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Existence results for impulsive nonlinear fractional differential equation with mixed boundary conditions

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Abstract

In this paper, the existence and uniqueness of solutions for an impulsive mixed boundary value problem of nonlinear differential equations of fractional order are obtained. Our results are based on some fixed point theorems. Some examples are also presented to illustrate the main results.

MSC: 34B15; 34A08

Keywords: fractional differential equations; impulse; mixed boundary value problem; fixed point theorem

1 Introduction

Recently, boundary value problems of nonlinear fractional differential equations have been addressed by several researchers. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, control theory, biology, economics, blood flow phenomena, signal and image processing, biophysics, aerodynamics, fitting of experimental data, *etc.* For details, see [1–13] and the references therein.

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for a better understanding of several real world problems in the applied sciences. Recently, the boundary value problems of impulsive differential equations of integer order have been studied extensively in the literature (see [1, 3–7, 9–13]). In [4, 13], Wang *et al.* gave a new concept of some impulsive differential equations with fractional derivative, which is a correction of that of piecewise continuous solutions used in [3, 7, 10–12].

This paper is strongly motivated by the above research papers. We investigate the existence and uniqueness of solutions for a mixed boundary value problem of nonlinear impulsive differential equations of fractional order given by

$$\begin{cases} {}^{C}D_{0^+}^{q}u(t) = f(t,u(t)), \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1,2,\dots,p, \\ u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \end{cases} \tag{1.1}$$

where ${}^{C}D_{0^+}^{q}$ is the Caputo fractional derivative of order $q \in (1,2)$, $f \in C(J \times R, R)$. $I_k, J_k \in C(R,R)$, $J = [0,1]$, $J' = J \setminus \{t_1, t_2, \ldots, t_p\}$, the $\{t_k\}$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, $p \in N$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$.

A function $u \in PC(J,R)$ is said to be a solution of problem (1.1) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k \in C([0, t_{k+1}], R)$ satisfies ${}^{C}D_{0^+}^{q} u(t) = f(t, u(t))$ a.e. on $(0, t_{k+1})$ with the restriction that $u_k(t)$ on $[0, t_k)$ is just $u_{k-1}(t)$ and the conditions $\Delta u(t_k) = I_k(u(t_k))$, $\Delta u'(t_k) = J_k(u(t_k))$, $k = 1, 2, \ldots, p$ with $u(0) + u'(0) = 0$, $u(1) + u'(1) = 0$.

The rest of this paper is organized as follows. In Section 2, we give some notations, recall some concepts and preparation results. In Section 3, we give the main results, the first result based on Banach contraction principle, the second result based on Krasnoselskii's fixed point theorem. Two examples are given in Section 4 to demonstrate the application of our main results.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Let $J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{p-1} = (t_{p-1}, t_p], J_p = (t_p, 1]$. We have

$$\begin{aligned} PC(J) = \bigl\{ & u : [0,1] \to R \mid u \in C(J'), \text{ and } u(t_k^+), u(t_k^-) \text{ exist, and} \\ & u(t_k^-) = u(t_k), 1 \le k \le p \bigr\}. \end{aligned}$$

Obviously, $PC(J)$ is a Banach space with the norm

$$\|u\|_{PC} = \sup_{0 \le t \le 1} |u(t)|.$$

Definition 2.1 The fractional integral of order q of a function $f : [0, \infty) \to R$ is defined as

$$I_{0+}^{q} f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} \, ds, \quad t > 0, q > 0, \tag{2.1}$$

provided the right side is point-wise defined on $(0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Caputo derivative of fractional order q for a function $f : [0, \infty) \to R$ is defined as

$${}^{C}D_{0^+}^{q} f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(s) - \sum_{k=0}^{n-1} \frac{s^k}{k!} f^{(k)}(0)}{(t-s)^{q-n+1}} \, ds, \quad t > 0, n = -[-q], \tag{2.2}$$

where $[q]$ denotes the integer part of the real number q.

Remark 2.1 In the case $f(t) \in C^n[0, +\infty)$, there is ${}^{C}D_{0^+}^{q} f(t) = I_{0+}^{n-q} f^{(n)}(t)$. That is to say that Definition 2.2 is just the usual Caputo's fractional derivative. In this paper, we consider an impulsive problem, so Definition 2.2 is appropriate.

Lemma 2.1 ([13]) *Let M be a closed, convex, and nonempty subset of a Banach space X, and A, B the operators such that*

(1) *$Ax + By \in M$ whenever $x, y \in M$;*

(2) *A is compact and continuous*;
(3) *B is a contraction mapping*.
Then there exists $z \in M$ *such that* $z = Az + Bz$.

