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New results of positive solutions for the Sturm-Liouville problem

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Abstract

Some inequalities are established to study the existence of positive solutions of the superlinear Sturm-Liouville problem, and new results are obtained. Usual limit conditions are not required to be bounded below, and the obtained results are demonstrated by an example.

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1 Introduction

We investigate the existence of positive solutions for the Sturm-Liouville problem

$$\big(p(t)z'(t)\big)' + f\big(t, z(t)\big) = 0 \quad \text{a.e. on } [0,1] \tag{1.1}$$

subject to the boundary conditions

$$\begin{cases} \alpha z(0) - \beta p(0)z'(0) = 0, \\ \gamma z(1) + \delta p(1)z'(1) = 0, \end{cases} \tag{1.2}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\Gamma := \gamma\beta + \alpha\gamma \int_0^1 \frac{1}{p(\mu)}\,d\mu + \alpha\delta > 0$.

Problem (1.1)-(1.2) has been used to model many phenomena in physics and engineering. Such problems arise in the study of gas dynamics, fluid mechanics, nuclear physics, chemically reacting systems, atomic calculations, the sources diffusion theory, and the thermal ignition theory (see [1–6]). In most of these applications, the physical interest lies in the existence of nonzero positive solutions.

The existence of nonzero positive solutions of (1.1)-(1.2) has been studied via the various methods. For the positone case or the semipositone case (that is, $f(t,z) \geq -h$ on $[0,1] \times [0,\infty)$, where $h \geq 0$ is a constant), the well-known fixed theorems in cone [7] were used to study the existence of nonzero positive solutions of (1.1)-(1.2); see, for example, [8–11] and the references therein. The case that f has a functional lower bound (that is, $f(t,z) \geq -h(t)$ on $[0,1] \times [0,\infty)$, where $h \in L_+[0,1]$) was considered [12], where f is required to satisfy that there exist $0 < a < b < 1$ such that

$$\int_a^b \liminf_{z\to\infty} f(t,z)/z\,dt = \infty. \tag{1.3}$$

Utilizing the first eigenvalues corresponding to the relevant linear operators, Li (Theorem 1, [13]) proved the existence of positive solutions of the Sturm-Liouville problem (1.1)-(1.2) for the sublinear case or the superlinear case, where some limits such as $f_\infty = \lim_{z\to\infty}\inf_{t\in[0,1]} f(t,z)/z$ and $f_0 = \lim_{z\to 0}\inf_{t\in[0,1]} f(t,z)/z$ are bounded below, and $p \in C^1[0,1]$. The well-known fixed theorems in cone [7] were used likewise in [13].

Under some strict conditions imposed on f, employing lower and upper solutions, variational methods and the global bifurcation theory of Rabinowitz, Benmezaï [14], Cui *et al.* [15], Tian and Ge [16], and Zhang *et al.* [17] studied the existence of multiple solutions and sign-changing solutions of (1.1)-(1.2), respectively, where f is a continuous function that is $o(|z|)$ near 0, $\lim_{z\to\infty} f'(z)$ and $\lim_{z\to-\infty} f'(z)$ exist and are finite [14]; or $f_0, f_\infty \in (0,\infty)$ and $p \in C^1[0,1]$ [15]; or $f(t,z)$ is Lipschitz continuous for z uniformly and $f_t'(t,z)$ exists [16]; or $p \in C^1[0,1]$, $f \in C^1([0,1]\times\mathbb{R}^1,\mathbb{R}^1)$, and $zf(t,z) \geq 0$ [17].

Different from methods used in the references mentioned, by investigating the property of nonzero solutions of an integral equation and utilizing the Leray-Schauder fixed point theorem in a Banach space, Yang and Zhou [18] proved an existence result for problem (1.1)-(1.2) under the sublinear condition, where p is not required to belong to $C^1[0,1]$ and f and f_∞ may not have any lower bound, that is, f and f_∞ may take $-\infty$. However, the authors did not studied the superlinear case with $f_0 = -\infty$ in [18].

In this paper, by establishing some inequalities (see, for example, Theorem 2.1 and Lemma 2.1) we shall prove new existence results of positive solutions for the superlinear Sturm-Liouville problem (1.1)-(1.2) concerning the first eigenvalues corresponding to the relevant linear operators. We do not assume that f satisfies (1.3), $f_0 > -\infty$ [13] (see Remark 3.1), and the strict restrictions such as in [15–17, 19]; p is also not required to belong to $C^1[0,1]$ as in [10, 11, 13, 16, 17, 20].

This paper is organized as follows. In Section 2, we make some preliminaries for studying the existence of positive solutions of (1.1)-(1.2). In Section 3, we prove the main results. Finally, we give an example to show that the existing results are not applicable to our case.

2 Preliminaries

We first prove some inequalities (Theorem 2.1 and Lemma 2.1), which play a key role in the study of the existence of positive solutions of (1.1)-(1.2).

We make the following assumptions on f and p:

(C_1) $f : [0,1]\times\mathbb{R}_+(\mathbb{R}_+ = [0,\infty)) \to \mathbb{R}$ is a Carathéodory function, that is, $f(\cdot,z)$ is measurable for each fixed $z \in \mathbb{R}_+$, $f(t,\cdot)$ is continuous for almost every (a.e.) $t \in [0,1]$, and for each $r > 0$, there exists $g_r \in L_+[0,1]$ such that

$$\left|f(t,z)\right| \leq g_r(t) \quad \text{for a.e. } t \in [0,1] \text{ and all } z \in [0,r],$$

where $L_+[0,1] = \{g \in L[0,1] : g(s) \geq 0 \text{ a.e. } [0,1]\}$.

(C_2) $f(t,0) \geq 0$ for a.e. $t \in [0,1]$.

(C_3) $p : [0,1] \to \mathbb{R}_+ \setminus \{0\}$, and $p \in C[0,1]$.

Remark 2.1 Standard condition (C_1) has been widely used, for example, in [19, 21]. The upper bound function g_r in (C_1) is independent of z and belongs to $L_+[0,1]$, which is more general than the conditions used previously in [19, 21]. The condition $f(t,z) \le C(1+z^{p-1})$ for *a.e.* $t \in [0,1]$ and all $z \in R_+$ was used in [19] ($n=1$), whereas [21] required g_r in $L_+^\infty[0,1]$.

It is easy to verify that $f_0 > -\infty$ [13] or $zf(t,z) \ge 0$ [17] or $f(t,z) = f_0(t,z) + h(t)z$ [10, 11, 13, 17, 20] ($f_0(t,z) \ge 0$ for $t \in [0,1]$, $z \ge 0$) implies that (C_2) holds; the inverse is false, and we do not require $p \in C^1[0,1]$ as in [10, 11, 13, 16, 17, 20]. Hence, conditions (C_2)-(C_3) are weaker than the usual assumptions.

A function z is said to be a positive solution of (1.1)-(1.2) if $z \in C^1[0,1]$ with $z(t) \ge 0$ on $[0,1]$, $z \not\equiv 0$, $p(t)z'(t) \in AC[0,1]$, and z satisfies (1.1)-(1.2), where $AC[0,1]$ is the space of all absolutely continuous functions on $[0,1]$.

