Henderson and Luca *Boundary Value Problems* (2016) 2016:61
DOI 10.1186/s13661-016-0569-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Positive solutions for a system of semipositone coupled fractional boundary value problems

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Abstract
We study the existence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with sign-changing nonlinearities, subject to coupled integral boundary conditions.

MSC: 34A08; 45G15

Keywords: Riemann-Liouville fractional differential equations; coupled integral boundary conditions; positive solutions; sign-changing nonlinearities

1 Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [1–6]). Integral boundary conditions arise in thermal conduction problems, semiconductor problems and hydrodynamic problems.

We consider the system of nonlinear fractional differential equations

$$(\mathrm{S})\qquad \begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, & t\in(0,1), n-1<\alpha\le n,\\ D_{0+}^{\beta}v(t)+\mu g(t,u(t),v(t))=0, & t\in(0,1), m-1<\beta\le m,\end{cases}$$

with the coupled integral boundary conditions

$$(\mathrm{BC})\qquad \begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, & u(1)=\int_0^1 v(s)\,dH(s),\\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, & v(1)=\int_0^1 u(s)\,dK(s),\end{cases}$$

where $n,m\in\mathbb{N}$, $n,m\ge 3$, D_{0+}^{α}, and D_{0+}^{β} denote the Riemann-Liouville derivatives of orders α and β, respectively, the integrals from (BC) are Riemann-Stieltjes integrals, and f, g are sign-changing continuous functions (that is, we have a so-called system of semipositone boundary value problems). These functions may be nonsingular or singular at $t=0$ and/or $t=1$. The boundary conditions above include multi-point and integral boundary conditions and sum of these in a single framework.

We present intervals for parameters λ and μ such that the above problem (S)-(BC) has at least one positive solution. By a positive solution of problem (S)-(BC) we mean a pair of functions $(u,v) \in C([0,1]) \times C([0,1])$ satisfying (S) and (BC) with $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,1]$ and $u(t) > 0$, $v(t) > 0$ for all $t \in (0,1)$. In the case when f and g are nonnegative, problem (S)-(BC) has been investigated in [7] by using the Guo-Krasnosel'skii fixed point theorem, and in [8] where $\lambda = \mu = 1$ and $f(t,u,v)$ and $g(t,u,v)$ are replaced by $\tilde{f}(t,v)$ and $\tilde{g}(t,u)$, respectively (denoted by $(\widetilde{\mathrm{S}})$). In [8], the authors study two cases: f and g are nonsingular and singular functions and they used some theorems from the fixed point index theory and the Guo-Krasnosel'skii fixed point theorem. The systems (S) and $(\widetilde{\mathrm{S}})$ with uncoupled boundary conditions

$$(\widetilde{\mathrm{BC}}) \qquad \begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u(1) = \int_0^1 u(s)\,dH(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, & v(1) = \int_0^1 v(s)\,dK(s), \end{cases}$$

were investigated in [9] (problem (S)-$(\widetilde{\mathrm{BC}})$ with f, g nonnegative), in [10] (problem $(\widetilde{\mathrm{S}})$-$(\widetilde{\mathrm{BC}})$ with f, g nonnegative, singular or not), and in [11] (problem (S)-$(\widetilde{\mathrm{BC}})$ with f, g sign-changing functions). We also mention paper [12], where the authors studied the existence and multiplicity of positive solutions for system (S) with $\alpha = \beta$, $\lambda = \mu$, and the boundary conditions $u^{(i)}(0) = v^{(i)}(0) = 0$, $i = 0,\dots,n-2$, $u(1) = av(\xi)$, $v(1) = bu(\eta)$, $\xi,\eta \in (0,1)$, with $\xi,\eta \in (0,1)$, $0 < ab\xi\eta < 1$, and f, g are sign-changing nonsingular or singular functions.

The paper is organized as follows. Section 2 contains some preliminaries and lemmas. The main results are presented in Section 3, and finally in Section 4 some examples are given to support the new results.

2 Auxiliary results

We present here the definitions of Riemann-Liouville fractional integral and Riemann-Liouville fractional derivative and then some auxiliary results that will be used to prove our main results.

Definition 2.1 The (left-sided) fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$\left(I_{0+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \quad t > 0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,dt$, $\alpha > 0$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha \geq 0$ for a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$\left(D_{0+}^{\alpha} f\right)(t) = \left(\frac{d}{dt}\right)^n \left(I_{0+}^{n-\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,ds, \quad t > 0,$$

where $n = \lfloor \alpha \rfloor + 1$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

The notation $\lfloor \alpha \rfloor$ stands for the largest integer not greater than α. If $\alpha = m \in \mathbb{N}$ then $D_{0+}^m f(t) = f^{(m)}(t)$ for $t > 0$, and if $\alpha = 0$ then $D_{0+}^0 f(t) = f(t)$ for $t > 0$.

We consider now the fractional differential system

$$\begin{cases} D_{0+}^{\alpha}u(t) + \tilde{x}(t) = 0, & t \in (0,1), n-1 < \alpha \le n, \\ D_{0+}^{\beta}v(t) + \tilde{y}(t) = 0, & t \in (0,1), m-1 < \beta \le m, \end{cases} \tag{1}$$

with the coupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u(1) = \int_0^1 v(s)\,dH(s), \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, & v(1) = \int_0^1 u(s)\,dK(s), \end{cases} \tag{2}$$

where $n, m \in \mathbb{N}$, $n, m \ge 3$, and $H, K : [0,1] \to \mathbb{R}$ are functions of bounded variation.

Lemma 2.1 ([7]) *If $H, K : [0,1] \to \mathbb{R}$ are functions of bounded variations, $\Delta = 1 - (\int_0^1 \tau^{\alpha-1}\,dK(\tau))(\int_0^1 \tau^{\beta-1}\,dH(\tau)) \neq 0$ and $\tilde{x}, \tilde{y} \in C(0,1) \cap L^1(0,1)$, then the pair of functions $(u, v) \in C([0,1]) \times C([0,1])$ given by*

$$\begin{cases} u(t) = \int_0^1 G_1(t,s)\tilde{x}(s)\,ds + \int_0^1 G_2(t,s)\tilde{y}(s)\,ds, & t \in [0,1], \\ v(t) = \int_0^1 G_3(t,s)\tilde{y}(s)\,ds + \int_0^1 G_4(t,s)\tilde{x}(s)\,ds, & t \in [0,1], \end{cases} \tag{3}$$

where

$$\begin{cases} G_1(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta}(\int_0^1 \tau^{\beta-1}\,dH(\tau))(\int_0^1 g_1(\tau,s)\,dK(\tau)), \\ G_2(t,s) = \frac{t^{\alpha-1}}{\Delta}\int_0^1 g_2(\tau,s)\,dH(\tau), \\ G_3(t,s) = g_2(t,s) + \frac{t^{\beta-1}}{\Delta}(\int_0^1 \tau^{\alpha-1}\,dK(\tau))(\int_0^1 g_2(\tau,s)\,dH(\tau)), \\ G_4(t,s) = \frac{t^{\beta-1}}{\Delta}\int_0^1 g_1(\tau,s)\,dK(\tau), \quad \forall t,s \in [0,1] \end{cases} \tag{4}$$

and

$$\begin{cases} g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \\ g_2(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ t^{\beta-1}(1-s)^{\beta-1}, & 0 \le t \le s \le 1, \end{cases} \end{cases} \tag{5}$$

is solution of problem (1)-(2).

Lemma 2.2 *The functions g_1, g_2 given by* (5) *have the properties*:

(a) $g_1, g_2 : [0,1] \times [0,1] \to \mathbb{R}_+$ *are continuous functions, and* $g_1(t,s) > 0$, $g_2(t,s) > 0$ *for all* $(t,s) \in (0,1) \times (0,1)$.
(b) $g_1(t,s) \le h_1(s)$, $g_2(t,s) \le h_2(s)$ *for all* $(t,s) \in [0,1] \times [0,1]$, *where* $h_1(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}$ *and* $h_2(s) = \frac{s(1-s)^{\beta-1}}{\Gamma(\beta-1)}$ *for all* $s \in [0,1]$.
(c) $g_1(t,s) \ge k_1(t)h_1(s)$, $g_2(t,s) \ge k_2(t)h_2(s)$ *for all* $(t,s) \in [0,1] \times [0,1]$, *where*

$$k_1(t) = \min\left\{\frac{(1-t)t^{\alpha-2}}{\alpha-1}, \frac{t^{\alpha-1}}{\alpha-1}\right\} = \begin{cases} \frac{t^{\alpha-1}}{\alpha-1}, & 0 \le t \le \frac{1}{2}, \\ \frac{(1-t)t^{\alpha-2}}{\alpha-1}, & \frac{1}{2} \le t \le 1, \end{cases}$$

$$k_2(t) = \min\left\{\frac{(1-t)t^{\beta-2}}{\beta-1}, \frac{t^{\beta-1}}{\beta-1}\right\} = \begin{cases} \frac{t^{\beta-1}}{\beta-1}, & 0 \le t \le \frac{1}{2}, \\ \frac{(1-t)t^{\beta-2}}{\beta-1}, & \frac{1}{2} \le t \le 1. \end{cases}$$

(d) *For any* $(t,s) \in [0,1] \times [0,1]$, *we have*

$$g_1(t,s) \leq \frac{(1-t)t^{\alpha-1}}{\Gamma(\alpha-1)} \leq \frac{t^{\alpha-1}}{\Gamma(\alpha-1)}, \qquad g_2(t,s) \leq \frac{(1-t)t^{\beta-1}}{\Gamma(\beta-1)} \leq \frac{t^{\beta-1}}{\Gamma(\beta-1)}.$$

For the proof of Lemma 2.2(a) and (b) see [13], for the proof of Lemma 2.2(c) see [11], and the proof of Lemma 2.2(d) is based on the relations $g_1(t,s) = g_1(1-s,1-t)$, $g_2(t,s) = g_2(1-s,1-t)$, and relations (b) above.

Lemma 2.3 ([7]) *If* $H,K:[0,1] \to \mathbb{R}$ *are nondecreasing functions, and* $\Delta > 0$, *then* G_i, $i=1,\ldots,4$ *given by* (4) *are continuous functions on* $[0,1] \times [0,1]$ *and satisfy* $G_i(t,s) \geq 0$ *for all* $(t,s) \in [0,1] \times [0,1]$, $i=1,\ldots,4$. *Moreover, if* $\tilde{x},\tilde{y} \in C(0,1) \cap L^1(0,1)$ *satisfy* $\tilde{x}(t) \geq 0$, $\tilde{y}(t) \geq 0$ *for all* $t \in (0,1)$, *then the solution* (u,v) *of problem* (1)-(2) *given by* (3) *satisfies* $u(t) \geq 0$, $v(t) \geq 0$ *for all* $t \in [0,1]$.

Lemma 2.4 *Assume that* $H,K:[0,1] \to \mathbb{R}$ *are nondecreasing functions,* $\Delta > 0$, $\int_0^1 \tau^{\alpha-1}(1-\tau)\,dK(\tau) > 0$, $\int_0^1 \tau^{\beta-1}(1-\tau)\,dH(\tau) > 0$. *Then the functions* G_i, $i=1,\ldots,4$ *satisfy the inequalities*:

(a$_1$) $G_1(t,s) \leq \sigma_1 h_1(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\sigma_1 = 1 + \frac{1}{\Delta}\bigl(K(1)-K(0)\bigr)\int_0^1 \tau^{\beta-1}\,dH(\tau) > 0.$$

(a$_2$) $G_1(t,s) \leq \delta_1 t^{\alpha-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\delta_1 = \frac{1}{\Gamma(\alpha-1)}\left[1 + \frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 (1-\tau)\tau^{\alpha-1}\,dK(\tau)\right)\right] > 0.$$

(a$_3$) $G_1(t,s) \geq \varrho_1 t^{\alpha-1} h_1(s)$, $(t,s) \in [0,1] \times [0,1]$, *where*

$$\varrho_1 = \frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 k_1(\tau)\,dK(\tau)\right) > 0.$$

(b$_1$) $G_2(t,s) \leq \sigma_2 h_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\sigma_2 = \frac{1}{\Delta}(H(1)-H(0)) > 0$.