Lemma 2.2 ([2]) *The set* $F \subset PC([0,T], R^n)$ *is relatively compact if and only if*:
(i) *F is bounded, that is,* $\|x\| \leq C$ *for each* $x \in F$ *and some* $C > 0$;
(ii) *F is quasi-equicontinuous in* $[0,T]$. *That is to say that for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $x \in F$; $k \in N$; $\tau_1, \tau_2 \in (t_{k-1}, t_k]$, *and* $|\tau_1 - \tau_2| < \delta$, *we have* $|x(\tau_1) - x(\tau_2)| < \epsilon$.

Lemma 2.3 ([13]) *For* $q > 0$, *the general solution of the fractional differential equation* ${}^C D_{0^+}^q u(t) = 0$ *is given by*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in R$, $i = 0, 1, 2, \ldots, n-1$, $n = -[-q]$.

In view of Lemma 2.3, it follows that

$$I_{0^+}^q \bigl({}^C D_{0^+}^q u\bigr)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in R$, $i = 0, 1, 2, \ldots, n-1$, $n = -[-q]$.

Lemma 2.4 *Let* $q \in (1,2)$ *and* $h : J \to R$ *be continuous. A function u given by*

$$u(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds + \frac{1-t}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds \\ \quad + \frac{1-t}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\,ds, \quad t \in [0, t_1]; \\ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds + \frac{1-t}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds \\ \quad + \frac{1-t}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\,ds + (2-t) \sum_{j=1}^p J_j(u(t_j))(1-t_j) \\ \quad + (2-t) \sum_{j=1}^p I_j(u(t_j)) - (t - t_j) \sum_{j=k+1}^p J_j(u(t_j)) - \sum_{j=k+1}^p I_j(u(t_j)), \\ \quad t \in (t_k, t_{k+1}], k = 1, 2, \ldots, p-1; \\ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds + \frac{1-t}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds \\ \quad + \frac{1-t}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\,ds + (2-t) \sum_{j=1}^p J_j(u(t_j))(1-t_j) \\ \quad + (2-t) \sum_{j=1}^p I_j(u(t_j)), \quad t \in (t_p, t_{p+1}], \end{cases} \tag{2.3}$$

is a unique solution of the following impulsive problem:

$$\begin{cases} {}^C D_{0^+}^q u(t) = h(t), \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1, 2, \ldots, p, \\ u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0. \end{cases} \tag{2.4}$$

Proof With Lemma 2.3, a general solution u of the equation ${}^C D_{0^+}^q u(t) = h(t)$ on each interval $(t_k, t_{k+1}]$ $(k = 0, 1, 2, \ldots, p)$ is given by

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds + a_k + b_k t, \quad \text{for } t \in (t_k, t_{k+1}], \tag{2.5}$$

where $t_0 = 0$ and $t_{p+1} = 1$. Then we have

$$u'(t) = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} h(s)\,ds + b_k, \quad \text{for } t \in (t_k, t_{k+1}]. \tag{2.6}$$

We have

$$\begin{aligned} &u(0) = a_0, \qquad u'(0) = b_0, \\ &u(1) = \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds + a_p + b_p, \\ &u'(1) = \frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\,ds + b_p. \end{aligned}$$

So applying the boundary conditions (2.4), we have

$$a_0 + b_0 = 0, \tag{2.7}$$

$$\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds + \frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\,ds + a_p + 2b_p = 0. \tag{2.8}$$

Furthermore, using the impulsive condition $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-) = J_k(u(t_k))$, we derive

$$b_k = b_{k-1} + J_k\big(u(t_k)\big), \tag{2.9}$$

$$b_k = b_p - \sum_{j=k+1}^{p} J_j\big(u(t_j)\big) \quad (k = 1, 2, \ldots, p-1). \tag{2.10}$$

In the same way, using the impulsive condition $\Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k))$, we derive

$$a_k + b_k t_k = a_{k-1} + b_{k-1} t_k + I_k\big(u(t_k)\big), \tag{2.11}$$

which by (2.9) implies that

$$a_k = a_{k-1} - J_k\big(u(t_k)\big) t_k + I_k\big(u(t_k)\big). \tag{2.12}$$

Thus

$$a_k = a_p + \sum_{j=k+1}^{p} J_j\big(u(t_j)\big) t_j - \sum_{j=k+1}^{p} I_j\big(u(t_j)\big) \quad (k = 0, 1, 2, \ldots, p-1). \tag{2.13}$$