Let $C[0,1]$ be continuous function space with norm $\|z\| = \max\{|z(t)| : t \in [0,1]\}$. It is well known that z is a positive solution of (1.1)-(1.2) if and only if $z \in C[0,1]$ with $z(t) \not\equiv 0$ and $z(t) \ge 0$ on $[0,1]$ satisfies the following integral equation [8, 9, 12]:

$$z(t) = \int_0^1 G(t,s) f\bigl(s, z(s)\bigr)\, ds \quad \text{for } t \in [0,1], \tag{2.1}$$

where $G(t,s)$ is the Green function to $-(p(t)z'(t))' = 0$ associated to the boundary conditions (1.2) defined by

$$G(t,s) = \frac{1}{\Gamma} \begin{cases} \omega_1(t)\omega_0(s), & s \le t, \\ \omega_1(s)\omega_0(t), & t < s, \end{cases} \tag{2.2}$$

where $\alpha, \beta, \gamma, \delta \ge 0$, Γ is in (1.2), and

$$\omega_0(s) = \beta + \alpha \int_0^s \frac{1}{p(\mu)}\, d\mu,$$

$$\omega_1(s) = \delta + \gamma \int_s^1 \frac{1}{p(\mu)}\, d\mu.$$

Let $g, h \in L_+[0,1]$ and $\int_0^1 h(s)\, ds > 0$. We define a few functions

$$\chi_a(t) = \int_0^a G(t,s) g(s)\, ds \quad \text{on } [0,1],$$

$$\chi_b(t) = \int_b^1 G(t,s) g(s)\, ds \quad \text{on } [0,1],$$

$$\chi_{a,b}(t) = \int_a^b G(t,s) h(s)\, ds \quad \text{on } [0,1],$$

where $0 < a < b < 1$ are constants.

First, we prove one of two inequalities.

Theorem 2.1 *Assume that* (C_3) *holds. Then there exist* $0 < a_0 < b_0 < 1$ *such that*

$$\chi_{a,b}(t) \ge \chi_a(t) + \chi_b(t) \quad \textit{on } [0,1]$$

for all $0 < a \le a_0$ *and* $b_0 \le b < 1$, *that is*, $\varphi_{a,b}(t) := \chi_{a,b}(t) - \chi_a(t) - \chi_b(t) \ge 0$ *on* $[0,1]$.

Proof The proof is divided into three steps.

Step 1. There exist $0 < \widetilde{a} < \widetilde{b} < 1$ such that $\varphi'_{a,b}(t) \geq 0$ on $[0,a]$ and $\varphi'_{a,b}(t) \leq 0$ on $[b,1]$ for all $0 < a \leq \widetilde{a}$ and $\widetilde{b} \leq b < 1$.

Since $\Gamma > 0$, we know that $\omega_0(s) > 0$ on $(0,1]$ and $\omega_1(s) > 0$ on $[0,1)$. By direct computation we have

$$\chi'_a(t) = \frac{1}{\Gamma p(t)} \begin{cases} -\gamma \int_0^t \omega_0(s)g(s)\,ds + \alpha \int_t^a \omega_1(s)g(s)\,ds & \text{for } 0 \leq t \leq a, \\ -\gamma \int_0^a \omega_0(s)g(s)\,ds & \text{for } t > a, \end{cases}$$

$$\bigl(p(t)\chi'_a(t)\bigr)' = \begin{cases} -g(t) & \text{for } 0 \leq t \leq a, \\ 0 & \text{for } t > a, \end{cases}$$

$$\chi'_b(t) = \frac{1}{\Gamma p(t)} \begin{cases} -\gamma \int_b^t \omega_0(s)g(s)\,ds + \alpha \int_t^1 \omega_1(s)g(s)\,ds & \text{for } b \leq t \leq 1, \\ \alpha \int_b^1 \omega_1(s)g(s)\,ds & \text{for } t < b, \end{cases}$$

$$\bigl(p(t)\chi'_b(t)\bigr)' = \begin{cases} -g(t) & \text{for } b \leq t \leq 1, \\ 0 & \text{for } t < b, \end{cases}$$

$$\chi'_{a,b}(t) = \frac{1}{\Gamma p(t)} \begin{cases} -\gamma \int_a^t \omega_0(s)h(s)\,ds + \alpha \int_t^b \omega_1(s)h(s)\,ds & \text{for } a \leq t \leq b, \\ \alpha \int_a^b \omega_1(s)h(s)\,ds & \text{for } t < a, \\ -\gamma \int_a^b \omega_0(s)h(s)\,ds & \text{for } t > b, \end{cases}$$

$$\bigl(p(t)\chi'_{a,b}(t)\bigr)' = \begin{cases} -h(t) & \text{for } a \leq t \leq b, \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$

Then $\chi_a, \chi_b, \chi_{a,b} \in C^1[0,1]$; hence, $\varphi_{a,b} \in C^1[0,1]$, and

$$\varphi'_{a,b}(t) = \frac{1}{\Gamma p(t)} \begin{cases} \gamma \int_0^t \omega_0(s)g(s)\,ds + \alpha H_1(t) & \text{for } 0 \leq t \leq a, \\ \gamma H_2(t) - \alpha \int_t^1 \omega_1(s)g(s)\,ds & \text{for } b \leq t \leq 1, \end{cases} \tag{2.3}$$

$$\bigl(p(t)\varphi'_{a,b}(t)\bigr)' = \begin{cases} -h(t) & \text{for } a \leq t \leq b, \\ g(t) & \text{for } t < a \text{ or } t > b, \end{cases} \tag{2.4}$$

where

$$H_1(t) = \int_a^b \omega_1(s)h(s)\,ds - \left[\int_t^a \omega_1(s)g(s)\,ds + \int_b^1 \omega_1(s)g(s)\,ds)\right],$$
$$H_2(t) = \left[\int_0^a \omega_0(s)g(s)\,ds + \int_b^t \omega_0(s)g(s)\,ds\right] - \int_a^b \omega_0(s)h(s)\,ds.$$

Since $\int_0^1 h(s)\,ds > 0$ and $\omega_0(s) > 0$ on $(0,1]$ and $\omega_1(s) > 0$ on $[0,1)$, there exist $c, d \in (0,1)$ such that $c < d$ and $\int_c^d \omega_i(s)h(s)\,ds > 0$ $(i = 0,1)$. The absolute continuity of the Lebesgue integral shows that there exist $0 < \widetilde{a} \leq c < d \leq \widetilde{b} < 1$ satisfying

$$\int_{\widetilde{a}}^{\widetilde{b}} \omega_0(s)h(s)\,ds > \int_0^{\widetilde{a}} \omega_0(s)g(s)\,ds + \int_{\widetilde{b}}^1 \omega_0(s)g(s)\,ds,$$
$$\int_{\widetilde{a}}^{\widetilde{b}} \omega_1(s)h(s)\,ds > \int_0^{\widetilde{a}} \omega_1(s)g(s)\,ds + \int_{\widetilde{b}}^1 \omega_1(s)g(s)\,ds.$$

Then, for $0 < a \le \widetilde{a}, \widetilde{b} \le b < 1$,

$$\begin{aligned}\int_a^b \omega_1(s)h(s)\,ds &\ge \int_{\widetilde{a}}^{\widetilde{b}} \omega_1(s)h(s)\,ds > \int_0^{\widetilde{a}} \omega_1(s)g(s)\,ds + \int_{\widetilde{b}}^1 \omega_1(s)g(s)\,ds \\ &\ge \int_t^a \omega_1(s)g(s)\,ds + \int_b^1 \omega_1(s)g(s)\,ds \quad \text{for } 0 \le t \le a, \\ \int_a^b \omega_0(s)h(s)\,ds &\ge \int_{\widetilde{a}}^{\widetilde{b}} \omega_0(s)h(s)\,ds > \int_0^{\widetilde{a}} \omega_0(s)g(s)\,ds + \int_{\widetilde{b}}^1 \omega_0(s)g(s)\,ds \\ &\ge \int_0^a \omega_0(s)g(s)\,ds + \int_b^t \omega_0(s)g(s)\,ds \quad \text{for } b \le t \le 1.\end{aligned}$$

From these inequalities we obtain $H_1(t) \ge 0$ on $[0,a]$ and $H_2(t) \le 0$ on $[b,1]$.

By $\Gamma > 0$ we see that $\alpha > 0$ if $\gamma = 0$ and $\gamma > 0$ if $\alpha = 0$ by (2.3), and then $\varphi'_{a,b}(t) \ge 0$ on $[0,a]$ and $\varphi'_{a,b}(t) \le 0$ on $[b,1]$ for all $0 < t \le a$ and $b \le t < 1$.

