 1$, $\rho = 1 - \sigma_2 - \sigma_3 + \sigma_2\sigma_3 - \sigma_1\sigma_4 > 0$, where

$$\sigma_1 = \int_0^1 \frac{\alpha + s}{1+\alpha+\beta} g_1(s)\,ds, \qquad \sigma_2 = \int_0^1 \frac{1+\beta-s}{1+\alpha+\beta} g_1(s)\,ds,$$
$$\sigma_3 = \int_0^1 \frac{\alpha + s}{1+\alpha+\beta} g_2(s)\,ds, \qquad \sigma_4 = \int_0^1 \frac{1+\beta-s}{1+\alpha+\beta} g_2(s)\,ds.$$

2 Preliminaries

In this section, we present the Avery-Peterson fixed point theorem and some lemmas.

Theorem 2.1 ([14]) *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functional on P. Let α be a nonnegative continuous concave functional on P, and let ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d,*

$$\alpha(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma,d)}$. Suppose that $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$ is a completely continuous operator and there exist positive numbers a, b, and c with $a < b$ such that

(C_1) $\{x \in P(\gamma,\theta,\alpha,b,c,d) \mid \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in P(\alpha,b;\theta,c;\gamma,d)$;
(C_2) $\alpha(Tx) > b$ *for* $x \in P(\alpha,b;\gamma,d)$ *with* $\theta(Tx) > c$;
(C_3) $0 \notin R(\gamma,\psi,a,d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma,\psi,a,d)$ *with* $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma,d)}$ such that $\gamma(x_i) \leq d$ for $i = 1,2,3$; $b < \alpha(x_1)$; $a < \psi(x_2)$ with $\alpha(x_2) < b$; $\psi(x_3) < a$.

Let $E = (C^1[0,1], \|\cdot\|)$ be the Banach space with the maximum norm

$$\|u\| = \max\Big\{\max_{0\leq t\leq 1}|u(t)|, \max_{0\leq t\leq 1}|u'(t)|\Big\}.$$

Denote by P

$$P = \big\{u \in E \mid u(t) \geq 0, \text{and } u(t) \text{is concave on } [0,1]\big\}.$$

Lemma 2.2 *If* (H_2) *holds, then for $p(t) \geq 0$, $t \in [0,1]$, the boundary value problem*

$$u''(t) + p(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) - \alpha u'(0) = \int_0^1 g_1(s)u(s)\,ds, \qquad u(1) + \beta u'(1) = \int_0^1 g_2(s)u(s)\,ds, \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)p(s)\,ds + \int_0^1 R(t,s)\int_0^1 G(s,\tau)p(\tau)\,d\tau\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{(\alpha+t)(1+\beta-s)}{1+\alpha+\beta}, & 0 \le t \le s \le 1, \\ \frac{(\alpha+s)(1+\beta-t)}{1+\alpha+\beta}, & 0 \le s \le t \le 1, \end{cases} \tag{2.3}$$

and

$$R(t,s) = \frac{[(1-\sigma_3)(1+\beta-t)+\sigma_4(\alpha+t)]g_1(s)+[\sigma_1(1+\beta-t)+(1-\sigma_2)(\alpha+t)]g_2(s)}{\rho(1+\alpha+\beta)}.$$

Remark 2.1 Here we point out that the form of $u(t)$ is different from the corresponding part of [5], but their proofs are similar, we omit them here.

It is obvious that $G(t,s) \ge 0$ for $(t,s) \in [0,1]\times[0,1]$ if $\alpha \ge 0$, $\beta \ge 0$.

Lemma 2.3 ([6]) *Let* $\alpha \ge 0$, $\beta \ge 0$. *Then for* $t,s \in [0,1]$, *we have*

$$\gamma_0 G(s,s) \le G(t,s) \le G(s,s),$$

where $0 < \gamma_0 < 1$.

$\forall u \in P$, we define

$$(Tu)(t) = \int_0^1 G(t,s)h(s)f\bigl(s,u(s),u'(s)\bigr)\,ds + \int_0^1 R(t,s)\int_0^1 G(s,\tau)h(\tau)f\bigl(\tau,u(\tau),u'(\tau)\bigr)\,d\tau\,ds. \tag{2.4}$$

By Lemma 2.2, $u(t)$ is a solution of BVP (1.1) if and only if u is a fixed point of T.