(b$_2$) $G_2(t,s) \leq \delta_2 t^{\alpha-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\delta_2 = \frac{1}{\Delta\Gamma(\beta-1)}\int_0^1 (1-\tau)\tau^{\beta-1}\,dH(\tau) > 0$.

(b$_3$) $G_2(t,s) \geq \varrho_2 t^{\alpha-1} h_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\varrho_2 = \frac{1}{\Delta}\int_0^1 k_2(\tau)\,dH(\tau) > 0$.

(c$_1$) $G_3(t,s) \leq \sigma_3 h_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\sigma_3 = 1 + \frac{1}{\Delta}\bigl(H(1)-H(0)\bigr)\int_0^1 \tau^{\alpha-1}\,dK(\tau) > 0.$$

(c$_2$) $G_3(t,s) \leq \delta_3 t^{\beta-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\delta_3 = \frac{1}{\Gamma(\beta-1)}\left[1 + \frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 (1-\tau)\tau^{\beta-1}\,dH(\tau)\right)\right] > 0.$$

(c$_3$) $G_3(t,s) \geq \varrho_3 t^{\beta-1} h_2(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where*

$$\varrho_3 = \frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 k_2(\tau)\,dH(\tau)\right) > 0.$$

(d$_1$) $G_4(t,s) \le \sigma_4 h_1(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\sigma_4 = \frac{1}{\Delta}(K(1) - K(0)) > 0$.
(d$_2$) $G_4(t,s) \le \delta_4 t^{\beta-1}$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\delta_4 = \frac{1}{\Delta\Gamma(\alpha-1)} \int_0^1 (1-\tau)\tau^{\alpha-1}\,dK(\tau) > 0$.
(d$_3$) $G_4(t,s) \ge \varrho_4 t^{\beta-1} h_1(s)$, $\forall (t,s) \in [0,1] \times [0,1]$, *where* $\varrho_4 = \frac{1}{\Delta} \int_0^1 k_1(\tau)\,dK(\tau) > 0$.

Proof From the assumptions of this lemma, we obtain

$$
\begin{aligned}
&\int_0^1 \tau^{\alpha-1}\,dK(\tau) \ge \int_0^1 \tau^{\alpha-1}(1-\tau)\,dK(\tau) > 0,\\
&\int_0^1 (1-\tau)\tau^{\alpha-2}\,dK(\tau) \ge \int_0^1 (1-\tau)\tau^{\alpha-1}\,dK(\tau) > 0,\\
&\int_0^1 k_1(\tau)\,dK(\tau) \ge \frac{1}{\alpha-1}\int_0^1 \tau^{\alpha-1}(1-\tau)\,dK(\tau) > 0,\\
&\int_0^1 \tau^{\beta-1}\,dH(\tau) \ge \int_0^1 \tau^{\beta-1}(1-\tau)\,dH(\tau) > 0,\\
&\int_0^1 (1-\tau)\tau^{\beta-2}\,dH(\tau) \ge \int_0^1 (1-\tau)\tau^{\beta-1}\,dH(\tau) > 0,\\
&\int_0^1 k_2(\tau)\,dH(\tau) \ge \frac{1}{\beta-1}\int_0^1 \tau^{\beta-1}(1-\tau)\,dH(\tau) > 0,\\
&K(1) - K(0) = \int_0^1 dK(\tau) \ge \int_0^1 \tau^{\alpha-1}(1-\tau)\,dK(\tau) > 0,\\
&H(1) - H(0) = \int_0^1 dH(\tau) \ge \int_0^1 \tau^{\beta-1}(1-\tau)\,dH(\tau) > 0.
\end{aligned}
$$

By using Lemma 2.2, we deduce, for all $(t,s) \in [0,1] \times [0,1]$:
(a$_1$)

$$
\begin{aligned}
G_1(t,s) &= g_1(t,s) + \frac{t^{\alpha-1}}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 g_1(\tau,s)\,dK(\tau)\right)\\
&\le h_1(s) + \frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 h_1(s)\,dK(\tau)\right)\\
&= h_1(s)\left[1 + \frac{1}{\Delta}\big(K(1) - K(0)\big)\int_0^1 \tau^{\beta-1}\,dH(\tau)\right] = \sigma_1 h_1(s).
\end{aligned}
$$

(a$_2$)

$$
\begin{aligned}
G_1(t,s) &\le \frac{t^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{t^{\alpha-1}}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 \frac{(1-\tau)\tau^{\alpha-1}}{\Gamma(\alpha-1)}\,dK(\tau)\right)\\
&= t^{\alpha-1}\frac{1}{\Gamma(\alpha-1)}\left[1 + \frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 (1-\tau)\tau^{\alpha-1}\,dK(\tau)\right)\right] = \delta_1 t^{\alpha-1}.
\end{aligned}
$$

(a$_3$)

$$
\begin{aligned}
G_1(t,s) &\ge \frac{t^{\alpha-1}}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 k_1(\tau)h_1(s)\,dK(\tau)\right)\\
&= t^{\alpha-1}h_1(s)\frac{1}{\Delta}\left(\int_0^1 \tau^{\beta-1}\,dH(\tau)\right)\left(\int_0^1 k_1(\tau)\,dK(\tau)\right) = \varrho_1 t^{\alpha-1}h_1(s).
\end{aligned}
$$

(b_1)

$$G_2(t,s) = \frac{t^{\alpha-1}}{\Delta}\int_0^1 g_2(\tau,s)\,dH(\tau) \le \frac{1}{\Delta}\int_0^1 h_2(s)\,dH(\tau) = \frac{1}{\Delta}\big(H(1)-H(0)\big)h_2(s) = \sigma_2 h_2(s).$$

(b_2)

$$G_2(t,s) \le \frac{t^{\alpha-1}}{\Delta}\int_0^1 \frac{(1-\tau)\tau^{\beta-1}}{\Gamma(\beta-1)}\,dH(\tau) = \delta_2 t^{\alpha-1}.$$

(b_3)

$$G_2(t,s) \ge \frac{t^{\alpha-1}}{\Delta}\int_0^1 k_2(\tau)h_2(s)\,dH(\tau) = \frac{t^{\alpha-1}}{\Delta}h_2(s)\int_0^1 k_2(\tau)\,dH(\tau) = \varrho_2 t^{\alpha-1}h_2(s).$$

(c_1)

$$\begin{aligned} G_3(t,s) &= g_2(t,s) + \frac{t^{\beta-1}}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 g_2(\tau,s)\,dH(\tau)\right) \\ &\le h_2(s) + \frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 h_2(s)\,dH(\tau)\right) \\ &= h_2(s)\left[1 + \frac{1}{\Delta}\big(H(1)-H(0)\big)\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right] = \sigma_3 h_2(s). \end{aligned}$$

(c_2)

$$\begin{aligned} G_3(t,s) &\le \frac{(1-t)t^{\beta-1}}{\Gamma(\beta-1)} + \frac{t^{\beta-1}}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 \frac{(1-\tau)\tau^{\beta-1}}{\Gamma(\beta-1)}\,dH(\tau)\right) \\ &\le \frac{t^{\beta-1}}{\Gamma(\beta-1)}\left[1 + \frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 (1-\tau)\tau^{\beta-1}\,dH(\tau)\right)\right] = \delta_3 t^{\beta-1}. \end{aligned}$$

(c_3)

$$\begin{aligned} G_3(t,s) &\ge \frac{t^{\beta-1}}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 k_2(\tau)h_2(s)\,dH(\tau)\right) \\ &= t^{\beta-1}h_2(s)\frac{1}{\Delta}\left(\int_0^1 \tau^{\alpha-1}\,dK(\tau)\right)\left(\int_0^1 k_2(\tau)\,dH(\tau)\right) = \varrho_3 t^{\beta-1}h_2(s). \end{aligned}$$

(d_1)

$$G_4(t,s) = \frac{t^{\beta-1}}{\Delta}\int_0^1 g_1(\tau,s)\,dK(\tau) \le \frac{1}{\Delta}\int_0^1 h_1(s)\,dK(\tau) = h_1(s)\frac{1}{\Delta}\big(K(1)-K(0)\big) = \sigma_4 h_1(s).$$

(d_2)

$$G_4(t,s) \le \frac{t^{\beta-1}}{\Delta}\int_0^1 \frac{(1-\tau)\tau^{\alpha-1}}{\Gamma(\alpha-1)}\,dK(\tau) = \delta_4 t^{\beta-1}.$$

(d_3)

$$G_4(t,s) \geq \frac{t^{\beta-1}}{\Delta}\int_0^1 k_1(\tau)h_1(s)\,dK(\tau) = t^{\beta-1}h_1(s)\frac{1}{\Delta}\int_0^1 k_1(s)\,dK(\tau) = \varrho_4 t^{\beta-1}h_1(s). \qquad \square$$

Lemma 2.5 *Assume that $H, K : [0,1] \to \mathbb{R}$ are nondecreasing functions, $\Delta > 0$, $\int_0^1 \tau^{\alpha-1}(1-\tau)\,dK(\tau) > 0$, $\int_0^1 \tau^{\beta-1}(1-\tau)\,dH(\tau) > 0$, and $\tilde{x}, \tilde{y} \in C(0,1) \cap L^1(0,1)$, $\tilde{x}(t) \geq 0$, $\tilde{y}(t) \geq 0$ for all $t \in (0,1)$. Then the solution (u,v) of problem (1)-(2) given by (3) satisfies the inequalities $u(t) \geq \gamma_1 t^{\alpha-1}u(t')$, $v(t) \geq \gamma_2 t^{\beta-1}v(t')$, for all $t, t' \in [0,1]$, where $\gamma_1 = \min\{\frac{\varrho_1}{\sigma_1}, \frac{\varrho_2}{\sigma_2}\} > 0$, $\gamma_2 = \min\{\frac{\varrho_3}{\sigma_3}, \frac{\varrho_4}{\sigma_4}\} > 0$.*

Proof By using Lemma 2.4, we obtain

$$\begin{aligned}
u(t) &= \int_0^1 G_1(t,s)\tilde{x}(s)\,ds + \int_0^1 G_2(t,s)\tilde{y}(s)\,ds \\
&\geq \int_0^1 \varrho_1 t^{\alpha-1}h_1(s)\tilde{x}(s)\,ds + \int_0^1 \varrho_2 t^{\alpha-1}h_2(s)\tilde{y}(s)\,ds \\
&= t^{\alpha-1}\left(\varrho_1\int_0^1 h_1(s)\tilde{x}(s)\,ds + \varrho_2\int_0^1 h_2(s)\tilde{y}(s)\,ds\right) \\
&\geq t^{\alpha-1}\left(\frac{\varrho_1}{\sigma_1}\int_0^1 G_1\bigl(t',s\bigr)\tilde{x}(s)\,ds + \frac{\varrho_2}{\sigma_2}\int_0^1 G_2\bigl(t',s\bigr)\tilde{y}(s)\,ds\right) \\
&\geq t^{\alpha-1}\min\left\{\frac{\varrho_1}{\sigma_1}, \frac{\varrho_2}{\sigma_2}\right\}\left(\int_0^1 G_1\bigl(t',s\bigr)\tilde{x}(s)\,ds + \int_0^1 G_2\bigl(t',s\bigr)\tilde{y}(s)\,ds\right) \\
&= \gamma_1 t^{\alpha-1}u\bigl(t'\bigr), \quad \forall t, t' \in [0,1], \text{where } \gamma_1 = \min\left\{\frac{\varrho_1}{\sigma_1}, \frac{\varrho_2}{\sigma_2}\right\} > 0.
\end{aligned}$$

In a similar way, we deduce

$$\begin{aligned}
v(t) &= \int_0^1 G_3(t,s)\tilde{y}(s)\,ds + \int_0^1 G_4(t,s)\tilde{x}(s)\,ds \\
&\geq \gamma_2 t^{\beta-1}v\bigl(t'\bigr), \quad \forall t, t' \in [0,1], \text{where } \gamma_2 = \min\left\{\frac{\varrho_3}{\sigma_3}, \frac{\varrho_4}{\sigma_4}\right\} > 0. \qquad \square
\end{aligned}$$

In the proof of our main results we shall use the nonlinear alternative of Leray-Schauder type and the Guo-Krasnosel'skii fixed point theorem presented below (see [14, 15]).