Step 2. There exist $0 < a_0 \le \widetilde{a}$ and $\widetilde{b} \le b_0 < 1$ satisfying $\varphi_{a,b}(0) \ge 0$ and $\varphi_{a,b}(1) \ge 0$ for $0 < a \le a_0$ and $b_0 \le b < 1$.

If $\beta = 0$, then we see that $G(0,s) = 0$, $\chi_a(0) = 0$, $\chi_b(0) = 0$, $\chi_{a,b}(0) = 0$, and $\varphi_{a,b}(0) = 0$. If $\delta = 0$, then we have $\chi_a(1) = 0$, $\chi_b(1) = 0$, $\chi_{a,b}(1) = 0$, and $\varphi_{a,b}(1) = 0$.

We prove the following facts:

(i) If $\beta > 0$, then there exist $0 < a_1 \le \widetilde{a}$ and $\widetilde{b} \le b_1 < 1$ satisfying $\varphi_{a,b}(0) \ge 0$ for $0 \le a \le a_1$ and $b_1 \le b \le 1$.

(ii) If $\delta > 0$, then there exist $0 < a_2 \le \widetilde{a}$ and $\widetilde{b} \le b_2 < 1$ satisfying $\varphi_{a,b}(1) \ge 0$ for $0 < a \le a_2$ and $b_2 \le b < 1$.

(i) Let $\beta > 0$. The equality $G(0,s) = \frac{\beta}{\Gamma}\omega_1(s)$ shows

$$\chi_{a,b}(0) = \int_a^b G(0,s)h(s)\,ds = \frac{\beta}{\Gamma}\int_a^b \omega_1(s)h(s)\,ds.$$

From $\int_{\widetilde{a}}^{\widetilde{b}} \omega_1(s)h(s)\,ds > 0$ and the absolute continuity of the Lebesgue integral we know that there exist $0 < a_1 \le \widetilde{a}$ and $\widetilde{b} \le b_1 < 1$ satisfying

$$\int_0^{a_1} G(0,s)g(s)\,ds + \int_{b_1}^1 G(0,s)g(s)\,ds \le \frac{\beta}{\Gamma}\int_{\widetilde{a}}^{\widetilde{b}} \omega_1(s)h(s)\,ds.$$

This implies

$$\begin{aligned}\chi_a(0) + \chi_b(0) &= \int_0^a G(0,s)g(s)\,ds + \int_b^1 G(0,s)g(s)\,ds \\ &\le \int_0^{a_1} G(0,s)g(s)\,ds + \int_{b_1}^1 G(0,s)g(s)\,ds \\ &\le \frac{\beta}{\Gamma}\int_{\widetilde{a}}^{\widetilde{b}} \omega_1(s)h(s)\,ds \\ &\le \frac{\beta}{\Gamma}\int_a^b \omega_1(s)h(s)\,ds = \int_a^b G(0,s)h(s)\,ds = \chi_{a,b}(0)\end{aligned}$$

for $0 < a \le a_1$ and $b_1 \le b < 1$, that is, $\varphi_{a,b}(0) \ge 0$ for $0 < a \le a_1$ and $b_1 \le b < 1$.

(ii) Let $\delta > 0$. The equality $G(1,s) = \frac{\delta}{\Gamma}\omega_0(s)$ implies

$$\chi_{a,b}(1) = \int_a^b G(1,s)h(s)\,ds = \frac{\delta}{\Gamma}\int_a^b \omega_0(s)h(s)\,ds.$$

By $\int_{\widetilde{a}}^{\widetilde{b}} \omega_0(s)h(s)\,ds > 0$ and the absolute continuity of the Lebesgue integral we know that there exist $0 < a_2 \le \widetilde{a}$ and $\widetilde{b} \le b_2 < 1$ satisfying

$$\int_0^{a_2} G(1,s)g(s)\,ds + \int_{b_2}^1 G(1,s)g(s)\,ds \le \frac{\delta}{\Gamma}\int_{\widetilde{a}}^{\widetilde{b}} \omega_0(s)h(s)\,ds.$$

This shows that

$$\begin{aligned} \chi_a(1) + \chi_b(1) &= \int_0^a G(1,s)g(s)\,ds + \int_b^1 G(1,s)g(s)\,ds \\ &\le \int_0^{a_2} G(1,s)g(s)\,ds + \int_{b_2}^1 G(1,s)g(s)\,ds \\ &\le \frac{\delta}{\Gamma}\int_{\widetilde{a}}^{\widetilde{b}} \omega_0(s)h(s)\,ds \\ &\le \frac{\delta}{\Gamma}\int_a^b \omega_0(s)h(s)\,ds = \int_a^b G(1,s)h(s)\,ds = \chi_{a,b}(1) \end{aligned}$$

for $0 < a \le a_2$ and $b_2 \le b < 1$, that is, $\varphi_{a,b}(1) \ge 0$ for $0 < a \le a_2$ and $b_2 \le b < 1$.

Let

$$a_0 = \begin{cases} \widetilde{a} & \text{if } \beta = 0, \delta = 0, \\ a_1 & \text{if } \beta > 0, \delta = 0, \\ a_2 & \text{if } \beta = 0, \delta > 0, \\ \min\{a_1, a_2\} & \text{if } \beta > 0, \delta > 0, \end{cases}$$

$$b_0 = \begin{cases} \widetilde{b} & \text{if } \beta = 0, \delta = 0, \\ b_1 & \text{if } \beta > 0, \delta = 0, \\ b_2 & \text{if } \beta = 0, \delta > 0, \\ \max\{b_1, b_2\} & \text{if } \beta > 0, \delta > 0. \end{cases}$$

Then $\varphi_{a,b}(0) \ge 0$ and $\varphi_{a,b}(1) \ge 0$ for $0 < a \le a_0$ and $b_0 \le b < 1$.

Step 3. $\varphi_{a,b}(t) \ge 0$ on $[0,1]$ for $0 \le a \le a_0$ and $b_0 \le b < 1$.

If there exists $t \in [0,1]$ such that $\varphi_{a,b}(t) < 0$, then let $\nu \in [0,1]$ satisfy

$$\varphi_{a,b}(\nu) = \min\bigl\{\varphi_{a,b}(t) : t \in [0,1]\bigr\} < 0.$$

Then $\nu \in (0,1)$ by Step 2 and $\varphi'_{a,b}(\nu) = 0$.

By Step 1, $\varphi'_{a,b}(t) \ge 0$ on $[0,a]$ and $\varphi'_{a,b}(t) \le 0$ on $[b,1]$ for all $0 < a \le a_0$ and $b_0 \le b < 1$. Hence, by Step 2, $\varphi_{a,b}(t) \ge 0$ on $[0,a]$ and $\varphi_{a,b}(t) \ge 0$ on $[b,1]$. This implies $\nu \in (a,b)$.

Let $\pi(t) = \int_a^t p(s)\varphi'_{a,b}(s)\,ds$ on $[a,b]$. By (2.4) we have

$$\pi''(t) = \bigl(p(t)\varphi'_{a,b}(t)\bigr)' = -h(t) \le 0 \quad \text{a.e. } (a,b),$$

and thus π' is decreasing on (a,b). This implies

$$p(t)\varphi'_{a,b}(t) = \pi'(t) \geq \pi'(\nu) = p(\nu)\varphi'_{a,b}(\nu) = 0 \quad \text{on } [a,\nu].$$

This, together with (C_3) ($p(t) > 0$ on $[0,1]$), shows that $\varphi'_{a,b}(t) \geq 0$ on $[a,\nu]$ and $\varphi_{a,b}(\nu) \geq \varphi_{a,b}(a) \geq 0$, which is a contradiction. □

Next, we define a function

$$f^*(t,y) = \begin{cases} f(t,y) & \text{if } y \geq 0, \\ f(t,0) & \text{if } y < 0. \end{cases}$$

Let $z \in C[0,1]$. We define the map A from $C[0,1]$ to $C[0,1]$ by

$$Az(t) = \int_0^1 G(t,s)f^*\bigl(s,z(s)\bigr)\,ds, \tag{2.5}$$

where $G(t,s)$ is as in (2.2).