Lemma 2.4 *If conditions* (H_1) *and* (H_2) *hold, then* $T: P \to P$ *is completely continuous.*

Proof In virtue of the definitions of T, $G(t,s)$, $R(t,s)$, we see, for each $u \in P$, that there is $Tu \ge 0$, $t \in [0,1]$. From $(Tu)''(t) = -h(t)f(t,u(t),u'(t)) \le 0$, we deduce that Tu is concave on $[0,1]$. Therefore, $T: P \to P$. A standard argument indicates that $T: P \to P$ is completely continuous. □

Lemma 2.5 ([15]) *If* $u \in P$, $\delta \in (0,\frac{1}{2})$, *then* $u(t) \ge \delta \max_{0\le t\le 1} u(t)$, $t \in [\delta, 1-\delta]$.

Lemma 2.6 *For* $u \in P$, *if* (H_2) *holds, then*

$$\max_{0\le t\le 1} u(t) \le \frac{1+\alpha}{1-\sigma_1-\sigma_2}\max_{0\le t\le 1}\bigl|u'(t)\bigr|.$$

Proof The fact that $u(t) = u(0) + \int_0^t u'(s)\,ds$ implies that

$$u(t) \le u(0) + \max_{0\le t\le 1} |u'(t)|.$$

Simultaneously,

$$u(0) = \alpha u'(0) + \int_0^1 g_1(s)u(s)\,ds \le \alpha \max_{0\le t\le 1} |u'(t)| + \max_{0\le t\le 1} u(t) \int_0^1 g_1(s)\,ds.$$

Hence,

$$\max_{0\le t\le 1} u(t) \le (1+\alpha) \max_{0\le t\le 1} |u'(t)| + \max_{0\le t\le 1} u(t) \int_0^1 g_1(s)\,ds,$$

i.e.,

$$\max_{0\le t\le 1} u(t) \le \frac{1+\alpha}{1-\int_0^1 g_1(s)\,ds} \max_{0\le t\le 1} |u'(t)| = \frac{1+\alpha}{1-\sigma_1-\sigma_2} \max_{0\le t\le 1} |u'(t)|.$$ □

3 Main result

Let

$$\gamma(u) = \max_{0\le t\le 1} |u'(t)|, \qquad \theta(u) = \psi(u) = \max_{0\le t\le 1} u(t), \qquad \alpha(u) = \min_{\delta\le t\le 1-\delta} u(t),$$

where γ and θ are nonnegative continuous convex functionals, ψ is a nonnegative continuous functional, α is a nonnegative continuous concave functional on the cone P.

With Lemmas 2.5 and 2.6, for all $u \in P$, we have

$$\delta\theta(u) \le \alpha(u) \le \theta(u) = \psi(u), \qquad \|u\| = \max\{\theta(u), \gamma(u)\} \le \frac{1+\alpha}{1-\sigma_1-\sigma_2}\gamma(u).$$

For convenience, put

$$m_1 = \min\{R(t,s) \mid t,s \in [0,1]\}, \qquad M_1 = \max\{R(t,s) \mid t,s \in [0,1]\},$$
$$m_2 = \min\left\{\left|\frac{\partial R(t,s)}{\partial t}\right| \;\middle|\; t,s \in [0,1]\right\}, \qquad M_2 = \max\left\{\left|\frac{\partial R(t,s)}{\partial t}\right| \;\middle|\; t,s \in [0,1]\right\},$$
$$L = \frac{2+\alpha+2\beta}{1+\alpha+\beta} \int_0^1 h(s)\,ds + M_2 \int_0^1 G(s,s)h(s)\,ds,$$
$$M = \delta\gamma_0(1+m_1) \int_\delta^{1-\delta} G(s,s)h(s)\,ds, \qquad N = (1+M_1) \int_0^1 G(s,s)h(s)\,ds.$$

Now, we are in the position to give our main result.

Theorem 3.1 *Let conditions* (H_1) *and* (H_2) *hold, and there exist positive numbers* a, b, d *with* $0 < a < b < \delta d$ *such that*

(A_1) $f(t,x,y) \le \frac{d}{L}$, *for* $(t,x,y) \in [0,1] \times [0, \frac{1+\alpha}{1-\sigma_1-\sigma_2}d] \times [-d,d]$,
(A_2) $f(t,x,y) > \frac{b}{M}$, *for* $(t,x,y) \in [\delta, 1-\delta] \times [b, \frac{b}{\delta}] \times [-d,d]$,
(A_3) $f(t,x,y) < \frac{a}{N}$, *for* $(t,x,y) \in [0,1] \times [0,a] \times [-d,d]$.