Theorem 2.1 *Let X be a Banach space with $\Omega \subset X$ closed and convex. Assume U is a relatively open subset of Ω with $0 \in U$, and let $S : \bar{U} \to \Omega$ be a completely continuous operator (continuous and compact). Then either*

(1) *S has a fixed point in $\bar{U}$, or*
(2) *there exist $u \in \partial U$ and $\nu \in (0,1)$ such that $u = \nu Su$.*

Theorem 2.2 *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and let $\mathcal{A} : C \cap (\bar{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either*

(i) *$\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, or*
(ii) *$\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.*

Then $\mathcal{A}$ has a fixed point in $C \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In this section, we investigate the existence and multiplicity of positive solutions for our problem (S)-(BC). We present now the assumptions that we shall use in the sequel.

(H1) $H, K : [0,1] \to \mathbb{R}$ are nondecreasing functions, $\Delta = 1 - (\int_0^1 \tau^{\alpha-1}\,dK(\tau)) \times (\int_0^1 \tau^{\beta-1}\,dH(\tau)) > 0$, and $\int_0^1 \tau^{\alpha-1}(1-\tau)\,dK(\tau) > 0$, $\int_0^1 \tau^{\beta-1}(1-\tau)\,dH(\tau) > 0$.

(H2) The functions $f, g \in C([0,1] \times [0,\infty) \times [0,\infty), (-\infty,+\infty))$ and there exist functions $p_1, p_2 \in C([0,1],[0,\infty))$ such that $f(t,u,v) \geq -p_1(t)$ and $g(t,u,v) \geq -p_2(t)$ for any $t \in [0,1]$ and $u, v \in [0,\infty)$.

(H3) $f(t,0,0) > 0$, $g(t,0,0) > 0$ for all $t \in [0,1]$.

(H4) The functions $f, g \in C((0,1) \times [0,\infty) \times [0,\infty), (-\infty,+\infty))$, f, g may be singular at $t = 0$ and/or $t = 1$, and there exist functions $p_1, p_2 \in C((0,1),[0,\infty))$, $\alpha_1, \alpha_2 \in C((0,1),[0,\infty))$, $\beta_1, \beta_2 \in C([0,1] \times [0,\infty) \times [0,\infty),[0,\infty))$ such that

$$-p_1(t) \leq f(t,u,v) \leq \alpha_1(t)\beta_1(t,u,v), \qquad -p_2(t) \leq g(t,u,v) \leq \alpha_2(t)\beta_2(t,u,v),$$

for all $t \in (0,1)$, $u, v \in [0,\infty)$, with $0 < \int_0^1 p_i(s)\,ds < \infty$, $0 < \int_0^1 \alpha_i(s)\,ds < \infty$, $i = 1,2$.

(H5) There exists $c \in (0,1/2)$ such that

$$f_\infty = \lim_{u+v\to\infty} \min_{t\in[c,1-c]} \frac{f(t,u,v)}{u+v} = \infty \quad \text{or} \quad g_\infty = \lim_{u+v\to\infty} \min_{t\in[c,1-c]} \frac{g(t,u,v)}{u+v} = \infty.$$

(H6) $\beta_{i\infty} = \lim_{u+v\to\infty} \max_{t\in[0,1]} \frac{\beta_i(t,u,v)}{u+v} = 0$, $i = 1,2$.

We consider the system of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha} x(t) + \lambda(f(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*) + p_1(t)) = 0, & 0 < t < 1, \\ D_{0+}^{\beta} y(t) + \mu(g(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*) + p_2(t)) = 0, & 0 < t < 1, \end{cases} \tag{6}$$

with the integral boundary conditions

$$\begin{cases} x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & x(1) = \int_0^1 y(s)\,dH(s), \\ y(0) = y'(0) = \cdots = y^{(m-2)}(0) = 0, & y(1) = \int_0^1 x(s)\,dK(s), \end{cases} \tag{7}$$

where $z(t)^* = z(t)$ if $z(t) \geq 0$, and $z(t)^* = 0$ if $z(t) < 0$. Here (q_1, q_2) with

$$q_1(t) = \lambda \int_0^1 G_1(t,s)p_1(s)\,ds + \mu \int_0^1 G_2(t,s)p_2(s)\,ds, \quad t \in [0,1],$$

$$q_2(t) = \mu \int_0^1 G_3(t,s)p_2(s)\,ds + \lambda \int_0^1 G_4(t,s)p_1(s)\,ds, \quad t \in [0,1],$$

is solution of the system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha} q_1(t) + \lambda p_1(t) = 0, & 0 < t < 1, \\ D_{0+}^{\beta} q_2(t) + \mu p_2(t) = 0, & 0 < t < 1, \end{cases} \tag{8}$$

with the integral boundary conditions

$$\begin{cases} q_1(0) = q_1'(0) = \cdots = q_1^{(n-2)}(0) = 0, & q_1(1) = \int_0^1 q_2(s)\,dH(s), \\ q_2(0) = q_2'(0) = \cdots = q_2^{(m-2)}(0) = 0, & q_2(1) = \int_0^1 q_1(s)\,dK(s). \end{cases} \tag{9}$$

Under the assumptions (H1) and (H2), or (H1) and (H4), we have $q_1(t) \geq 0$, $q_2(t) \geq 0$ for all $t \in [0,1]$.

We shall prove that there exists a solution (x,y) for the boundary value problem (6)-(7) with $x(t) \geq q_1(t)$ and $y(t) \geq q_2(t)$ on $[0,1]$, $x(t) > q_1(t)$, $y(t) > q_2(t)$ on $(0,1)$. In this case (u,v) with $u(t) = x(t) - q_1(t)$ and $v(t) = y(t) - q_2(t)$, $t \in [0,1]$ represents a positive solution of boundary value problem (S)-(BC).

By using Lemma 2.1 (relations (3)), a solution of the system

$$\begin{cases} x(t) = \lambda \int_0^1 G_1(t,s)(f(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s))\,ds \\ \qquad + \mu \int_0^1 G_2(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\,ds, \quad t \in [0,1], \\ y(t) = \mu \int_0^1 G_3(t,s)(g(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_2(s))\,ds \\ \qquad + \lambda \int_0^1 G_4(t,s)(f(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*) + p_1(s))\,ds, \quad t \in [0,1], \end{cases}$$

is a solution for problem (6)-(7).

We consider the Banach space $X = C([0,1])$ with the supremum norm $\|\cdot\|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cones

$$P_1 = \{x \in X, x(t) \geq \gamma_1 t^{\alpha-1}\|x\|, \forall t \in [0,1]\},$$
$$P_2 = \{y \in X, y(t) \geq \gamma_2 t^{\beta-1}\|y\|, \forall t \in [0,1]\},$$

where γ_1, γ_2 are defined in Section 2 (Lemma 2.5), and $P = P_1 \times P_2 \subset Y$.

For $\lambda, \mu > 0$, we introduce the operators $Q_1, Q_2 : Y \to X$ and $\mathcal{Q} : Y \to Y$ defined by $\mathcal{Q}(x,y) = (Q_1(x,y), Q_2(x,y))$, $(x,y) \in Y$ with

$$\begin{aligned} Q_1(x,y)(t) &= \lambda \int_0^1 G_1(t,s)\big(f\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big) + p_1(s)\big)\,ds \\ &\quad + \mu \int_0^1 G_2(t,s)\big(g\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big) + p_2(s)\big)\,ds, \quad t \in [0,1], \\ Q_2(x,y)(t) &= \mu \int_0^1 G_3(t,s)\big(g\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big) + p_2(s)\big)\,ds \\ &\quad + \lambda \int_0^1 G_4(t,s)\big(f\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big) + p_1(s)\big)\,ds, \quad t \in [0,1]. \end{aligned}$$

It is clear that if (x,y) is a fixed point of operator $\mathcal{Q}$, then (x,y) is a solution of problem (6)-(7).

Lemma 3.1 *If* (H1) *and* (H2), *or* (H1) *and* (H4) *hold, then operator* $\mathcal{Q} : P \to P$ *is a completely continuous operator.*

Proof The operators Q_1 and Q_2 are well defined. To prove this, let $(x,y) \in P$ be fixed with $\|(x,y)\|_Y = \widetilde{L}$. Then we have

$$\big[x(s)-q_1(s)\big]^* \leq x(s) \leq \|x\| \leq \big\|(x,y)\big\|_Y = \widetilde{L}, \quad \forall s \in [0,1],$$
$$\big[y(s)-q_2(s)\big]^* \leq y(s) \leq \|y\| \leq \big\|(x,y)\big\|_Y = \widetilde{L}, \quad \forall s \in [0,1].$$

If (H1) and (H2) hold, then we deduce easily that $Q_1(x,y)(t) < \infty$ and $Q_2(x,y)(t) < \infty$ for all $t \in [0,1]$.

If (H1) and (H4) hold, we deduce, for all $t \in [0,1]$,

$$\begin{aligned}
Q_1(x,y)(t) &\le \lambda\sigma_1 \int_0^1 h_1(s)\big[\alpha_1(s)\beta_1\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_1(s)\big]\,ds \\
&\quad + \mu\sigma_2 \int_0^1 h_2(s)\big[\alpha_2(s)\beta_2\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_2(s)\big]\,ds \\
&\le M\left(\lambda\sigma_1\int_0^1 h_1(s)\big(\alpha_1(s)+p_1(s)\big)\,ds + \mu\sigma_2\int_0^1 h_2(s)\big(\alpha_2(s)+p_2(s)\big)\,ds\right) < \infty, \\
Q_2(x,y)(t) &\le \mu\sigma_3 \int_0^1 h_2(s)\big[\alpha_2(s)\beta_2\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_2(s)\big]\,ds \\
&\quad + \lambda\sigma_4 \int_0^1 h_1(s)\big[\alpha_1(s)\beta_1\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_1(s)\big]\,ds \\
&\le M\left(\mu\sigma_3\int_0^1 h_2(s)\big(\alpha_2(s)+p_2(s)\big)\,ds + \lambda\sigma_4\int_0^1 h_1(s)\big(\alpha_1(s)+p_1(s)\big)\,ds\right) < \infty,
\end{aligned}$$

where $M = \max\{\max_{t\in[0,1],u,v\in[0,\widetilde{L}]}\beta_1(t,u,v), \max_{t\in[0,1],u,v\in[0,\widetilde{L}]}\beta_2(t,u,v), 1\}$.

Besides, by Lemma 2.5, we conclude that

$$Q_1(x,y)(t) \ge \gamma_1 t^{\alpha-1}\big\|Q_1(x,y)\big\|, \qquad Q_2(x,y)(t) \ge \gamma_2 t^{\beta-1}\big\|Q_2(x,y)\big\|, \quad \forall t \in [0,1],$$

and so $Q_1(x,y), Q_2(x,y) \in P$.