We prove a key fact.

Theorem 2.2 *Assume that* (C_1)-(C_2) *hold. Let* $0 < a < b < 1$, $w_0 \in C[0,1]$ *with* $w_0(t) \geq 0$ *on* $[0,1]$, *and* $w_*(t) = \int_a^b G(t,s)w_0(s)\,ds$. *If* $z = \nu Az + \mu w_*$ *has a solution* $z \in C[0,1]$ *for some* $\nu > 0$ *and* $\mu \geq 0$, *then* $z(t) \geq 0$ *for* $t \in [0,1]$.

Proof Let

$$w_1(t) = \begin{cases} w_0(t) & \text{if } a \leq t \leq b, \\ 0 & \text{if } 0 \leq t < a \text{ or } b < t \leq 1. \end{cases}$$

Then $w_*(t) = \int_0^1 G(t,s)w_1(s)\,ds$ and $z(t) = \nu\int_0^1 G(t,s)[f^*(s,z(s)) + \frac{\mu}{\nu}w_1(s)]\,ds$. Let $f_0(s,z) = f^*(s,z) + \frac{\mu}{\nu}w_1(s)$. Then $f_0(s,0) \geq 0$ a.e. for $s \in [0,1]$. A very similar argument to that of Theorem 2.1(1)-(4) in [18] shows that $z(t) \geq 0$ on $[0,1]$, and the details are omitted. □

We continue with some preliminaries. Let $g_0 \in L_+[0,1]$ be such that

$$f(t,z) + g_0(t) \geq 0 \quad \text{a.e. } [0,1] \text{ and for all } z \in \mathbb{R}_+. \tag{2.6}$$

Notation

$$w(t) = \int_0^1 G(t,s)g_0(s)\,ds. \tag{2.7}$$

Let $z \in C[0,1]$ satisfy

$$z(t) = Az(t) + \mu w_*(t) \tag{2.8}$$

and

$$\alpha(t) = z(t) + w(t), \tag{2.9}$$

where A is defined by (2.5), $\mu \geq 0$, and $w_*(t)$ has the properties as in Theorem 2.2.

Let $\|\alpha\| = \max\{|\alpha(t)| : t \in [0,1]\}$. We prove other inequalities.

Lemma 2.1 *Assume that* (C_1), (C_2), *and* (C_3) *hold. Let* $\rho > 0$ *and* $\|\alpha\| > (\frac{P_0}{p_0} + 1)(\rho + \|w\|)$. *Then there exist* $a_1, b_1 \in [0,1]$ *with* $a_1 < b_1$ *such that* $z(t) \geq \rho$ *on* $[a_1, b_1]$ *and*

$$a_1 \leq \frac{P_0(\rho + \|w\|)}{p_0(\|\alpha\| - \rho - \|w\|)}, \tag{2.10}$$

$$b_1 \geq 1 - \frac{P_0(\rho + \|w\|)}{p_0(\|\alpha\| - \rho - \|w\|)}, \tag{2.11}$$

where

$$p_0 = \min\bigl\{p(t) : t \in [0,1]\bigr\}, \qquad P_0 = \max\bigl\{p(t) : t \in [0,1]\bigr\}.$$

In order to prove Lemma 2.1, we need to prove the following propositions.

Proposition 2.1 *Let* $\theta : [0,1] \to R$ *be continuous, and* $\theta'(t)$ *exist for* $t \in (0,1)$ *and be decreasing on* $(0,1)$. *Then* θ *is concave down on* $[0,1]$.

Proof Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and $\lambda \in (0,1)$. By the differential mean-value theorem and the decrease in θ' there exist $\xi_1 \in (t_1, \lambda t_1 + (1-\lambda)t_2)$ and $\xi_2 \in (\lambda t_1 + (1-\lambda)t_2, t_2)$ such that

$$\begin{aligned}
&\theta\bigl(\lambda t_1 + (1-\lambda)t_2\bigr) - \bigl[\lambda\theta(t_1) + (1-\lambda)\theta(t_2)\bigr] \\
&\quad = \lambda\bigl[\theta\bigl(\lambda t_1 + (1-\lambda)t_2\bigr) - \theta(t_1)\bigr] + (1-\lambda)\bigl[\theta\bigl(\lambda t_1 + (1-\lambda)t_2\bigr) - \theta(t_2)\bigr] \\
&\quad = \lambda(1-\lambda)\theta'(\xi_1)(t_2 - t_1) - \lambda(1-\lambda)\theta'(\xi_2)(t_2 - t_1) \\
&\quad = \lambda(1-\lambda)\bigl[\theta'(\xi_1) - \theta'(\xi_2)\bigr](t_2 - t_1) \geq 0.
\end{aligned}$$

Hence, θ is concave down on $[0,1]$. □

Let

$$\begin{aligned}
&\xi(t) = \int_0^t p(s)\alpha'(s)\,ds \quad \text{on } [0,1], \\
&\eta(t) = -\int_t^1 p(s)\alpha'(s)\,ds \quad \text{on } [0,1].
\end{aligned}$$

Proposition 2.2 *Let* (C_2) *hold, and* $\widetilde{t} \in [0,1]$ *be such that* $\alpha(\widetilde{t}) = \max\{\alpha(t), t \in [0,1]\}$. *Then the following assertions hold.*

(1) $\alpha(t) \geq 0$ *on* $[0,1]$, $\alpha(\widetilde{t}) = \|\alpha\|$, *and* $\alpha \in C^1[0,1]$.

(2) $\xi(t)$ *and* $\eta(t)$ *are concave down on* $[0,1]$.

(3) (i) *If* $\widetilde{t} < 1$, *then* $\alpha(t)$ *is decreasing on* $[\widetilde{t}, 1]$.

(ii) *If* $\widetilde{t} > 0$, *then* $\alpha(t)$ *is increasing on* $[0, \widetilde{t}]$.

(iii) *If* $0 < \widetilde{t} < 1$, *then* $\alpha(t)$ *is increasing on* $[0, \widetilde{t}]$ *and decreasing on* $[\widetilde{t}, 1]$.

Proof (1) Letting $\nu = 1$, Theorem 2.2 shows $z(t) \geq 0$ on $[0,1]$. This implies $\alpha(t) \geq 0$ on $[0,1]$, $f^* = f$, and $\alpha(\widetilde{t}) = \|\alpha\|$. The result $\alpha \in C^1[0,1]$ follows from (2.8) and (2.9).

(2) From (2.5) and (2.8) we have

$$\begin{aligned}\xi''(t) = \eta''(t) &= \big(p(t)\alpha'(t)\big)' \\ &= \big(p(t)z'(t)\big)' + \big(p(t)w'(t)\big)' \\ &= -\big(f\big(t,z(t)\big) + g_0(t)\big) + \mu\big(p(t)w_*'(t)\big)' \leq 0 \quad \text{a.e. } [0,1].\end{aligned}$$

Condition (C_1) implies $\xi'' \in L[0,1]$ and $\eta''(t) \in L[0,1]$. Hence, $\xi'(t) \in AC[0,1]$ and $\eta'(t) \in AC[0,1]$. For $0 \leq t_1 \leq t_2 \leq 1$, we have

$$\xi'(t_2) - \xi'(t_1) = \int_{t_1}^{t_2} \xi''(s)\,ds \leq 0,$$

that is, $\xi'(t)$ is decreasing on $[0,1]$. By Proposition 2.1, $\xi(t)$ is concave down on $[0,1]$.

A similar argument shows that $\eta'(t)$ is decreasing on $[0,1]$ and $\eta(t)$ is concave down on $[0,1]$.