Then BVP (1.1) *has at least three positive solutions* u_1, u_2, *and* $u_3 \in \overline{P(\gamma,d)}$ *satisfying*

$$\max_{0\le t\le 1}\left|u_i'(t)\right| \le d, \quad i = 1,2,3,$$

and

$$\min_{\delta\le t\le 1-\delta} u_1(t) > b, \qquad \max_{0\le t\le 1} u_2(t) > a, \quad \text{with} \min_{\delta\le t\le 1-\delta} u_2(t) < b, \qquad \max_{0\le t\le 1} u_3(t) < a.$$

Proof Now we prove that T satisfies the conditions of the Avery-Peterson fixed point theorem which will give the existence of three fixed points of T.

We first of all show that $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$. If $u \in \overline{P(\gamma,d)}$, then

$$\gamma(u) = \max_{0\le t\le 1}\left|u'(t)\right| \le d.$$

In view of Lemma 2.6, we have

$$\max_{0\le t\le 1} u(t) \le \frac{1+\alpha}{1-\sigma_1-\sigma_2}d,$$

then (A_1) implies that $f(t,u(t),u'(t)) \le \frac{d}{L}$. By the concavity of Tu on $[0,1]$, we have

$$\begin{aligned}
\gamma(Tu) &= \max_{0\le t\le 1}\left|(Tu)'(t)\right| = \max\left\{\left|(Tu)'(0)\right|,\left|(Tu)'(1)\right|\right\} \\
&\le \left|\int_0^1 \frac{1+\beta-s}{1+\alpha+\beta} h(s)f\left(s,u(s),u'(s)\right)ds\right| + \left|\int_0^1 h(s)f\left(s,u(s),u'(s)\right)ds\right| \\
&\quad + \left|\int_0^1 \frac{\partial R(t,s)}{\partial t}\int_0^1 G(s,\tau)h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\,ds\right| \\
&\le \left(1+\frac{1+\beta}{1+\alpha+\beta}\right)\int_0^1 h(s)f\left(s,u(s),u'(s)\right)ds \\
&\quad + \int_0^1\left|\frac{\partial R(t,s)}{\partial t}\right|\int_0^1 G(\tau,\tau)h(\tau)f\left(\tau,u(\tau),u'(\tau)\right)d\tau\,ds \\
&\le \frac{2+\alpha+2\beta}{1+\alpha+\beta}\frac{d}{L}\int_0^1 h(s)\,ds + M_2\frac{d}{L}\int_0^1 G(s,s)h(s)\,ds \\
&= \frac{d}{L}\cdot L = d.
\end{aligned}$$

Thus, $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

Second, we confirm the condition (C_1) of Theorem 2.1. By choosing $u(t) \equiv \frac{b}{\delta}$, $0 \le t \le 1$, we get

$$\alpha(u) = \frac{b}{\delta} > b, \qquad \theta(u) = \frac{b}{\delta}, \qquad \gamma(u) = 0 < d.$$

Therefore $\{u \in P(\gamma,\theta,\alpha,b,\frac{b}{\delta},d) \mid \alpha(u) > b\} \ne \emptyset$. Hence, if $u \in \{P(\gamma,\theta,\alpha,b,\frac{b}{\delta},d) \mid \alpha(u) > b\}$, then $b \le u(t) \le \frac{b}{\delta}$, $|u'(t)| \le d$, $\delta \le t \le 1-\delta$. By (A_2), we have $f(t,u(t),u'(t)) > \frac{b}{M}$, $\delta \le t \le$

$1-\delta$. Combining the definition of α with Lemma 2.5, we obtain

$$\begin{aligned}
\alpha(Tu) &= \min_{\delta\leq t\leq 1-\delta}(Tu)(t) \geq \delta \max_{0\leq t\leq 1}(Tu)(t) \\
&= \delta \max_{0\leq t\leq 1}\biggl[\int_0^1 G(t,s)h(s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\quad + \int_0^1 R(t,s)\int_0^1 G(s,\tau)h(\tau)f\bigl(\tau,u(\tau),u'(\tau)\bigr)\,d\tau\,ds\biggr] \\
&\geq \gamma_0\delta\biggl[1+\int_0^1 R(t,s)\,ds\biggr]\int_0^1 G(s,s)h(s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\geq \gamma_0\delta(1+m_1)\int_\delta^{1-\delta} G(s,s)h(s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&> \gamma_0\delta(1+m_1)\frac{b}{M}\int_\delta^{1-\delta} G(s,s)h(s)\,ds \\
&= \frac{b}{M}\cdot M = b.
\end{aligned}$$

This shows that condition (C_1) of Theorem 2.1 is satisfied.