By using standard arguments, we deduce that operator $\mathcal{Q}: P \to P$ is a completely continuous operator (a compact operator, that is, one that maps bounded sets into relatively compact sets and is continuous). $\square$

Theorem 3.1 *Assume that* (H1)-(H3) *hold. Then there exist constants* $\lambda_0 > 0$ *and* $\mu_0 > 0$ *such that, for any* $\lambda \in (0,\lambda_0]$ *and* $\mu \in (0,\mu_0]$, *the boundary value problem* (S)-(BC) *has at least one positive solution.*

Proof Let $\delta \in (0,1)$ be fixed. From (H2) and (H3), there exists $R_0 \in (0,1]$ such that

$$f(t,u,v) \ge \delta f(t,0,0), \qquad g(t,u,v) \ge \delta g(t,0,0), \quad \forall t \in [0,1], u,v \in [0,R_0]. \tag{10}$$

We define

$$\begin{aligned}
&\bar{f}(R_0) = \max_{t\in[0,1],u,v\in[0,R_0]}\big\{f(t,u,v)+p_1(t)\big\} \ge \max_{t\in[0,1]}\big\{\delta f(t,0,0)+p_1(t)\big\} > 0, \\
&\bar{g}(R_0) = \max_{t\in[0,1],u,v\in[0,R_0]}\big\{g(t,u,v)+p_2(t)\big\} \ge \max_{t\in[0,1]}\big\{\delta g(t,0,0)+p_2(t)\big\} > 0, \\
&c_1 = \sigma_1\int_0^1 h_1(s)\,ds, \qquad c_2 = \sigma_2\int_0^1 h_2(s)\,ds, \\
&c_3 = \sigma_3\int_0^1 h_2(s)\,ds, \qquad c_4 = \sigma_4\int_0^1 h_1(s)\,ds, \\
&\lambda_0 = \max\left\{\frac{R_0}{8c_1\bar{f}(R_0)}, \frac{R_0}{8c_4\bar{f}(R_0)}\right\}, \qquad \mu_0 = \max\left\{\frac{R_0}{8c_2\bar{g}(R_0)}, \frac{R_0}{8c_3\bar{g}(R_0)}\right\}.
\end{aligned}$$

We will show that, for any $\lambda \in (0, \lambda_0]$ and $\mu \in (0, \mu_0]$, problem (6)-(7) has at least one positive solution.

So, let $\lambda \in (0, \lambda_0]$ and $\mu \in (0, \mu_0]$ be arbitrary, but fixed for the moment. We define the set $U = \{(x, y) \in P, \|(u, v)\|_Y < R_0\}$. We suppose that there exist $(x, y) \in \partial U$ ($\|(x, y)\|_Y = R_0$ or $\|x\| + \|y\| = R_0$) and $\nu \in (0, 1)$ such that $(x, y) = \nu \mathcal{Q}(x, y)$ or $x = \nu Q_1(x, y)$, $y = \nu Q_2(x, y)$.

We deduce that

$$\begin{aligned}
&\big[x(t) - q_1(t)\big]^* = x(t) - q_1(t) \le x(t) \le R_0, \quad \text{if } x(t) - q_1(t) \ge 0, \\
&\big[x(t) - q_1(t)\big]^* = 0, \quad \text{for } x(t) - q_1(t) < 0, \forall t \in [0, 1], \\
&\big[y(t) - q_2(t)\big]^* = y(t) - q_2(t) \le y(t) \le R_0, \quad \text{if } y(t) - q_2(t) \ge 0, \\
&\big[y(t) - q_2(t)\big]^* = 0, \quad \text{for } y(t) - q_2(t) < 0, \forall t \in [0, 1].
\end{aligned}$$

Then by Lemma 2.4, for all $t \in [0, 1]$, we obtain

$$\begin{aligned}
x(t) &= \nu Q_1(x, y)(t) \le Q_1(x, y)(t) \\
&\le \lambda \sigma_1 \int_0^1 h_1(s) \bar{f}(R_0)\, ds + \mu \sigma_2 \int_0^1 h_2(s) \bar{g}(R_0)\, ds \\
&\le \lambda_0 c_1 \bar{f}(R_0) + \mu_0 c_2 \bar{g}(R_0) \le \frac{R_0}{8} + \frac{R_0}{8} = \frac{R_0}{4}, \\
y(t) &= \nu Q_2(x, y)(t) \le Q_2(x, y)(t) \\
&\le \mu \sigma_3 \int_0^1 h_2(s) \bar{g}(R_0)\, ds + \lambda \sigma_4 \int_0^1 h_1(s) \bar{f}(R_0)\, ds \\
&\le \mu_0 c_3 \bar{g}(R_0) + \lambda_0 c_4 \bar{f}(R_0) \le \frac{R_0}{8} + \frac{R_0}{8} = \frac{R_0}{4}.
\end{aligned}$$

Hence $\|x\| \le \frac{R_0}{4}$ and $\|y\| \le \frac{R_0}{4}$. Then $R_0 = \|(x, y)\|_Y = \|x\| + \|y\| \le \frac{R_0}{4} + \frac{R_0}{4} = \frac{R_0}{2}$, which is a contradiction.

Therefore, by Theorem 2.1 (with $\Omega = P$), we deduce that $\mathcal{Q}$ has a fixed point $(x_0, y_0) \in \bar{U} \cap P$. That is, $(x_0, y_0) = \mathcal{Q}(x_0, y_0)$ or $x_0 = Q_1(x_0, y_0)$, $y_0 = Q_2(x_0, y_0)$, and $\|x_0\| + \|y_0\| \le R_0$ with $x_0(t) \ge \gamma_1 t^{\alpha-1} \|x_0\|$ and $y_0(t) \ge \gamma_2 t^{\beta-1} \|y_0\|$ for all $t \in [0, 1]$.

Moreover, by (10), we conclude

$$\begin{aligned}
x_0(t) &= Q_1(x_0, y_0)(t) \\
&\ge \lambda \int_0^1 G_1(t, s)\big(\delta f(t, 0, 0) + p_1(s)\big)\, ds + \mu \int_0^1 G_2(t, s)\big(\delta g(t, 0, 0) + p_2(s)\big)\, ds \\
&\ge \lambda \int_0^1 G_1(t, s) p_1(s)\, ds + \mu \int_0^1 G_2(t, s) p_2(s)\, ds = q_1(t), \quad \forall t \in [0, 1], \\
x_0(t) &> \lambda \int_0^1 G_1(t, s) p_1(s)\, ds + \mu \int_0^1 G_2(t, s) p_2(s)\, ds = q_1(t), \quad \forall t \in (0, 1), \\
y_0(t) &= Q_2(x_0, y_0)(t) \\
&\ge \mu \int_0^1 G_3(t, s)\big(\delta g(t, 0, 0) + p_2(s)\big)\, ds + \lambda \int_0^1 G_4(t, s)\big(\delta f(t, 0, 0) + p_1(s)\big)\, ds
\end{aligned}$$

$$
\begin{aligned}
&\ge \mu \int_0^1 G_3(t,s)p_2(s)\,ds + \lambda \int_0^1 G_4(t,s)p_1(s)\,ds = q_2(t), \quad \forall t \in [0,1],\\
y_0(t) &> \mu \int_0^1 G_3(t,s)p_2(s)\,ds + \lambda \int_0^1 G_4(t,s)p_1(s)\,ds = q_2(t), \quad \forall t \in (0,1).
\end{aligned}
$$

Therefore $x_0(t) \ge q_1(t)$, $y_0(t) \ge q_2(t)$ for all $t \in [0,1]$, and $x_0(t) > q_1(t)$, $y_0(t) > q_2(t)$ for all $t \in (0,1)$. Let $u_0(t) = x_0(t) - q_1(t)$ and $v_0(t) = y_0(t) - q_2(t)$ for all $t \in [0,1]$. Then $u_0(t) \ge 0$, $v_0(t) \ge 0$ for all $t \in [0,1]$, $u_0(t) > 0$, $v_0(t) > 0$ for all $t \in (0,1)$. Therefore (u_0, v_0) is a positive solution of (S)-(BC). □

Theorem 3.2 *Assume that* (H1), (H4), *and* (H5) *hold. Then there exist* $\lambda^* > 0$ *and* $\mu^* > 0$ *such that, for any* $\lambda \in (0, \lambda^*]$ *and* $\mu \in (0, \mu^*]$, *the boundary value problem* (S)-(BC) *has at least one positive solution.*

Proof We choose a positive number

$$
\begin{aligned}
R_1 > \max\biggl\{&1, \frac{2}{\gamma_1}\int_0^1 \bigl(\delta_1 p_1(s) + \delta_2 p_2(s)\bigr)\,ds, \frac{2}{\gamma_2}\int_0^1 \bigl(\delta_3 p_2(s) + \delta_4 p_1(s)\bigr)\,ds,\\
&\frac{2}{\gamma_1\gamma_2}\left(\int_0^1 s^{\beta-1}\,dH(s)\right)^{-1}\int_0^1 \bigl(\delta_1 p_1(s) + \delta_2 p_2(s)\bigr)\,ds,\\
&\frac{2}{\gamma_1\gamma_2}\left(\int_0^1 s^{\alpha-1}\,dK(s)\right)^{-1}\int_0^1 \bigl(\delta_3 p_2(s) + \delta_4 p_1(s)\bigr)\,ds\biggr\},
\end{aligned}
$$

and we define the set $\Omega_1 = \{(x,y) \in P, \|(x,y)\|_Y < R_1\}$.

We introduce

$$
\begin{aligned}
\lambda^* = \min\biggl\{&1, \frac{R_1}{4\sigma_1 M_1}\left(\int_0^1 h_1(s)\bigl(\alpha_1(s) + p_1(s)\bigr)\,ds\right)^{-1},\\
&\frac{R_1}{4\sigma_4 M_1}\left(\int_0^1 h_1(s)\bigl(\alpha_1(s) + p_1(s)\bigr)\,ds\right)^{-1}\biggr\},\\
\mu^* = \min\biggl\{&1, \frac{R_1}{4\sigma_2 M_2}\left(\int_0^1 h_2(s)\bigl(\alpha_2(s) + p_2(s)\bigr)\,ds\right)^{-1},\\
&\frac{R_1}{4\sigma_3 M_2}\left(\int_0^1 h_2(s)\bigl(\alpha_2(s) + p_2(s)\bigr)\,ds\right)^{-1}\biggr\},
\end{aligned}
$$

with

$$
\begin{aligned}
M_1 &= \max\Bigl\{\max_{\substack{t\in[0,1]\\ u,v\ge 0,\, u+v\le R_1}} \beta_1(t,u,v), 1\Bigr\},\\
M_2 &= \max\Bigl\{\max_{\substack{t\in[0,1]\\ u,v\ge 0,\, u+v\le R_1}} \beta_2(t,u,v), 1\Bigr\}.
\end{aligned}
$$

Let $\lambda \in (0, \lambda^*]$ and $\mu \in (0, \mu^*]$. Then, for any $(x,y) \in P \cap \partial\Omega_1$ and $s \in [0,1]$, we have

$$
\begin{aligned}
&\bigl[x(s) - q_1(s)\bigr]^* \le x(s) \le \|x\| \le R_1,\\
&\bigl[y(s) - q_2(s)\bigr]^* \le y(s) \le \|y\| \le R_1.
\end{aligned}
$$