(3) (i) If $\widetilde{t} < 1$, then

$$\alpha'(\widetilde{t}) = \lim_{t\to\widetilde{t}+} \frac{\alpha(\widetilde{t}) - \alpha(t)}{\widetilde{t} - t} \leq 0,$$

and $\eta'(\widetilde{t}) = p(\widetilde{t})\alpha'(\widetilde{t}) \leq 0$.

From the decrease of η' in t we see that $p(t)\alpha'(t) = \eta'(t) \leq \eta'(\widetilde{t}) \leq 0$ for $t > \widetilde{t}$, and by (C_3) $\alpha'(t) \leq 0$ for $t > \widetilde{t}$. This implies that $\alpha(t)$ is decreasing on $[\widetilde{t},1]$.

(ii) If $\widetilde{t} > 0$, then

$$\alpha'(\widetilde{t}) = \lim_{t\to\widetilde{t}-} \frac{\alpha(\widetilde{t}) - \alpha(t)}{\widetilde{t} - t} \geq 0,$$

and $\xi'(\widetilde{t}) = p(\widetilde{t})\alpha'(\widetilde{t}) \geq 0$.

Since ξ' is decreasing in $[0,1]$, we see that $p(t)\alpha'(t) = \xi'(t) \geq \xi'(\widetilde{t}) \geq 0$ and $\alpha'(t) \geq 0$ on $[0,\widetilde{t}]$ by (C_3). Hence, $\alpha(t)$ is increasing on $[0,\widetilde{t}]$.

(iii) The result follows from (i) and (ii). □

Proposition 2.3 (i) $p_0(\alpha(t) - \alpha(0)) \leq \xi(t) \leq P_0\alpha(t)$ *on* $[0,\widetilde{t}]$ *if* $\widetilde{t} > 0$.

(ii) $p_0(\alpha(t) - \alpha(1)) \leq \eta(t) \leq P_0\alpha(t)$ *on* $[\widetilde{t},1]$ *if* $\widetilde{t} < 1$.

Proof (i) By Proposition 2.2(3), part (ii), $\alpha'(s) \geq 0$ on $[0,\widetilde{t}]$, and, for $t \in [0,\widetilde{t}]$, we have

$$\xi(t) = \int_0^t p(s)\alpha'(s)\,ds \geq p_0 \int_0^t \alpha'(s)\,ds = p_0\big(\alpha(t) - \alpha(0)\big)$$

and

$$\xi(t) = \int_0^t p(s)\alpha'(s)\,ds \leq P_0 \int_0^t \alpha'(s)\,ds = P_0\big(\alpha(t) - \alpha(0)\big) \leq P_0\alpha(t).$$

(ii) From Proposition 2.2(3), part (i), $\alpha'(s) \leq 0$, and, for $t \in [\widetilde{t},1]$, we have

$$\eta(t) = \int_t^1 p(s)\big(-\alpha'(s)\big)\,ds \geq p_0 \int_t^1 \big(-\alpha'(s)\big)\,ds = p_0\big(\alpha(t) - \alpha(1)\big)$$

and

$$\eta(t) = \int_t^1 p(s)\bigl(-\alpha'(s)\bigr)\,ds \le P_0 \int_t^1 \bigl(-\alpha'(s)\bigr)\,ds = P_0\bigl(\alpha(t) - \alpha(1)\bigr) \le P_0\alpha(t). \qquad \square$$

Proposition 2.4 *If* $\|\alpha\| > (\frac{P_0}{p_0} + 1)(\rho + \|w\|)$ *and* $\alpha(0) \le \rho + \|w\|$, *then there exists* $t_0 \in (0, \widetilde{t})$ *such that* $\xi(t_0) = P_0(\rho + \|w\|)$ *and* $t_0 \le \frac{P_0(\rho+\|w\|)}{p_0(\|\alpha\|-\rho-\|w\|)}$.

Proof By Proposition 2.3(i) we see that $\xi(\widetilde{t}) \ge p_0(\|\alpha\| - \alpha(0))$. Noticing that $\alpha(0) \le \rho + \|w\|$, we have $\xi(\widetilde{t}) > P_0(\rho + \|w\|)$ and $\widetilde{t} > 0$. The result $\xi(t_0) = P_0(\rho + \|w\|)$ follows from $\xi(0) = 0$.

By Proposition 2.2(2), $\xi(t)$ is concave down on $[0, \widetilde{t}]$. This implies $\xi(t) \ge \frac{\xi(\widetilde{t})}{\widetilde{t}} t$ for $t \in [0, \widetilde{t}]$. Then

$$P_0\bigl(\rho + \|w\|\bigr) = \xi(t_0) \ge \frac{\xi(\widetilde{t})}{\widetilde{t}} t_0.$$

This, together with Proposition 2.3(i) and Proposition 2.2(1), implies

$$t_0 \le \frac{P_0(\rho + \|w\|)\widetilde{t}}{\xi(\widetilde{t})} \le \frac{P_0(\rho + \|w\|)}{\xi(\widetilde{t})} \le \frac{P_0(\rho + \|w\|)}{p_0(\alpha(\widetilde{t}) - \alpha(0))} \le \frac{P_0(\rho + \|w\|)}{p_0(\|\alpha\| - \rho - \|w\|)}. \qquad \square$$

Proposition 2.5 *If* $\|\alpha\| > (\frac{P_0}{p_0} + 1)(\rho + \|w\|)$ *and* $\alpha(1) \le \rho + \|w\|$, *then there exists* $t_1 \in (\widetilde{t}, 1)$ *such that* $\eta(t_1) = P_0(\rho + \|w\|)$ *and* $t_1 \ge 1 - \frac{P_0(\rho+\|w\|)}{p_0(\|\alpha\|-\rho-\|w\|)}$.

Proof From Proposition 2.3(ii) we see that $\eta(\widetilde{t}) \ge p_0(\alpha(\widetilde{t}) - \alpha(1))$. Noticing that $\alpha(1) \le \rho + \|w\|$, we have $\eta(\widetilde{t}) > P_0(\rho + \|w\|)$ and $\widetilde{t} < 1$. The result $\eta(t_1) = P_0(\rho + \|w\|)$ follows from $\eta(1) = 0$.

By Proposition 2.2(2), $\eta(t)$ is concave down on $[\widetilde{t}, 1]$. This implies $\eta(t) \ge \frac{\eta(\widetilde{t})}{1-\widetilde{t}}(1 - t)$ for $t \in [\widetilde{t}, 1]$. Then

$$P_0\bigl(\rho + \|w\|\bigr) = \eta(t_1) \ge \frac{\eta(\widetilde{t})}{1 - \widetilde{t}}(1 - t_1).$$

This, together with Proposition 2.3(ii) and Proposition 2.2(1), implies

$$1 - t_1 \le \frac{P_0(\rho + \|w\|)(1 - \widetilde{t})}{\eta(\widetilde{t})} \le \frac{P_0(\rho + \|w\|)}{\eta(\widetilde{t})} \le \frac{P_0(\rho + \|w\|)}{p_0(\alpha(\widetilde{t}) - \alpha(1))} \le \frac{P_0(\rho + \|w\|)}{p_0(\|\alpha\| - \rho - \|w\|)},$$

that is, $t_1 \ge 1 - \frac{P_0(\rho+\|w\|)}{p_0(\|\alpha\|-\rho-\|w\|)}$. $\square$

Proof of Lemma 2.1 Noticing that $\|\alpha\| > \rho + \|w\|$ and utilizing Proposition 2.2(3), we have the following fact:

(P) if $\widetilde{t} \in [a, b]$, $\alpha(a) \ge \rho + \|w\|$, and $\alpha(b) \ge \rho + \|w\|$, then $z(t) \ge \rho$ on $[a, b]$.