Third, if $u \in P(\gamma,\alpha,b,d)$ and $\theta(Tu) > \frac{b}{\delta}$, then

$$\alpha(Tu) = \min_{\delta\leq t\leq 1-\delta}(Tu)(t) \geq \delta \max_{0\leq t\leq 1}(Tu)(t) = \delta\theta(Tu) > \delta\cdot\frac{b}{\delta} = b.$$

Thus, condition (C_2) of Theorem 2.1 follows.

Finally, we show that (C_3) of Theorem 2.1 holds. Clearly, $\psi(0) = 0 < a$, so $0 \notin R(\gamma,\psi,a,d)$. Suppose that $u \in R(\gamma,\psi,a,d)$ with $\psi(u) = a$, then $0 \leq u(t) \leq a$, $t \in [0,1]$. By (A_3), we get

$$\begin{aligned}
\psi(Tu) &= \max_{0\leq t\leq 1}(Tu)(t) \\
&= \max_{0\leq t\leq 1}\biggl[\int_0^1 G(t,s)h(s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&\quad + \int_0^1 R(t,s)\int_0^1 G(s,\tau)h(\tau)f\bigl(\tau,u(\tau),u'(\tau)\bigr)\,d\tau\,ds\biggr] \\
&\leq \max_{0\leq t\leq 1}\biggl[1+\int_0^1 R(t,s)\,ds\biggr]\int_0^1 G(s,s)h(s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&= (1+M_1)\int_0^1 G(s,s)h(s)f\bigl(s,u(s),u'(s)\bigr)\,ds \\
&< (1+M_1)\frac{a}{N}\int_0^1 G(s,s)h(s)\,ds \\
&= \frac{a}{N}\cdot N = a.
\end{aligned}$$

Condition (C_3) of Theorem 2.1 is also satisfied.

Therefore, Theorem 2.1 implies that BVP (1.1) has at least three positive solutions u_1, u_2, and u_3 such that

$$\max_{0\leq t\leq 1}\bigl|u_i'(t)\bigr| \leq d,\quad i=1,2,3,$$

and

$$\min_{\delta\le t\le 1-\delta} u_1(t) > b, \qquad \max_{0\le t\le 1} u_2(t) > a, \qquad \min_{\delta\le t\le 1-\delta} u_2(t) < b, \qquad \max_{0\le t\le 1} u_3(t) < a.$$

The proof of Theorem 3.1 is complete. □

In the following we give an example to illustrate our result.

4 Example

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} u''(t) + f(t,u(t),u'(t)) = 0, & 0 < t < 1, \\ u(0) - 2u'(0) = \frac{1}{2}\int_0^1 u(s)\,ds, & u(1) + 2u'(1) = \int_0^1 su(s)\,ds, \end{cases} \tag{4.1}$$

where

$$f(t,x,y) = \begin{cases} \frac{1}{100}t + 75x^{10} + \frac{1}{100}(\frac{y}{6\times 10^8})^2, & x \le 3, \\ \frac{1}{100}t + 75\times 3^{10} + \frac{1}{100}(\frac{y}{6\times 10^8})^2, & x > 3. \end{cases} \tag{4.2}$$

Let $\delta = \frac{1}{3}$, $a = \frac{1}{2}$, $b = 1$, $d = 6\times 10^8$, after a direct calculation, we get $\sigma_1 = \frac{1}{4}$, $\sigma_2 = \frac{1}{4}$, $\sigma_3 = \frac{4}{15}$, $\sigma_4 = \frac{7}{30}$, $\rho = \frac{59}{120}$, $\gamma_0 = \frac{2}{15}$, $m_1 = \frac{26}{59}$, $M_1 = \frac{92}{59}$, $M_2 = \frac{6}{59}$, $m_2 = 0$, $L = \frac{509}{295}$, $M = \frac{11458}{430110}$, $N = \frac{5587}{1770}$. Then $f(t,x,y)$ satisfies

$$f(t,x,y) \le \frac{d}{L} = 3.48\times 10^8, \quad \text{for } (t,x,y) \in [0,1]\times\left[0, 3.6\times 10^9\right]\times\left[-6\times 10^8, 6\times 10^8\right];$$

$$f(t,x,y) > \frac{b}{M} = 37.6, \quad \text{for } (t,x,y) \in \left[\frac{1}{3},\frac{2}{3}\right]\times[1,3]\times\left[-6\times 10^8, 6\times 10^8\right];$$

$$f(t,x,y) < \frac{a}{N} = 0.18, \quad \text{for } (t,x,y) \in [0,1]\times\left[0,\frac{1}{2}\right]\times\left[-6\times 10^8, 6\times 10^8\right].$$