Then, for any $(x,y) \in P \cap \partial\Omega_1$, we obtain

$$\begin{aligned}
\|Q_1(x,y)\| &\le \lambda\sigma_1 \int_0^1 h_1(s)\big[\alpha_1(s)\beta_1\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_1(s)\big]\,ds \\
&\quad + \mu\sigma_2 \int_0^1 h_2(s)\big[\alpha_2(s)\beta_2\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_2(s)\big]\,ds \\
&\le \lambda^*\sigma_1 M_1 \int_0^1 h_1(s)\big(\alpha_1(s)+p_1(s)\big)\,ds + \mu^*\sigma_2 M_2 \int_0^1 h_2(s)\big(\alpha_2(s)+p_2(s)\big)\,ds \\
&\le \frac{R_1}{4}+\frac{R_1}{4}=\frac{R_1}{2}=\frac{\|(x,y)\|_Y}{2}, \\
\|Q_2(x,y)\| &\le \mu\sigma_3 \int_0^1 h_2(s)\big[\alpha_2(s)\beta_2\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_2(s)\big]\,ds \\
&\quad + \lambda\sigma_4 \int_0^1 h_1(s)\big[\alpha_1(s)\beta_1\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_1(s)\big]\,ds \\
&\le \mu^*\sigma_3 M_2 \int_0^1 h_2(s)\big(\alpha_2(s)+p_2(s)\big)\,ds + \lambda^*\sigma_4 M_1 \int_0^1 h_1(s)\big(\alpha_1(s)+p_1(s)\big)\,ds \\
&\le \frac{R_1}{4}+\frac{R_1}{4}=\frac{R_1}{2}=\frac{\|(x,y)\|_Y}{2}.
\end{aligned}$$

Therefore

$$\big\|\mathcal{Q}(x,y)\big\|_Y = \big\|Q_1(x,y)\big\| + \big\|Q_2(x,y)\big\| \le \big\|(x,y)\big\|_Y, \quad \forall (x,y) \in P \cap \partial\Omega_1. \tag{11}$$

On the other hand, we choose a constant $L > 0$ such that

$$\begin{aligned}
&\lambda L\varrho_1\gamma_1 c^{2(\alpha-1)} \int_c^{1-c} h_1(s)\,ds \ge 4, \lambda L\varrho_4\gamma_2 c^{2(\beta-1)} \int_c^{1-c} h_1(s)\,ds \ge 4, \\
&\mu L\varrho_2\gamma_1 c^{2(\alpha-1)} \int_c^{1-c} h_2(s)\,ds \ge 4, \mu L\varrho_3\gamma_2 c^{2(\beta-1)} \int_c^{1-c} h_2(s)\,ds \ge 4.
\end{aligned}$$

From (H5), we deduce that there exists a constant $M_0 > 0$ such that

$$f(t,u,v) \ge L(u+v) \quad \text{or} \quad g(t,u,v) \ge L(u+v), \quad \forall t \in [c,1-c], u,v \ge 0, u+v \ge M_0. \tag{12}$$

Now we define

$$\begin{aligned}
R_2 = \max\bigg\{&2R_1, \frac{4M_0}{\gamma_1 c^{\alpha-1}}, \frac{4M_0}{\gamma_2 c^{\beta-1}}, \frac{4}{\gamma_1}\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds, \\
&\frac{4}{\gamma_2}\int_0^1 \big(\delta_3 p_2(s)+\delta_4 p_1(s)\big)\,ds\bigg\} > 0,
\end{aligned}$$

and let $\Omega_2 = \{(x,y) \in P, \|(x,y)\|_Y < R_2\}$.

We suppose that $f_\infty = \infty$, that is, $f(t,u,v) \ge L(u+v)$ for all $t \in [c,1-c]$ and $u,v \ge 0$, $u+v \ge M_0$. Then, for any $(x,y) \in P \cap \partial\Omega_2$, we have $\|(x,y)\|_Y = R_2$ or $\|x\| + \|y\| = R_2$. We deduce that $\|x\| \ge \frac{R_2}{2}$ or $\|y\| \ge \frac{R_2}{2}$.

We suppose that $\|x\| \geq \frac{R_2}{2}$. Then, for any $(x,y) \in P \cap \partial\Omega_2$, we obtain

$$\begin{aligned} x(t) - q_1(t) &= x(t) - \lambda \int_0^1 G_1(t,s)p_1(s)\,ds - \mu \int_0^1 G_2(t,s)p_2(s)\,ds \\ &\geq x(t) - t^{\alpha-1}\left(\delta_1 \int_0^1 p_1(s)\,ds + \delta_2 \int_0^1 p_2(s)\,ds\right) \\ &\geq x(t) - \frac{x(t)}{\gamma_1\|x\|} \int_0^1 \big(\delta_1 p_1(s) + \delta_2 p_2(s)\big)\,ds \\ &= x(t)\left[1 - \frac{1}{\gamma_1\|x\|} \int_0^1 \big(\delta_1 p_1(s) + \delta_2 p_2(s)\big)\,ds\right] \\ &\geq x(t)\left[1 - \frac{2}{\gamma_1 R_2} \int_0^1 \big(\delta_1 p_1(s) + \delta_2 p_2(s)\big)\,ds\right] \geq \frac{1}{2}x(t) \geq 0, \quad \forall t \in [0,1]. \end{aligned}$$

Therefore, we conclude

$$\begin{aligned} \big[x(t) - q_1(t)\big]^* &= x(t) - q_1(t) \geq \frac{1}{2}x(t) \geq \frac{1}{2}\gamma_1 t^{\alpha-1}\|x\| \\ &\geq \frac{1}{4}\gamma_1 t^{\alpha-1}R_2 \geq \frac{1}{4}\gamma_1 c^{\alpha-1}R_2 \geq M_0, \quad \forall t \in [c,1-c]. \end{aligned}$$

Hence

$$\big[x(t)-q_1(t)\big]^* + \big[y(t)-q_2(t)\big]^* \geq \big[x(t)-q_1(t)\big]^* = x(t)-q_1(t) \geq M_0, \quad \forall t \in [c,1-c]. \tag{13}$$

Then, for any $(x,y) \in P \cap \partial\Omega_2$ and $t \in [c,1-c]$, by (12) and (13), we deduce

$$\begin{aligned} f\big(t,\big[x(t) - q_1(t)\big]^*,\big[y(t) - q_2(t)\big]^*\big) &\geq L\big(\big[x(t) - q_1(t)\big]^* + \big[y(t) - q_2(t)\big]^*\big) \\ &\geq L\big[x(t) - q_1(t)\big]^* \geq \frac{L}{2}x(t), \quad \forall t \in [c,1-c]. \end{aligned}$$

It follows that, for any $(x,y) \in P \cap \partial\Omega_2$, $t \in [c,1-c]$, we obtain

$$\begin{aligned} Q_1(x,y)(t) &\geq \lambda \int_0^1 G_1(t,s)\big(f\big(s,\big[x(s) - q_1(s)\big]^*,\big[y(s) - q_2(s)\big]^*\big) + p_1(s)\big)\,ds \\ &\geq \lambda \int_c^{1-c} G_1(t,s)\big(f\big(s,\big[x(s) - q_1(s)\big]^*,\big[y(s) - q_2(s)\big]^*\big) + p_1(s)\big)\,ds \\ &\geq \lambda \int_c^{1-c} G_1(t,s)L\big(\big[x(s) - q_1(s)\big]^*\big)\,ds \geq \lambda \int_c^{1-c} G_1(t,s)\frac{1}{4}L\gamma_1 c^{\alpha-1}R_2\,ds \\ &\geq \lambda \int_c^{1-c} \varrho_1 t^{\alpha-1}h_1(s)\frac{1}{4}L\gamma_1 c^{\alpha-1}R_2\,ds \\ &\geq \lambda c^{2(\alpha-1)}\frac{1}{4}\varrho_1 L\gamma_1 R_2 \int_c^{1-c} h_1(s)\,ds \geq R_2. \end{aligned}$$

Then $\|Q_1(x,y)\| \geq \|(x,y)\|_Y$ and

$$\big\|\mathcal{Q}(x,y)\big\|_Y \geq \big\|(x,y)\big\|_Y, \quad \forall (x,y) \in P \cap \partial\Omega_2. \tag{14}$$

If $\|y\| \geq \frac{R_2}{2}$, then by a similar approach, we obtain again relation (14).

We suppose now that $g_\infty = \infty$, that is, $g(t,u,v) \geq L(u+v)$, for all $t \in [c,1-c]$ and $u,v \geq 0$, $u+v \geq M_0$. Then, for any $(x,y) \in P \cap \partial\Omega_2$, we have $\|(x,y)\|_Y = R_2$. Hence $\|x\| \geq \frac{R_2}{2}$ or $\|y\| \geq \frac{R_2}{2}$.

If $\|x\| \geq \frac{R_2}{2}$, then for any $(x,y) \in P \cap \partial\Omega_2$ we deduce in a similar manner as above that $x(t) - q_1(t) \geq \frac{1}{2}x(t)$ for all $t \in [0,1]$ and

$$\begin{aligned}
Q_1(x,y)(t) &\geq \mu \int_0^1 G_2(t,s)\bigl(g\bigl(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\bigr)+p_2(s)\bigr)\,ds \\
&\geq \mu \int_c^{1-c} G_2(t,s)\bigl(g\bigl(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\bigr)+p_2(s)\bigr)\,ds \\
&\geq \mu \int_c^{1-c} G_2(t,s)L\bigl([x(s)-q_1(s)]^*\bigr)\,ds \geq \mu \int_c^{1-c} G_2(t,s)\frac{1}{4}L\gamma_1 c^{\alpha-1}R_2\,ds \\
&\geq \mu \int_c^{1-c} \varrho_2 t^{\alpha-1}h_2(s)\frac{1}{4}L\gamma_1 c^{\alpha-1}R_2\,ds \geq \mu c^{2(\alpha-1)}\frac{1}{4}\varrho_2 L\gamma_1 R_2 \int_c^{1-c} h_2(s)\,ds \\
&\geq R_2, \quad \forall t \in [c,1-c].
\end{aligned}$$

Hence we obtain relation (14).

If $\|y\| \geq \frac{R_2}{2}$, then in a similar way as above, we deduce again relation (14).

Therefore, by Theorem 2.2, relations (11) and (14), we conclude that $\mathcal{Q}$ has a fixed point $(x_1,y_1) \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$, that is, $R_1 \leq \|(x_1,y_1)\|_Y \leq R_2$. Since $\|(x_1,y_1)\|_Y \geq R_1$, then $\|x_1\| \geq \frac{R_1}{2}$ or $\|y_1\| \geq \frac{R_1}{2}$.

We suppose first that $\|x_1\| \geq \frac{R_1}{2}$. Then we deduce

$$\begin{aligned}
x_1(t) - q_1(t) &= x_1(t) - \lambda \int_0^1 G_1(t,s)p_1(s)\,ds - \mu \int_0^1 G_2(t,s)p_2(s)\,ds \\
&\geq x_1(t) - t^{\alpha-1}\left(\delta_1 \int_0^1 p_1(s)\,ds + \delta_2 \int_0^1 p_2(s)\,ds\right) \\
&\geq x_1(t) - \frac{x_1(t)}{\gamma_1\|x_1\|}\int_0^1 \bigl(\delta_1 p_1(s)+\delta_2 p_2(s)\bigr)\,ds \\
&\geq \left[1 - \frac{2}{\gamma_1 R_1}\int_0^1 \bigl(\delta_1 p_1(s)+\delta_2 p_2(s)\bigr)\,ds\right]x_1(t) \\
&\geq \left[1 - \frac{2}{\gamma_1 R_1}\int_0^1 \bigl(\delta_1 p_1(s)+\delta_2 p_2(s)\bigr)\,ds\right]\gamma_1 t^{\alpha-1}\|x_1\| \\
&\geq \frac{R_1}{2}\left[1 - \frac{2}{\gamma_1 R_1}\int_0^1 \bigl(\delta_1 p_1(s)+\delta_2 p_2(s)\bigr)\,ds\right]\gamma_1 t^{\alpha-1} \\
&= \Lambda_1 t^{\alpha-1}, \quad \forall t \in [0,1],
\end{aligned}$$

and so $x_1(t) \geq q_1(t) + \Lambda_1 t^{\alpha-1}$ for all $t \in [0,1]$, where $\Lambda_1 = \frac{\gamma_1 R_1}{2} - \int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds > 0$.