In fact, if $\widetilde{t} = a$, then by Proposition 2.2(3), part (i), $\eta(t)$ is decreasing on $[a, b]$. If $\widetilde{t} = b$, then Proposition 2.2(3), part (ii), implies that $\alpha(t)$ is increasing on $[a, b]$. If $a < \widetilde{t} < b$, then by Proposition 2.2(3), part (iii), $\eta(t)$ is decreasing on $[\widetilde{t}, b]$, and α is increasing on $[a, \widetilde{t}]$. Hence, $\alpha(t) \ge \rho + \|w\|$ on $[a, b]$, and

$$z(t) = \alpha(t) - w(t) \ge \rho + \bigl(\|w\| - w(t)\bigr) \ge \rho \quad \text{on } [a, b].$$

The rest is divided into four cases.

Case 1. $\alpha(0) \geq \rho + \|w\|$ and $\alpha(1) \geq \rho + \|w\|$.

The result follows from (P).

(2) $\alpha(0) \geq \rho + \|w\|$, $\alpha(1) < \rho + \|w\|$.

Since $\alpha(1) < \rho + \|w\|$, then $\widetilde{t} < 1$. Proposition 2.5 shows that there exists $t_1 \in (\widetilde{t}, 1)$ such that $\eta(t_1) = P_0(\rho + \|w\|)$ and $t_1 \geq 1 - \frac{P_0(\rho+\|w\|)}{p_0(\|\alpha\|-\rho-\|w\|)}$. By Proposition 2.3(ii), $\alpha(t_1) \geq \rho + \|w\|$. (P) implies $z(t) \geq \rho$ on $[0, t_1]$.

(3) $\alpha(0) < \rho + \|w\|$, $\alpha(1) \geq \rho + \|w\|$.

Since $\alpha(0) < \rho + \|w\|$, we have $\widetilde{t} > 0$. By Proposition 2.4, there exists $t_0 \in (\widetilde{t}, 1)$ such that $\xi(t_0) = P_0(\rho + \|w\|)$ and $t_0 \leq \frac{P_0(\rho+\|w\|)}{p_0(\|\alpha\|-\rho-\|w\|)}$. By Proposition 2.3(i), $\alpha(t_0) \geq \rho + \|w\|$. The result $z(t) \geq \rho$ on $[t_0, 1]$ follows from (P).

(4) $\alpha(0) < \rho + \|w\|$, $\alpha(1) < \rho + \|w\|$.

Since $\alpha(0) < \rho + \|w\|$ and $\alpha(1) < \rho + \|w\|$, we have $0 < \widetilde{t} < 1$. By Propositions 2.4 and 2.5 there exist $t_0 \in (0, \widetilde{t})$ and $t_1 \in (\widetilde{t}, 1)$ such that $\eta(t_1) = P_0(\rho + \|w\|) = \xi(t_0)$ and

$$\begin{aligned} t_0 &\leq \frac{P_0(\rho + \|w\|)}{p_0(\|\alpha\| - \rho - \|w\|)}, \\ t_1 &\geq 1 - \frac{P_0(\rho + \|w\|)}{p_0(\|\alpha\| - \rho - \|w\|)}. \end{aligned}$$

The inequality $z(t) \geq \rho$ on $[t_1, t_2]$ follows from (P).

Let

$$a_1 = \begin{cases} 0 & \text{if } \alpha(0) \geq \rho + \|w\| \text{ if } \alpha(1) \geq \rho + \|w\|, \\ 0 & \text{if } \alpha(0) \geq \rho + \|w\| \text{ if } \alpha(1) < \rho + \|w\|, \\ t_0 & \text{if } \alpha(0) < \rho + \|w\| \text{ if } \alpha(1) \geq \rho + \|w\|, \\ t_0 & \text{if } \alpha(0) < \rho + \|w\| \text{ if } \alpha(1) < \rho + \|w\|, \end{cases}$$

$$b_1 = \begin{cases} 1 & \text{if } \alpha(0) \geq \rho + \|w\|, \alpha(1) \geq \rho + \|w\|, \\ t_1 & \text{if } \alpha(0) \geq \rho + \|w\|, \alpha(1) < \rho + \|w\|, \\ 1 & \text{if } \alpha(0) < \rho + \|w\|, \alpha(1) \geq \rho + \|w\|, \\ t_1 & \text{if } \alpha(0) < \rho + \|w\|, \alpha(1) < \rho + \|w\|. \end{cases}$$

Then $z(t) \geq \rho$ on $[a_1, b_1]$.

Let

$$K = \bigl\{z \in C[0,1] : z(t) \geq 0 \text{ on } [0,1]\bigr\}.$$

Then K is the standard positive cone of $C[0,1]$, and K is a total cone. It defines the partial order $\leq$ of $C[0,1]$ by $x \leq y$ if and only if $y - x \in K$.

Let $g \in L_+[0,1]$ with $\int_0^1 g(s)\,ds > 0$ and $z \in C[0,1]$. We define two linear maps by

$$\begin{aligned} L_g z(t) &= \int_0^1 G(t,s)g(s)z(s)\,ds, \\ L_g^{(n)} z(t) &= \int_{\frac{1}{n}}^{1-\frac{1}{n}} G(t,s)g(s)z(s)\,ds, \end{aligned}$$

where $1/n \leq a_0$, $b_0 \leq 1 - 1/n$, $\int_{\frac{1}{n}}^{1-\frac{1}{n}} g(s)\,ds > 0$, and a_0 and b_0 are as in Theorem 2.1.

It is easy to know that L_g and $L_g^{(n)}$ are compact in $C[0,1]$ and map K into K. Let $r(L_g)$, $r(L_g^{(n)})$ denote the radiuses of the spectra of L_g and $L_g^{(n)}$, respectively. Since $0 < \int_{\frac{1}{n}}^{1-\frac{1}{n}} g(s)\,ds \leq \int_0^1 g(s)\,ds < \infty$, it is easy to verify that $0 < r(L_g^{(n)}), r(L_g) < \infty$ [22]. □

Notation

$$\mu_1(L_g) = \frac{1}{r(L_g)}, \qquad \mu_1\bigl(L_g^{(n)}\bigr) = \frac{1}{r(L_g^{(n)})}.$$

When $g \equiv 1$, $\mu_1(L_g)$ is written usually as μ_1.

It was proved by Nussbaum ([23], Lemma 2) that the radius of the spectrum is continuous, that is, if $L, L_m : X \to X$ are compact linear operators and $\lim_{m\to\infty} \|L_m - L\| = 0$, then $\lim_{m\to\infty} r(L_m) = r(L)$. We use this result to prove the following lemma.

Lemma 2.2 *For any $\varepsilon > 0$, there exists $n_0 > 0$ such that $\mu_1(L_g) + \varepsilon \geq \mu_1(L_g^{(n)})$ for $n \geq n_0$.*

Proof It is easy to verify that $\lim_{n\to\infty} \|L_g^{(n)} - L_g\| = 0$. Then $\lim_{n\to\infty} r(L_g^{(n)}) = r(L_g)$, and then $\lim_{n\to\infty} \mu_1(L_g^{(n)}) = \mu_1(L_g)$. The result follows. □

Lemma 2.3 ([7], Theorem 19.2) *Let K be a total cone in a real Banach space X, and let L be a compact linear operator with $L(K) \subseteq K$. If $r(L) > 0$, then there is $\varphi \in K \setminus \{\theta\}$ such that $L\varphi = r(L)\varphi$.*

We shall use the following known result (see, for example, [7]), which can be proved by using Leray-Schauder degree theory for compact maps in Banach spaces.

Lemma 2.4 *Let X be a real Banach space, Ω_1 and Ω_2 be two bounded open sets of X, and $\theta \in \Omega_1 \subset \Omega_2$, where θ is zero element of X. Assume that F: $\overline{\Omega_2 \setminus \Omega_1} \to X$ is compact and satisfies*

(1) *$x \neq \mu Fx$ for $x \in \partial\Omega_1$ and $0 < \mu \leq 1$.*

(2) *There exists $y_0 \in X \setminus \{\theta\}$ such that $x \neq Fx + \mu y_0$ for $x \in \partial\Omega_2$ and $\mu \geq 0$.*

Then F has a fixed point in $\Omega_2 \setminus \overline{\Omega_1}$.