Combining (2.7), (2.8), (2.10) with (2.13) yields

$$\begin{aligned} a_p = {}& \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds + \frac{1}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\,ds \\ & - 2 \sum_{j=1}^{p} J_j\big(u(t_j)\big)(t_j - 1) + 2 \sum_{j=1}^{p} I_j\big(u(t_j)\big), \end{aligned} \tag{2.14}$$

$$
\begin{aligned}
b_p = {} & -\frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)\,ds - \frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}h(s)\,ds \\
& + \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)(t_j-1) - \sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr).
\end{aligned} \tag{2.15}
$$

Furthermore, by (2.10), (2.13), (2.14), (2.15) we have

$$
\begin{aligned}
a_k = {} & \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)\,ds + \frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}h(s)\,ds \\
& - 2\sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)(t_j-1) + 2\sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr) \\
& + \sum_{j=k+1}^{p} J_j\bigl(u(t_j)\bigr)t_j - \sum_{j=k+1}^{p} I_j\bigl(u(t_j)\bigr) \quad (k = 0,1,2,\dots,p-1),
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
b_k = {} & -\frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)\,ds - \frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}h(s)\,ds \\
& + \sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)(t_j-1) - \sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr) \\
& - \sum_{j=k+1}^{p} J_j\bigl(u(t_j)\bigr) \quad (k = 0,1,2,\dots,p-1).
\end{aligned} \tag{2.17}
$$

Hence for $k = 0,1,2,\dots,p-1$, (2.16) and (2.17) imply

$$
\begin{aligned}
a_k + b_k t = {} & \frac{1-t}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)\,ds + \frac{1-t}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}h(s)\,ds \\
& + (2-t)\sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)(1-t_j) + (2-t)\sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr) \\
& - (t-t_j)\sum_{j=k+1}^{p} J_j\bigl(u(t_j)\bigr) - \sum_{j=k+1}^{p} I_j\bigl(u(t_j)\bigr).
\end{aligned} \tag{2.18}
$$

For $k = p$, (2.14) and (2.15) imply

$$
\begin{aligned}
a_k + b_k t = {} & \frac{1-t}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)\,ds + \frac{1-t}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}h(s)\,ds \\
& + (2-t)\sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)(1-t_j) + (2-t)\sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr).
\end{aligned} \tag{2.19}
$$

Now it is clear that (2.5), (2.18), (2.19) imply that (2.3) holds.

Conversely, assume that u satisfies (2.3). By a direct computation, it follows that the solution given by (2.3) satisfies (2.4). □

3 Main results

This section deals with the existence and uniqueness of solutions to problem (1.1).

Theorem 3.1 *Let $f : J \times R \to R$ be a continuous function. Suppose there exist positive constants L_1, L_2, L_3, M_2, M_3 such that*

(A1) $|f(t,x) - f(t,y)| \leq L_1|x-y|$, *for all* $t \in J$, $x, y \in R$;

(A2) $|I_k(x) - I_k(y)| \leq L_2|x-y|$, $|J_k(x) - J_k(y)| \leq L_3|x-y|$, $|I_k(x)| \leq M_2$, $|J_k(x)| \leq M_3$, $x, y \in R$, $k = 1, 2, \ldots, p$,

with

$$L_1 \leq \frac{\Gamma(q+1)}{2(2+q)}, \qquad L_1\left[\frac{2}{\Gamma(q+1)} + \frac{1}{\Gamma(q)}\right] + 3p(L_2 + L_3) < 1.$$

Then problem (1.1) *has a unique solution on J.*

Proof Define an operator $T : PC(J) \to PC(J)$

$$\begin{aligned}
(Tu)(t) :={}& \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds + \frac{1-t}{\Gamma(q)}\int_0^1 (1-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds \\
&+ \frac{1-t}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2} f\bigl(s,u(s)\bigr)\,ds + (2-t)\sum_{j=1}^{p} J_j\bigl(u(t_j)\bigr)(1-t_j) \\
&+ (2-t)\sum_{j=1}^{p} I_j\bigl(u(t_j)\bigr) - (t-t_j)\sum_{j=k+1}^{p} J_j\bigl(u(t_j)\bigr) - \sum_{j=k+1}^{p} I_j\bigl(u(t_j)\bigr), \\
& t \in (t_k, t_{k+1}], k = 0, 1, 2, \ldots, p.
\end{aligned}$$

Let $\sup_{t\in J} |f(t,0)| = M$, and $B_r = \{u \in PC(J,R) \mid \|u\|_{PC} \leq r\}$, where

$$r \geq 2\left[\frac{2+q}{\Gamma(q+1)}M + 3p(M_2 + M_3)\right].$$

Step 1. We show that $TB_r \subset B_r$.