Then $y_1(1) = \int_0^1 x_1(s)\,dK(s) \geq \Lambda_1 \int_0^1 s^{\alpha-1}\,dK(s) > 0$ and

$$\begin{aligned}
\|y_1\| \geq y_1(1) &= \int_0^1 x_1(s)\,dK(s) \geq \int_0^1 \gamma_1 s^{\alpha-1}\|x_1\|\,dK(s) \\
&\geq \frac{\gamma_1 R_1}{2}\int_0^1 s^{\alpha-1}\,dK(s) > 0.
\end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
y_1(t)-q_2(t) &= y_1(t)-\mu\int_0^1 G_3(t,s)p_2(s)\,ds-\lambda\int_0^1 G_4(t,s)p_1(s)\,ds \\
&\ge y_1(t)-t^{\beta-1}\int_0^1\big(\delta_3p_2(s)+\delta_4p_1(s)\big)\,ds \\
&\ge y_1(t)-\frac{y_1(t)}{\gamma_2\|y_1\|}\int_0^1\big(\delta_3p_2(s)+\delta_4p_1(s)\big)\,ds \\
&\ge y_1(t)\left[1-\frac{2}{\gamma_1\gamma_2R_1}\left(\int_0^1 s^{\alpha-1}\,dK(s)\right)^{-1}\int_0^1\big(\delta_3p_2(s)+\delta_4p_1(s)\big)\,ds\right] \\
&\ge \gamma_2t^{\beta-1}\|y_1\|\left[1-\frac{2}{\gamma_1\gamma_2R_1}\left(\int_0^1 s^{\alpha-1}\,dK(s)\right)^{-1}\int_0^1\big(\delta_3p_2(s)+\delta_4p_1(s)\big)\,ds\right] \\
&\ge \frac{\gamma_1\gamma_2R_1}{2}t^{\beta-1}\int_0^1 s^{\alpha-1}\,dK(s) \\
&\quad\times\left[1-\frac{2}{\gamma_1\gamma_2R_1}\left(\int_0^1 s^{\alpha-1}\,dK(s)\right)^{-1}\int_0^1\big(\delta_3p_2(s)+\delta_4p_1(s)\big)\,ds\right] \\
&= \Lambda_2t^{\beta-1},\quad \forall t\in[0,1],
\end{aligned}
$$

where $\Lambda_2=\frac{\gamma_1\gamma_2R_1}{2}\int_0^1 s^{\alpha-1}\,dK(s)-\int_0^1(\delta_3p_2(s)+\delta_4p_1(s))\,ds>0$.

Hence $y_1(t)\ge q_2(t)+\Lambda_2t^{\beta-1}$ for all $t\in[0,1]$.

If $\|y_1\|\ge\frac{R_1}{2}$, then by a similar approach, we deduce that $y_1(t)\ge q_2(t)+\Lambda_3t^{\beta-1}$ and $x_1(t)\ge q_1(t)+\Lambda_4t^{\alpha-1}$ for all $t\in[0,1]$, where $\Lambda_3=\frac{\gamma_2R_1}{2}-\int_0^1(\delta_3p_2(s)+\delta_4p_1(s))\,ds>0$ and $\Lambda_4=\frac{\gamma_1\gamma_2R_1}{2}\int_0^1 s^{\beta-1}\,dH(s)-\int_0^1(\delta_1p_1(s)+\delta_2p_2(s))\,ds>0$.

Let $u_1(t)=x_1(t)-q_1(t)$ and $v_1(t)=y_1(t)-q_2(t)$ for all $t\in[0,1]$. Then (u_1,v_1) is a positive solution of (S)-(BC) with $u_1(t)\ge\Lambda_5t^{\alpha-1}$ and $v_1(t)\ge\Lambda_6t^{\beta-1}$ for all $t\in[0,1]$, where $\Lambda_5=\min\{\Lambda_1,\Lambda_4\}$ and $\Lambda_6=\min\{\Lambda_2,\Lambda_3\}$. This completes the proof of Theorem 3.2. □

Theorem 3.3 *Assume that* (H1), (H3), (H5), *and*

(H4′) *The functions* $f,g\in C([0,1]\times[0,\infty)\times[0,\infty),(-\infty,+\infty))$ *and there exist functions* $p_1,p_2,\alpha_1,\alpha_2\in C([0,1],[0,\infty))$, $\beta_1,\beta_2\in C([0,1]\times[0,\infty)\times[0,\infty),[0,\infty))$ *such that*

$$
-p_1(t)\le f(t,u,v)\le\alpha_1(t)\beta_1(t,u,v),\qquad -p_2(t)\le g(t,u,v)\le\alpha_2(t)\beta_2(t,u,v),
$$

for all $t\in[0,1]$, $u,v\in[0,\infty)$, *with* $\int_0^1 p_i(s)\,ds>0$, $i=1,2$,

hold. Then the boundary value problem (S)-(BC) *has at least two positive solutions for* $\lambda>0$ *and* $\mu>0$ *sufficiently small.*

Proof Because assumption (H4′) implies assumptions (H2) and (H4), we can apply Theorems 3.1 and 3.2. Therefore, we deduce that, for $0<\lambda\le\min\{\lambda_0,\lambda^*\}$ and $0<\mu\le\min\{\mu_0,\mu^*\}$, problem (S)-(BC) has at least two positive solutions (u_0,v_0) and (u_1,v_1) with $\|(u_0+q_1,v_0+q_2)\|_Y\le1$ and $\|(u_1+q_1,v_1+q_2)\|_Y>1$. □

Theorem 3.4 *Assume that* $\lambda=\mu$, *and* (H1), (H4), *and* (H6) *hold. In addition if*

(H7) *there exists* $c \in (0,1/2)$ *such that*

$$f_\infty^i = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} f(t,u,v) > L_0 \quad or \quad g_\infty^i = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} g(t,u,v) > L_0,$$

where

$$\begin{aligned}L_0 = {} & \max\left\{\frac{4}{\gamma_1}\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds, \frac{4}{\gamma_2}\int_0^1 \big(\delta_3 p_2(s)+\delta_4 p_1(s)\big)\,ds,\right.\\ & \frac{4}{\gamma_1\gamma_2}\left(\int_0^1 s^{\alpha-1}\,dK(s)\right)^{-1}\int_0^1 \big(\delta_3 p_2(s)+\delta_4 p_1(s)\big)\,ds,\\ & \left.\frac{4}{\gamma_1\gamma_2}\left(\int_0^1 s^{\beta-1}\,dH(s)\right)^{-1}\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds\right\}\\ & \times\left(\min\left\{c^{\alpha-1}\varrho_1\int_c^{1-c} h_1(s)\,ds, c^{\alpha-1}\varrho_2\int_c^{1-c} h_2(s)\,ds\right\}\right)^{-1},\end{aligned}$$

then there exists $\lambda_* > 0$ *such that for any* $\lambda \geq \lambda_*$ *problem* (S)-(BC) (*with* $\lambda = \mu$) *has at least one positive solution.*

Proof By (H7) we conclude that there exists $M_3 > 0$ such that

$$f(t,u,v) \geq L_0 \quad \text{or} \quad g(t,u,v) \geq L_0, \quad \forall t \in [c,1-c], u,v \geq 0, u+v \geq M_3.$$

We define

$$\lambda_* = \max\left\{\frac{M_3}{c^{\alpha-1}}\left(\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds\right)^{-1}, \frac{M_3}{c^{\beta-1}}\left(\int_0^1 \big(\delta_3 p_2(s)+\delta_4 p_1(s)\big)\,ds\right)^{-1}\right\}.$$

We assume now $\lambda \geq \lambda_*$. Let

$$\begin{aligned}R_3 = {} & \max\left\{\frac{4\lambda}{\gamma_1}\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds, \frac{4\lambda}{\gamma_2}\int_0^1 \big(\delta_3 p_2(s)+\delta_4 p_1(s)\big)\,ds,\right.\\ & \frac{4\lambda}{\gamma_1\gamma_2}\left(\int_0^1 s^{\alpha-1}\,dK(s)\right)^{-1}\int_0^1 \big(\delta_3 p_2(s)+\delta_4 p_1(s)\big)\,ds,\\ & \left.\frac{4\lambda}{\gamma_1\gamma_2}\left(\int_0^1 s^{\beta-1}\,dH(s)\right)^{-1}\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds\right\},\end{aligned}$$

and $\Omega_3 = \{(x,y) \in P, \|(x,y)\|_Y < R_3\}$.

We suppose first that $f_\infty^i > L_0$, that is, $f(t,u,v) \geq L_0$ for all $t \in [c,1-c]$ and $u,v \geq 0$, $u+v \geq M_3$. Let $(x,y) \in P \cap \partial\Omega_3$. Then $\|(x,y)\|_Y = R_3$, so $\|x\| \geq R_3/2$ or $\|y\| \geq R_3/2$. We assume that $\|x\| \geq R_3/2$. Then for all $t \in [0,1]$ we deduce

$$\begin{aligned}x(t) - q_1(t) &\geq \gamma_1 t^{\alpha-1}\|x\| - \lambda t^{\alpha-1}\delta_1\int_0^1 p_1(s)\,ds - \lambda t^{\alpha-1}\delta_2\int_0^1 p_2(s)\,ds\\ &\geq t^{\alpha-1}\left[\frac{\gamma_1 R_3}{2} - \lambda\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds\right]\\ &\geq t^{\alpha-1}\left[2\lambda\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds - \lambda\int_0^1 \big(\delta_1 p_1(s)+\delta_2 p_2(s)\big)\,ds\right]\end{aligned}$$

$$= t^{\alpha-1}\lambda\int_0^1 \big(\delta_1 p_1(s) + \delta_2 p_2(s)\big)\,ds$$
$$\geq t^{\alpha-1}\lambda_*\int_0^1 \big(\delta_1 p_1(s) + \delta_2 p_2(s)\big)\,ds \geq \frac{M_3}{c^{\alpha-1}}t^{\alpha-1} \geq 0.$$

Therefore, for any $(x,y) \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we have

$$\big[x(t) - q_1(t)\big]^* + \big[y(t) - q_2(t)\big]^* \geq \big[x(t) - q_1(t)\big]^* = x(t) - q_1(t) \geq \frac{M_3}{c^{\alpha-1}}t^{\alpha-1} \geq M_3. \tag{15}$$

Hence, for any $(x,y) \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we conclude

$$\begin{aligned} Q_1(x,y)(t) &\geq \lambda\int_0^1 G_1(t,s)\big[f\big(s, \big[x(s) - q_1(s)\big]^*, \big[y(s) - q_2(s)\big]^*\big) + p_1(s)\big]\,ds \\ &\geq \lambda\varrho_1 t^{\alpha-1}\int_c^{1-c} h_1(s) f\big(s, \big[x(s) - q_1(s)\big]^*, \big[y(s) - q_2(s)\big]^*\big)\,ds \\ &\geq \lambda L_0\varrho_1 t^{\alpha-1}\int_c^{1-c} h_1(s)\,ds \geq \lambda L_0\varrho_1 c^{\alpha-1}\int_c^{1-c} h_1(s)\,ds \geq R_3 = \big\|(x,y)\big\|_Y. \end{aligned}$$

Therefore we obtain $\|Q_1(x,y)\| \geq R_3$ for all $(x,y) \in P \cap \partial\Omega_3$, and so

$$\big\|\mathcal{Q}(x,y)\big\|_Y \geq R_3 = \big\|(x,y)\big\|_Y, \quad \forall (x,y) \in P \cap \partial\Omega_3. \tag{16}$$

If $\|y\| \geq R_3/2$, then by a similar approach we deduce again relation (16).

We suppose now that $g_\infty^i > L_0$, that is, $g(t,u,v) \geq L_0$ for all $t \in [c, 1-c]$ and $u, v \geq 0$, $u + v \geq M_3$. Let $(x,y) \in P \cap \partial\Omega_3$. Then $\|(x,y)\|_Y = R_3$, so $\|x\| \geq R_3/2$ or $\|y\| \geq R_3/2$. If $\|x\| \geq R_3/2$, then we obtain in a similar manner as in the first case above ($f_\infty^i > L_0$) that $x(t) - q_1(t) \geq \frac{M_3}{c^{\alpha-1}}t^{\alpha-1} \geq 0$ for all $t \in [0,1]$.