3 New results of positive solutions of (1.1)-(1.2)

In this section, we utilize the inequalities established in Theorem 2.1 and Lemma 2.1 to prove new existence results of positive solutions of (1.1)-(1.2).

Theorem 3.1 *Assume that* (C_1)-(C_3) *and the following conditions hold.*

(i) *There exist $r_0 > 0$, $\phi \in L_+[0,1]$ with $\int_0^1 \phi(s)\,ds > 0$ and $\varepsilon \in (0, \mu_1(L_\phi))$ such that*

$$f(t,z) \leq \bigl(\mu_1(L_\phi) - \varepsilon\bigr)\phi(t)z \quad \text{for a.e. } t \in [0,1] \text{ and all } z \in [0, r_0]. \tag{3.1}$$

(ii) *There exist $\rho_0 > 0$, $\psi \in L_+[0,1]$ with $\int_0^1 \psi(s)\,ds > 0$ and $\varepsilon_1 > 0$ such that*

$$f(t,z) \geq \bigl(\mu_1(L_\psi) + \varepsilon_1\bigr)\psi(t)z \quad \text{for a.e. } t \in [0,1] \text{ and all } z \in [\rho_0, \infty). \tag{3.2}$$

Then (1.1)-(1.2) *has a positive solution.*

Proof By (C_1) there exists $g_{\rho_0} \in L_+[0,1]$ such that

$$\bigl|f(t,z)\bigr| \leq g_{\rho_0}(t) \quad \text{for a.e. } t \in [0,1] \text{ and all } z \in [0,\rho_0].$$

Let $g_0(t) = g_{\rho_0}(t)$. By (ii), we see that $f(t,z) + g_0(t) \geq 0$ a.e. [0,1] and all $z \in [0,\infty)$, that is, f satisfies (2.6).

Set $g(t) = g_0(t)$ and $h(t) = \frac{\varepsilon_1}{2}\rho_0\psi(t)$ in Theorem 2.1. Then there exist $0 < a_0 < b_0 < 1$ such that $\varphi_{a,b}(t) \geq 0$ on $[0,1]$ for all $0 < a \leq a_0$ and $b_0 \leq b < 1$.

By Lemma 2.2 there exists $n_0 > 0$ such that $1/n_0 \leq a_0$, $b_0 \leq 1 - 1/n_0$, $\mu_1(L_\psi) + \varepsilon_1/2 \geq \mu_1(L_\psi^{(n_0)}) > 0$, and $(\frac{n_0 P_0}{p_0} + 1)(\rho_0 + \|w\|) > r_0 + \|w\|$. From the result mentioned we see that $\varphi_{\frac{1}{n_0},1-\frac{1}{n_0}}(t) \geq 0$ on $[0,1]$.

Let $R_0 = (\frac{n_0 P_0}{p_0} + 1)(\rho_0 + \|w\|)$ and

$$\begin{aligned} \Omega_1 &= \bigl\{z \in C[0,1], \|z\| < r_0\bigr\}, \\ \Omega_2 &= \bigl\{z \in C[0,1], \|z + w\| < R_0\bigr\}. \end{aligned}$$

Then $\theta \in \Omega_1 \subset \Omega_2$, where w is as in (2.7).

Without loss of generality, we may assume that A has no fixed point in $\partial\Omega_2$ (otherwise, if A has a fixed point z in $\partial\Omega_2$, then by Theorem 2.2 we know that $z(t) \geq 0$ on $[0,1]$, $z(t) \not\equiv 0$, and $f^*(s,z(s)) = f(s,z(s))$, so that the result is already proved). The rest is divided into three steps.

Step 1. We prove that, for $z \in \partial\Omega_1$ and $0 < \mu \leq 1$,

$$z \neq \mu Az. \tag{3.3}$$

Suppose on the contrary that there exist $z \in \partial\Omega_1$ and $0 < \mu \leq 1$ such that $z = \mu Az$. Putting $w_* \equiv 0$ on $[0,1]$, Theorem 2.2 shows that $z(t) \geq 0$ and $z(t) \not\equiv 0$. This, together with (i), implies

$$\begin{aligned} z &= \mu Az = \mu \int_0^1 G(t,s) f^*\bigl(s,z(s)\bigr)\,ds \\ &= \mu \int_0^1 G(t,s) f\bigl(s,z(s)\bigr)\,ds \\ &\leq \mu\bigl(\mu_1(L_\phi) - \varepsilon\bigr) \int_0^1 G(t,s)\phi(s)z(s)\,ds \\ &\leq \bigl(\mu_1(L_\phi) - \varepsilon\bigr) \int_0^1 G(t,s)\phi(s)z(s)\,ds \\ &= \bigl(\mu_1(L_\phi) - \varepsilon\bigr) L_\phi z = Sz, \end{aligned}$$

where $S = (\mu_1(L_\phi) - \varepsilon)L_\phi$.

Since $S(K) \subseteq K$ and $r(S) < 1$, we have that $(I - S)^{-1}$ exists and is increasing [11, 18]. From the previous inequality we have $z \leq (I - S)^{-1}\theta = \theta$, which is a contradiction. Hence, (3.3) holds.

Step 2. Let $Tz = \mu_1(L_\psi^{(n_0)})L_\psi^{(n_0)}z$. Then $T(K) \subseteq K$ and $r(T) = 1$. Lemma 2.3 shows that there exists $z_* \in K \setminus \{\theta\}$ such that $Tz_* = z_*$. By direct computation we obtain

$$p(t)z_*'(t) = \frac{\mu_1(L_\psi^{(n_0)})}{\Gamma}\begin{cases} \alpha\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} \underline{w}_1(s)\psi(s)z_*(s)\,ds \\ -\gamma\int_{\frac{1}{n_0}}^{t} \underline{w}_0(s)\psi(s)z_*(s)\,ds, & 0 \le t < 1/n_0, \\ +\alpha\int_t^{1-\frac{1}{n_0}} \underline{w}_1(s)\psi(s)z_*(s)\,ds, & 1/n_0 \le t \le 1-1/n_0, \\ -\gamma\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} \underline{w}_0(s)\psi(s)z_*(s)\,ds, & 1-1/n_0 < t \le 1, \end{cases}$$

and

$$\bigl(p(t)z_*'(t)\bigr)' = \mu_1\bigl(L_\psi^{(n_0)}\bigr)\begin{cases} 0, & 0 \le t < 1/n_0 \text{ or } 1-1/n_0 < t \le 1, \\ -\psi(t)z_*(t), & 1-1/n_0 \le t \le 1. \end{cases}$$

From this, we know $p(t)z_*'(t) \in AC[0,1]$ and $(p(t)z_*'(t))' \le 0$ a.e. $[0,1]$.

We prove that, for $z \in \partial\Omega_2$ and $\mu \ge 0$,

$$z \ne Az + \mu z_*. \tag{3.4}$$

In fact, if there exist $z \in \partial\Omega_2$ (that is, $\|\alpha\| = \|z+w\| = R_0$, α in (2.9)) and $\mu \ge 0$ such that $z = Az + \mu z_*$, then $\mu > 0$ since A has no fixed point in $\partial\Omega_2$. Lemma 2.1 implies that there exist a_1 and b_1 satisfying (2.10), (2.11), and $z(t) \ge \rho_0$ on $[a_1, b_1]$.

Since $\frac{1}{n_0} = \frac{P_0(\rho_0+\|w\|)}{p_0(\|\alpha\|-\rho_0-\|w\|)}$, we have $0 < a_1 \le 1/n_0 \le a_0$, $b_0 \le 1-1/n_0 \le b_1 < 1$, and by Lemma 2.1 we get $z(t) \ge \rho_0$ on $[1/n_0, 1-1/n_0]$.