All conditions of Theorem 3.1 are satisfied. By Theorem 3.1, BVP (4.1) has at least three positive solutions u_1, u_2, u_3 such that

$$\max_{0\le t\le 1}\left|u_i'(t)\right| \le 6\times 10^8, \quad i = 1,2,3, \tag{4.3}$$

and

$$\min_{\frac{1}{3}\le t\le\frac{2}{3}} u_1(t) > 1, \qquad \max_{0\le t\le 1} u_2(t) > \frac{1}{2}, \qquad \min_{\frac{1}{3}\le t\le\frac{2}{3}} u_2(t) < 1, \qquad \max_{0\le t\le 1} u_3(t) < \frac{1}{2}. \tag{4.4}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Acknowledgements
The authors thank the referees for their valuable comments and suggestions. This research is supported by the Importation and Development of High-Caliber Talents Project of Beijing Municipal Institutions (CIT and TCD201504041) and NNSF (No. 91420202 and 61372088).

Received: 16 December 2015 Accepted: 27 February 2016 Published online: 04 March 2016

References

1. Zhao, J, Geng, F, Zhao, J, Ge, W: Positive solutions to a new kind Sturm-Liouville-like four-point boundary value problem. Appl. Math. Comput. **217**, 811-819 (2010)
2. Guo, Y, Ge, W: Positive solutions for three-point boundary value problems with dependence on the first order derivatives. J. Math. Anal. Appl. **290**, 291-301 (2004)
3. Zhang, Y: Existence and multiplicity results for a class of generalized one-dimensional *p*-Laplacian problem. Nonlinear Anal. **72**, 748-756 (2010)
4. Ji, D, Yang, Y, Ge, W: Triple positive pseudo-symmetric solutions to a four-point boundary value problem with *p*-Laplacian. Appl. Math. Lett. **212**, 68-274 (2008)
5. Jiang, J, Liu, L, Wu, Y: Second-order nonlinear singular Sturm-Liouville problems with integral boundary conditions. Appl. Math. Comput. **215**, 1573-1582 (2009)
6. Boucherif, A: Second-order boundary value problems with integral boundary conditions. Nonlinear Anal. **70**, 364-371 (2009)
7. Sun, Y: Three symmetric positive solutions for second-order nonlocal boundary value problems. Acta Math. Appl. Sinica (Engl. Ser.) **27**, 233-242 (2011)
8. Zhang, X, Feng, M, Ge, W: Symmetric positive solutions for *p*-Laplacian fourth-order differential equations with integral boundary conditions. J. Comput. Appl. Math. **222**, 561-573 (2008)
9. Zhang, X, Ge, W: Positive solutions for a class of boundary-value problems with integral boundary conditions. Comput. Math. Appl. **58**, 203-215 (2009)
10. Zhang, X, Ge, W: Symmetric positive solutions of boundary value problems with integral boundary conditions. Appl. Math. Comput. **219**, 3553-3564 (2012)
11. Boucherif, A, Bouguima, S, Al-malki, N, Benbouziane, Z: Third order differential equations with integral boundary conditions. Nonlinear Anal. **71**, e1736-e1743 (2009)
12. Yang, Z: Positive solutions of a second-order integral boundary value problem. J. Math. Anal. Appl. **321**, 751-765 (2006)
13. Guo, Y, Liu, Y, Liang, Y: Positive solutions for the third-order boundary value problems with the second derivatives. Bound. Value Probl. **2012**, 34 (2012)
14. Avery, R, Peterson, A: Three positive fixed points of nonlinear operators on ordered Banach spaces. Comput. Math. Appl. **42**, 313-322 (2001)
15. Liu, B: Positive solutions of three-point boundary value problems for the one-dimensional *p*-Laplacian with infinitely many singularities. Appl. Math. Lett. **17**, 655-661 (2004)