Therefore, for any $(x,y) \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we deduce inequalities (15).

Hence, for any $(x,y) \in P \cap \partial\Omega_3$ and $t \in [c, 1-c]$, we conclude

$$\begin{aligned} Q_1(x,y)(t) &\geq \lambda\int_0^1 G_2(t,s)\big[g\big(s, \big[x(s) - q_1(s)\big]^*, \big[y(s) - q_2(s)\big]^*\big) + p_2(s)\big]\,ds \\ &\geq \lambda\varrho_2 t^{\alpha-1}\int_c^{1-c} h_2(s) g\big(s, \big[x(s) - q_1(s)\big]^*, \big[y(s) - q_2(s)\big]^*\big)\,ds \\ &\geq \lambda L_0\varrho_2 t^{\alpha-1}\int_c^{1-c} h_2(s)\,ds \geq \lambda L_0\varrho_2 c^{\alpha-1}\int_c^{1-c} h_2(s)\,ds \geq R_3 = \big\|(x,y)\big\|_Y. \end{aligned}$$

Therefore we obtain $\|Q_1(x,y)\| \geq R_3$, and so $\|\mathcal{Q}(x,y)\|_Y \geq R_3 = \|(x,y)\|_Y$ for all $(x,y) \in P \cap \partial\Omega_3$, that is, we have relation (16).

By a similar approach we obtain relation (16) if $\|y\| \geq R_3/2$.

On the other hand, we consider the positive number

$$\begin{aligned} \varepsilon = \min\bigg\{ &\frac{1}{8\lambda\sigma_1}\bigg(\int_0^1 h_1(s)\alpha_1(s)\,ds\bigg)^{-1}, \frac{1}{8\lambda\sigma_2}\bigg(\int_0^1 h_2(s)\alpha_2(s)\,ds\bigg)^{-1}, \\ &\frac{1}{8\lambda\sigma_3}\bigg(\int_0^1 h_2(s)\alpha_2(s)\,ds\bigg)^{-1}, \frac{1}{8\lambda\sigma_4}\bigg(\int_0^1 h_1(s)\alpha_1(s)\,ds\bigg)^{-1}\bigg\}. \end{aligned}$$

Then by (H6) we deduce that there exists $M_4 > 0$ such that

$$\beta_i(t,u,v) \le \varepsilon(u+v), \quad \forall t \in [0,1], u,v \ge 0, u+v \ge M_4, i=1,2.$$

Therefore we obtain

$$\beta_i(t,u,v) \le M_5 + \varepsilon(u+v), \quad \forall t \in [0,1], u,v \ge 0, i=1,2,$$

where $M_5 = \max_{i=1,2}\{\max_{t\in[0,1],u,v\ge 0,u+v\le M_4} \beta_i(t,u,v)\}$.

We define now

$$\begin{aligned} R_4 = \max\bigg\{ & 2R_3, 8\lambda\sigma_1 \max\{M_5,1\} \int_0^1 h_1(s)\big(\alpha_1(s)+p_1(s)\big)\,ds, \\ & 8\lambda\sigma_2 \max\{M_5,1\} \int_0^1 h_2(s)\big(\alpha_2(s)+p_2(s)\big)\,ds, \\ & 8\lambda\sigma_3 \max\{M_5,1\} \int_0^1 h_2(s)\big(\alpha_2(s)+p_2(s)\big)\,ds, \\ & 8\lambda\sigma_4 \max\{M_5,1\} \int_0^1 h_1(s)\big(\alpha_1(s)+p_1(s)\big)\,ds\bigg\}, \end{aligned}$$

and let $\Omega_4 = \{(x,y) \in P, \|(x,y)\|_Y < R_4\}$.

For any $(x,y) \in P \cap \partial\Omega_4$, we have

$$\begin{aligned} Q_1(x,y)(t) &\le \lambda \int_0^1 \sigma_1 h_1(s)\big[\alpha_1(s)\beta_1\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_1(s)\big]\,ds \\ &\quad + \lambda \int_0^1 \sigma_2 h_2(s)\big[\alpha_2(s)\beta_2\big(s,\big[x(s)-q_1(s)\big]^*,\big[y(s)-q_2(s)\big]^*\big)+p_2(s)\big]\,ds \\ &\le \lambda\sigma_1 \int_0^1 h_1(s)\big[\alpha_1(s)\big(M_5+\varepsilon\big(\big[x(s)-q_1(s)\big]^*+\big[y(s)-q_2(s)\big]^*\big)\big)+p_1(s)\big]\,ds \\ &\quad + \lambda\sigma_2 \int_0^1 h_2(s)\big[\alpha_2(s)\big(M_5+\varepsilon\big(\big[x(s)-q_1(s)\big]^*+\big[y(s)-q_2(s)\big]^*\big)\big)+p_2(s)\big]\,ds \\ &\le \lambda\sigma_1 \max\{M_5,1\}\int_0^1 h_1(s)\big(\alpha_1(s)+p_1(s)\big)\,ds + \lambda\sigma_1\varepsilon R_4 \int_0^1 h_1(s)\alpha_1(s)\,ds \\ &\quad + \lambda\sigma_2 \max\{M_5,1\}\int_0^1 h_2(s)\big(\alpha_2(s)+p_2(s)\big)\,ds + \lambda\sigma_2\varepsilon R_4 \int_0^1 h_2(s)\alpha_2(s)\,ds \\ &\le \frac{R_4}{8}+\frac{R_4}{8}+\frac{R_4}{8}+\frac{R_4}{8} = \frac{R_4}{2} = \frac{\|(x,y)\|_Y}{2}, \quad \forall t \in [0,1], \end{aligned}$$

and so $\|Q_1(x,y)\| \le \frac{\|(x,y)\|_Y}{2}$ for all $(x,y) \in P \cap \partial\Omega_4$.

In a similar way we obtain $Q_2(x,y)(t) \le \frac{\|(x,y)\|_Y}{2}$ for all $t \in [0,1]$, and so $\|Q_2(x,y)\| \le \frac{\|(x,y)\|_Y}{2}$ for all $(x,y) \in P \cap \partial\Omega_4$.

Therefore, we deduce

$$\big\|\mathcal{Q}(x,y)\big\|_Y \le \big\|(x,y)\big\|_Y, \quad \forall (x,y) \in P \cap \partial\Omega_4. \tag{17}$$

By Theorem 2.2, (16), and (17), we conclude that $\mathcal{Q}$ has a fixed point $(x_1,y_1) \in P \cap (\bar{\Omega}_4 \setminus \Omega_3)$. Since $\|(x_1,y_1)\| \ge R_3$ then $\|x_1\| \ge R_3/2$ or $\|y_1\| \ge R_3/2$.

We suppose that $\|x_1\| \geq R_3/2$. Then $x_1(t) - q_1(t) \geq \frac{M_3}{c^{\alpha-1}} t^{\alpha-1}$ for all $t \in [0,1]$. Besides

$$y_1(1) = \int_0^1 x_1(s)\,dK(s) \geq \gamma_1 \|x_1\| \int_0^1 s^{\alpha-1}\,dK(s) \geq \frac{\gamma_1 R_3}{2} \int_0^1 s^{\alpha-1}\,dK(s) > 0,$$

and then

$$\|y_1\| \geq y_1(1) = \int_0^1 x_1(s)\,dK(s) \geq \frac{\gamma_1 R_3}{2} \int_0^1 s^{\alpha-1}\,dK(s) > 0.$$

Therefore, we deduce that, for all $t \in [0,1]$,

$$\begin{aligned}
y_1(t) - q_2(t) &\geq y_1(t) - \lambda\delta_3 \int_0^1 t^{\beta-1} p_2(s)\,ds - \lambda\delta_4 \int_0^1 t^{\beta-1} p_1(s)\,ds \\
&\geq \gamma_2 t^{\beta-1} \|y_1\| - \lambda t^{\beta-1} \int_0^1 \bigl(\delta_3 p_2(s) + \delta_4 p_1(s)\bigr)\,ds \\
&\geq \frac{\gamma_1\gamma_2 R_3}{2} t^{\beta-1} \int_0^1 s^{\alpha-1}\,dK(s) - \lambda t^{\beta-1} \int_0^1 \bigl(\delta_3 p_2(s) + \delta_4 p_1(s)\bigr)\,ds \\
&\geq \lambda t^{\beta-1} \int_0^1 \bigl(\delta_3 p_2(s) + \delta_4 p_1(s)\bigr)\,ds \\
&\geq \lambda_* t^{\beta-1} \int_0^1 \bigl(\delta_3 p_2(s) + \delta_4 p_1(s)\bigr)\,ds \geq \frac{M_3}{c^{\beta-1}} t^{\beta-1}.
\end{aligned}$$

If $\|y_1\| \geq R_3/2$, then by a similar approach we conclude again that $x_1(t) - q_1(t) \geq \frac{M_3}{c^{\alpha-1}} t^{\alpha-1}$ and $y_1(t) - q_2(t) \geq \frac{M_3}{c^{\beta-1}} t^{\beta-1}$ for all $t \in [0,1]$.

Let $u_1(t) = x_1(t) - q_1(t)$ and $v_1(t) = y_1(t) - q_2(t)$ for all $t \in [0,1]$. Then $u_1(t) \geq \widetilde{\Lambda}_1 t^{\alpha-1}$ and $v_1(t) \geq \widetilde{\Lambda}_2 t^{\beta-1}$ for all $t \in [0,1]$, where $\widetilde{\Lambda}_1 = \frac{M_3}{c^{\alpha-1}}$, $\widetilde{\Lambda}_2 = \frac{M_3}{c^{\beta-1}}$. Hence we deduce that (u_1, v_1) is a positive solution of (S)-(BC), which completes the proof of Theorem 3.4. □

In a similar manner as we proved Theorem 3.4, we obtain the following theorems.

Theorem 3.5 *Assume that* $\lambda = \mu$, *and* (H1), (H4), *and* (H6) *hold. In addition if*

(H7′) *there exists* $c \in (0,1/2)$ *such that*

$$f_\infty^i = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} f(t,u,v) > \widetilde{L}_0 \quad \text{or} \quad g_\infty^i = \liminf_{\substack{u+v\to\infty\\ u,v\geq 0}} \min_{t\in[c,1-c]} g(t,u,v) > \widetilde{L}_0,$$

where

$$\begin{aligned}
\widetilde{L}_0 = \max\biggl\{ &\frac{4}{\gamma_1} \int_0^1 \bigl(\delta_1 p_1(s) + \delta_2 p_2(s)\bigr)\,ds, \frac{4}{\gamma_2} \int_0^1 \bigl(\delta_3 p_2(s) + \delta_4 p_1(s)\bigr)\,ds, \\
&\frac{4}{\gamma_1\gamma_2} \left(\int_0^1 s^{\alpha-1}\,dK(s)\right)^{-1} \int_0^1 \bigl(\delta_3 p_2(s) + \delta_4 p_1(s)\bigr)\,ds, \\
&\frac{4}{\gamma_1\gamma_2} \left(\int_0^1 s^{\beta-1}\,dH(s)\right)^{-1} \int_0^1 \bigl(\delta_1 p_1(s) + \delta_2 p_2(s)\bigr)\,ds \biggr\} \\
&\times \left(\min\left\{ c^{\beta-1} \varrho_3 \int_c^{1-c} h_2(s)\,ds, c^{\beta-1} \varrho_4 \int_c^{1-c} h_1(s)\,ds \right\} \right)^{-1},
\end{aligned}$$

then there exists $\lambda'_* > 0$ *such that for any* $\lambda \geq \lambda'_*$ *problem* (S)-(BC) (*with* $\lambda = \mu$) *has at least one positive solution*.