By Theorem 2.2, letting $\nu = 1$, we see that $z(t) \ge 0$ on $[0,1]$ and, by (ii),

$$\begin{aligned} z(t) &= \int_0^1 G(t,s)f\bigl(s,z(s)\bigr)\,ds + \mu z_*(t), \\ &= \int_0^{\frac{1}{n_0}} G(t,s)f\bigl(s,z(s)\bigr)\,ds + \int_{1-\frac{1}{n_0}}^1 G(t,s)f\bigl(s,z(s)\bigr)\,ds \\ &\quad + \int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)f\bigl(s,z(s)\bigr)\,ds + \mu z_*(t) \\ &\ge -\int_0^{\frac{1}{n_0}} G(t,s)g_0(s)\,ds - \int_{1-\frac{1}{n_0}}^1 G(t,s)g_0(s)\,ds \\ &\quad + \bigl(\mu_1(L_\psi) + \varepsilon_1\bigr)\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z(s)\,ds + \mu z_*(t) \\ &= -\chi_{\frac{1}{n_0}}(t) - \chi_{1-\frac{1}{n_0}}(t) + \bigl(\mu_1(L_\psi) + \varepsilon_1/2\bigr)\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z(s)\,ds \\ &\quad + \varepsilon_1/2\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z(s)\,ds + \mu z_*(t) \end{aligned}$$

and

$$\begin{aligned} z(t) &\ge -\chi_{\frac{1}{n_0}}(t) - \chi_{1-\frac{1}{n_0}}(t) + \chi_{\frac{1}{n_0},1-\frac{1}{n_0}}(t) \\ &\quad + \mu_1\bigl(L_\psi^{(n_0)}\bigr)\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z(s)\,ds + \mu z_*(t) \end{aligned}$$

$$= \varphi_{\frac{1}{n_0},1-\frac{1}{n_0}}(t) + \mu_1\bigl(L_\psi^{(n_0)}\bigr)\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z(s)\,ds + \mu z_*(t)$$

$$\geq \mu_1\bigl(L_\psi^{(n_0)}\bigr)\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z(s)\,ds + \mu z_*(t).$$

Then $z(t) \geq \mu z_*(t)$ for $t \in [0,1]$.

Let

$$\mu^* = \sup\bigl\{\sigma : z(t) \geq \sigma z_*(t), 0 \leq t \leq 1\bigr\}.$$

Then $0 < \mu \leq \mu^* < \infty$ and $z(t) \geq \mu^* z_*(t)$ for $0 \leq t \leq 1$.

On the other hand, for $t \in [0,1]$, we have

$$z(t) \geq \mu_1\bigl(L_\psi^{(n_0)}\bigr)\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z(s)\,ds + \mu z^*(t)$$

$$\geq \mu^*\mu_1\bigl(L_\psi^{(n_0)}\bigr)\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z_*(s)\,ds + \mu z_*(t)$$

$$= \mu^*\mu_1\bigl(L_\psi^{(n_0)}\bigr)\int_{\frac{1}{n_0}}^{1-\frac{1}{n_0}} G(t,s)\psi(s)z_*(s)\,ds + \mu z_*(t)$$

$$= \mu^* Tz_*(t) + \mu z_*(t) = \mu^* z_*(t) + \mu z_*(t) = \bigl(\mu^* + \mu\bigr)z_*(t).$$

From the definition of μ^* we have $\mu^* \geq \mu^* + \mu > \mu^*$, which is a contradiction. Hence, (3.4) holds.

Step 3. Condition (C_1) implies that A is compact from $C[0,1]$ to $C[0,1]$. By Lemma 2.4, A has a fixed point z in $\Omega_2 \setminus \overline{\Omega_1}$. From Theorem 2.2 we obtain $z(t) \geq 0$ on $[0,1]$ and then $f^* = f$. This, together with (2.1), implies that z is a positive solution of (1.1)-(1.2). □

Let E be a fixed subset of $[0,1]$ of measure zero, and

$$\overline{f}(z) = \sup_{t\in[0,1]\setminus E} f(t,z), \qquad \underline{f}(z) = \inf_{t\in[0,1]\setminus E} f(t,z).$$

Notation

$$f^0 = \limsup_{z\to 0+} \overline{f}(z)/z, \qquad f_\infty = \liminf_{z\to\infty} \underline{f}(z)/z.$$

Utilizing Theorem 3.1, we have the following:

Corollary 3.1 *Let* (C_1)-(C_3) *and* $f^0 < \mu_1 < f_\infty$ *hold. Then* (1.1)-(1.2) *has at least one positive solution.*

Proof By $f^0 < \mu_1$, for any $\varepsilon \in (0,\mu_1)$, there exists $r_0 > 0$ such that $f(t,z) \leq (\mu_1 - \varepsilon)z$ for $0 \leq z \leq r_0$. Since $f_\infty > \mu_1$, there exist $\varepsilon_0 > 0$ and $\rho_0 > 0$ such that $f(t,z) \geq (\mu_1 + \varepsilon_0)z$ for $z \geq \rho_0$. Let $\psi(t) = 1$ and $\phi(t) = 1$. The result follows from Theorem 3.1. □

Remark 3.1 $f^0 < \mu_1 < f_\infty$ corresponds to the superlinear condition [13]. However, [13] needed $f^0 > -\infty$, whereas we need neither the assumption $f^0 > -\infty$ nor $p \in C^1[0,1]$ in this paper. Hence, Theorem 3.1 includes the superlinear case, and Corollary 3.1 improves Theorem 1 in [13].

Example 3.1 Let $f(t,z) = t^{\frac{1}{2}}(cz - z^{1/2})$, where $c > 0$ is a constant. Then f satisfies (C_1)-(C_2). Let $\phi(t) = \psi(t) = t^{\frac{1}{2}}$ and $r_0 = \frac{1}{c^2}$. Then $f(t,z) \leq 0$ for $t \in [0,1]$, $z \in [0,r_0]$, and (i) in Theorem 3.1 holds obviously. Let $c > \mu_1(L_\psi)$, $\varepsilon_1 = \frac{c-\mu_1(L_\psi)}{2} > 0$, and $\rho_0 = \frac{1}{\varepsilon_1^2}$. Then

$$f(t,z) = t^{\frac{1}{2}}\left(cz - z^{1/2}\right) = t^{\frac{1}{2}}\left[\left(\mu_1(L_\psi) + \varepsilon_1\right)z + \left(\varepsilon_1 z - z^{1/2}\right)\right] \geq \left(\mu_1(L_\psi) + \varepsilon_1\right)\psi(t)z$$

for $t \in [0,1]$ and $z \in [\rho_0, \infty)$, and (ii) in Theorem 3.1 holds. By Theorem 3.1 problem (1.1)-(1.2) has one positive solution for any $0 < \mu_1(L_\psi) < c$.

Remark 3.2 In Example 3.1, the superlinear condition (Theorem 1(F_1) in [13]) is false since $f^0 = -\infty$, f does not satisfy the strict conditions as in [13–17], $\lim_{z\to\infty} \min_{a\leq t\leq b} f(t, z)/z = a^{\frac{1}{2}}c < \infty$ [8], and $\int_a^b \liminf_{z\to\infty} f(t,z)/z\,dt = \frac{2}{3}(b^{\frac{3}{2}} - a^{\frac{3}{2}})c < \infty$ [12] for all $0 < a < b < 1$, p is not required to belong to $C^1[0,1]$ [10, 11, 13, 16, 17, 20]. Hence, the existing results can be not utilized to treat Example 3.1. So the results obtained in this paper fill in the gap in the study of problem (1.1)-(1.2).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to the main results. GC drafted the manuscript. HB improved the final version. All authors read and approved the final manuscript.

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