Theorem 3.6 *Assume that* $\lambda = \mu$, *and* (H1), (H4), *and* (H6) *hold*. *In addition if*

(H8) *there exists* $c \in (0,1/2)$ *such that*

$$\hat{f}_\infty = \lim_{\substack{u+v\to\infty \\ u,v\geq 0}} \min_{t\in[c,1-c]} f(t,u,v) = \infty \quad \text{or} \quad \hat{g}_\infty = \lim_{\substack{u+v\to\infty \\ u,v\geq 0}} \min_{t\in[c,1-c]} g(t,u,v) = \infty,$$

then there exists $\tilde{\lambda}_* > 0$ *such that for any* $\lambda \geq \tilde{\lambda}_*$ *problem* (S)-(BC) (*with* $\lambda = \mu$) *has at least one positive solution*.

4 Examples

Let $\alpha = 5/2$ ($n = 3$), $\beta = 7/3$ ($m = 3$), $H(t) = t^2$, $K(t) = t^3$. Then $\int_0^1 u(s)\,dK(s) = 3\int_0^1 s^2 u(s)\,ds$ and $\int_0^1 v(s)\,dH(s) = 2\int_0^1 sv(s)\,ds$.

We consider the system of fractional differential equations

$$(\mathrm{S}_0) \quad \begin{cases} D_{0+}^{5/2}u(t) + \lambda f(t,u(t),v(t)) = 0, & t\in(0,1), \\ D_{0+}^{7/3}v(t) + \mu g(t,u(t),v(t)) = 0, & t\in(0,1), \end{cases}$$

with the boundary conditions

$$(\mathrm{BC}_0) \quad \begin{cases} u(0) = u'(0) = 0, & u(1) = 2\int_0^1 sv(s)\,ds, \\ v(0) = v'(0) = 0, & v(1) = 3\int_0^1 s^2u(s)\,ds. \end{cases}$$

Then we obtain $\Delta = 1 - (\int_0^1 s^{\alpha-1}\,dK(s))(\int_0^1 s^{\beta-1}\,dH(s)) = \frac{3}{5} > 0$, $\int_0^1 \tau^{\alpha-1}(1-\tau)\,dK(\tau) = \frac{4}{33} > 0$, $\int_0^1 \tau^{\beta-1}(1-\tau)\,dH(\tau) = \frac{9}{65} > 0$. The functions H and K are nondecreasing, and so assumption (H1) is satisfied. Besides, we deduce

$$g_1(t,s) = \frac{4}{3\sqrt{\pi}} \begin{cases} t^{3/2}(1-s)^{3/2} - (t-s)^{3/2}, & 0\leq s\leq t\leq 1, \\ t^{3/2}(1-s)^{3/2}, & 0\leq t\leq s\leq 1, \end{cases}$$

$$g_2(t,s) = \frac{1}{\Gamma(7/3)} \begin{cases} t^{4/3}(1-s)^{4/3} - (t-s)^{4/3}, & 0\leq s\leq t\leq 1, \\ t^{4/3}(1-s)^{4/3}, & 0\leq t\leq s\leq 1, \end{cases}$$

$$G_1(t,s) = g_1(t,s) + 3t^{3/2}\int_0^1 \tau^2 g_1(\tau,s)\,ds, \qquad G_2(t,s) = \frac{10}{3}t^{3/2}\int_0^1 \tau g_2(\tau,s)\,d\tau,$$

$$G_3(t,s) = g_2(t,s) + \frac{20}{9}t^{4/3}\int_0^1 \tau g_2(\tau,s)d\tau, \qquad G_4(t,s) = 5t^{4/3}\int_0^1 \tau^2 g_1(\tau,s)\,d\tau.$$

We also obtain $h_1(s) = \frac{2}{\sqrt{\pi}}s(1-s)^{3/2}$, $h_2(s) = \frac{1}{\Gamma(4/3)}s(1-s)^{4/3}$,

$$k_1(t) = \begin{cases} \frac{2}{3}t^{3/2}, & 0\leq t\leq 1/2, \\ \frac{2}{3}(1-t)t^{1/2}, & 1/2\leq t\leq 1, \end{cases} \qquad k_2(t) = \begin{cases} \frac{3}{4}t^{4/3}, & 0\leq t\leq 1/2, \\ \frac{3}{4}(1-t)t^{1/3}, & 1/2\leq t\leq 1. \end{cases}$$

In addition, we have $\sigma_1 = 2$, $\delta_1 = \frac{74}{33\sqrt{\pi}}$, $\varrho_1 = \frac{8\sqrt{2}-1}{63\sqrt{2}}$, $\sigma_2 = \frac{5}{3}$, $\delta_2 = \frac{3}{13\Gamma(4/3)}$, $\varrho_2 = \frac{36\sqrt[3]{2}-9}{112\sqrt[3]{2}}$, $\sigma_3 = \frac{19}{9}$, $\delta_3 = \frac{15}{13\Gamma(4/3)}$, $\varrho_3 = \frac{12\sqrt[3]{2}-3}{56\sqrt[3]{2}}$, $\sigma_4 = \frac{5}{3}$, $\delta_4 = \frac{40}{99\sqrt{\pi}}$, $\varrho_4 = \frac{40\sqrt{2}-5}{189\sqrt{2}}$, $\gamma_1 = \frac{8\sqrt{2}-1}{126\sqrt{2}} \approx 0.0578801$, $\gamma_2 = \frac{9(12\sqrt[3]{2}-3)}{1064\sqrt[3]{2}} \approx 0.08136286$.

Example 1 We consider the functions

$$f(t,u,v) = \frac{(u+v)^2}{\sqrt{t(1-t)}} + \ln t, \qquad g(t,u,v) = \frac{2+\sin(u+v)}{\sqrt{t(1-t)}} + \ln(1-t), \quad t \in (0,1), u,v \geq 0.$$

We have $p_1(t) = -\ln t$, $p_2(t) = -\ln(1-t)$, $\alpha_1(t) = \alpha_2(t) = \frac{1}{\sqrt{t(1-t)}}$ for all $t \in (0,1)$, $\beta_1(t,u,v) = (u+v)^2$, $\beta_2(t,u,v) = 2 + \sin(u+v)$ for all $t \in [0,1]$, $u,v \geq 0$, $\int_0^1 p_1(t)\,dt = 1$, $\int_0^1 p_2(t)\,dt = 1$, $\int_0^1 \alpha_i(t)\,dt = \pi$, $i = 1,2$. Therefore, assumption (H4) is satisfied. In addition, for $c \in (0,1/2)$ fixed, assumption (H5) is also satisfied ($f_\infty = \infty$).

After some computations, we deduce $\int_0^1 (\delta_1 p_1(s) + \delta_2 p_2(s))\,ds \approx 1.52357852$, $\int_0^1 (\delta_3 p_2(s) + \delta_4 p_1(s))\,ds \approx 1.520086$, $\int_0^1 h_1(s)(\alpha_1(s) + p_1(s))\,ds \approx 0.42548534$, $\int_0^1 h_2(s)(\alpha_2(s) + p_2(s))\,ds \approx 0.44092924$. We choose $R_1 = 1080$, which satisfies the condition from the beginning of the proof of Theorem 3.2. Then $M_1 = R_1^2$, $M_2 = 3$, $\lambda^* \approx 2.7202 \cdot 10^{-4}$, and $\mu^* = 1$. By Theorem 3.2, we conclude that (S_0)-(BC_0) has at least one positive solution for any $\lambda \in (0,\lambda^*]$ and $\mu \in (0,\mu^*]$.

Example 2 We consider the functions

$$f(t,u,v) = (u+v)^2 + \cos u, \qquad g(t,u,v) = (u+v)^{1/2} + \cos v, \quad t \in [0,1], u,v \geq 0.$$

We have $p_1(t) = p_2(t) = 1$ for all $t \in [0,1]$, and then assumption (H2) is satisfied. Besides, assumption (H3) is also satisfied, because $f(t,0,0) = 1$ and $g(t,0,0) = 1$ for all $t \in [0,1]$.

Let $\delta = \frac{1}{2} < 1$ and $R_0 = 1$. Then

$$f(t,u,v) \geq \delta f(t,0,0) = \frac{1}{2}, \qquad g(t,u,v) \geq \delta g(t,0,0) = \frac{1}{2}, \quad \forall t \in [0,1], u,v \in [0,1].$$

In addition,

$$\bar{f}(R_0) = \bar{f}(1) = \max_{t\in[0,1],u,v\in[0,1]} \bigl\{f(t,u,v) + p_1(t)\bigr\} \approx 5.5403023,$$

$$\bar{g}(R_0) = \bar{g}(1) = \max_{t\in[0,1],u,v\in[0,1]} \bigl\{g(t,u,v) + p_2(t)\bigr\} \approx 3.10479256.$$

We also obtain $c_1 \approx 0.25791523$, $c_2 \approx 0.23996711$, $c_3 \approx 0.30395834$, $c_4 \approx 0.21492936$, and then $\lambda_0 = \max\{\frac{R_0}{8c_1\bar{f}(R_0)}, \frac{R_0}{8c_4\bar{f}(R_0)}\} \approx 0.10497377$ and $\mu_0 = \max\{\frac{R_0}{8c_2\bar{g}(R_0)}, \frac{R_0}{8c_3\bar{g}(R_0)}\} \approx 0.1677744$.

By Theorem 3.1, for any $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$, we deduce that problem (S_0)-(BC_0) has at least one positive solution.

Because assumption (H4′) is satisfied ($\alpha_1(t) = \alpha_2(t) = 1$, $\beta_1(t,u,v) = (u+v)^2+1$, $\beta_2(t,u,v) = (u+v)^{1/2} + 1$ for all $t \in [0,1]$, $u,v \geq 0$) and assumption (H5) is also satisfied ($f_\infty = \infty$), by Theorem 3.3 we conclude that problem (S_0)-(BC_0) has at least two positive solutions for λ and μ sufficiently small.

Example 3 We consider $\lambda = \mu$ and the functions

$$f(t,u,v) = \frac{(u+v)^a}{\sqrt[3]{t^2(1-t)}} - \frac{1}{\sqrt{t}}, \qquad g(t,u,v) = \frac{\ln(1+u+v)}{\sqrt[3]{t(1-t)^2}} - \frac{1}{\sqrt{1-t}}, \quad t \in (0,1), u,v \geq 0,$$

where $a \in (0,1)$.

Here we have $p_1(t) = \frac{1}{\sqrt{t}}$, $p_2(t) = \frac{1}{\sqrt{1-t}}$, $\alpha_1(t) = \frac{1}{\sqrt[3]{t^2(1-t)}}$, $\alpha_2(t) = \frac{1}{\sqrt[3]{t(1-t)^2}}$ for all $t \in (0,1)$, $\beta_1(t,u,v) = (u+v)^a$, $\beta_2(t,u,v) = \ln(1+u+v)$ for all $t \in [0,1]$, $u,v \geq 0$. For $c \in (0,1/2)$ fixed, the assumptions (H4), (H6), and (H8) are satisfied ($\beta_{i\infty} = 0$ for $i = 1,2$ and $\hat{f}_\infty = \infty$).

Then by Theorem 3.6, we deduce that there exists $\tilde{\lambda}_* > 0$ such that for any $\lambda \geq \tilde{\lambda}_*$ our problem (S_0)-(BC_0) (with $\lambda = \mu$) has at least one positive solution.

Competing interests

The authors declare that no competing interests exist.

Authors' contributions

The authors contributed equally to this paper. Both authors read and approved the final manuscript.

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Acknowledgements

The authors thank the referees for their valuable comments and suggestions. The work of R Luca was supported by the CNCS grant PN-II-ID-PCE-2011-3-0557, Romania.

Received: 8 October 2015 Accepted: 2 March 2016 Published online: 08 March 2016

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