For $u \in B_r$, $t \in J$, we have

$$\begin{aligned}
&\bigl|(Tu)(t)\bigr| \\
&\quad\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,ds + \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,ds \\
&\qquad+ \frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}\bigl|f\bigl(s,u(s)\bigr)\bigr|\,ds + 2\sum_{j=1}^{p}\bigl|J_j\bigl(u(t_j)\bigr)\bigr| \\
&\qquad+ 2\sum_{j=1}^{p}\bigl|I_j\bigl(u(t_j)\bigr)\bigr| + \sum_{j=k+1}^{p}\bigl|J_j\bigl(u(t_j)\bigr)\bigr| + \sum_{j=k+1}^{p}\bigl|I_j\bigl(u(t_j)\bigr)\bigr| \\
&\quad\leq \frac{1}{\Gamma(q)}\left[\int_0^t (t-s)^{q-1}\bigl|f\bigl(s,u(s)\bigr) - f(s,0)\bigr|\,ds + \int_0^t (t-s)^{q-1}\bigl|f(s,0)\bigr|\,ds\right] \\
&\qquad+ \frac{1}{\Gamma(q)}\left[\int_0^1 (1-s)^{q-1}\bigl|f\bigl(s,u(s)\bigr) - f(s,0)\bigr|\,ds + \int_0^1 (1-s)^{q-1}\bigl|f(s,0)\bigr|\,ds\right] \\
&\qquad+ \frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2}\bigl|f\bigl(s,u(s)\bigr) - f(s,0)\bigr|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}\bigl|f(s,0)\bigr|\,ds+2\sum_{j=1}^{p}\bigl|J_j\bigl(u(t_j)\bigr)\bigr| \\
&+2\sum_{j=1}^{p}\bigl|I_j\bigl(u(t_j)\bigr)\bigr|+\sum_{j=k+1}^{p}\bigl|J_j\bigl(u(t_j)\bigr)\bigr|+\sum_{j=k+1}^{p}\bigl|I_j\bigl(u(t_j)\bigr)\bigr| \\
&\leq\frac{L_1 r}{\Gamma(q+1)}+\frac{M}{\Gamma(q+1)}+\frac{L_1 r}{\Gamma(q+1)}+\frac{M}{\Gamma(q+1)}+\frac{L_1 r}{\Gamma(q)}+\frac{M}{\Gamma(q)} \\
&+2pM_3+2pM_2+pM_3+pM_2 \\
&=L_1\frac{2+p}{\Gamma(q+1)}r+\frac{2+p}{\Gamma(q+1)}M+3p(M_2+M_3).
\end{aligned}
$$

Since

$$
L_1\leq\frac{\Gamma(q+1)}{2(2+q)},\qquad r\geq 2\biggl[\frac{2+q}{\Gamma(q+1)}M+3p(M_2+M_3)\biggr],
$$

we have

$$
\bigl|(Tu)(t)\bigr|\leq r,\quad TB_r\subset B_r.
$$

Step 2. T is a contraction mapping.

For $x,y\in B_r$ and $t\in J$, we have

$$
\begin{aligned}
&\bigl|(Tx)(t)-(Ty)(t)\bigr| \\
&=\Biggl|\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f\bigl(s,x(s)\bigr)\,ds+\frac{1-t}{\Gamma(q)}\int_0^1(1-s)^{q-1}f\bigl(s,x(s)\bigr)\,ds \\
&+\frac{1-t}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}f\bigl(s,x(s)\bigr)\,ds+(2-t)\sum_{j=1}^{p}J_j\bigl(x(t_j)\bigr)(t_j-1) \\
&+(2-t)\sum_{j=1}^{p}I_j\bigl(x(t_j)\bigr)-(t-t_j)\sum_{j=k+1}^{p}J_j\bigl(x(t_j)\bigr)-\sum_{j=k+1}^{p}I_j\bigl(x(t_j)\bigr) \\
&-\Biggl[\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f\bigl(s,y(s)\bigr)\,ds+\frac{1-t}{\Gamma(q)}\int_0^1(1-s)^{q-1}f\bigl(s,y(s)\bigr)\,ds \\
&+\frac{1-t}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}f\bigl(s,y(s)\bigr)\,ds+(2-t)\sum_{j=1}^{p}J_j\bigl(y(t_j)\bigr)(t_j-1) \\
&+(2-t)\sum_{j=1}^{p}I_j\bigl(y(t_j)\bigr)-(t-t_j)\sum_{j=k+1}^{p}J_j\bigl(y(t_j)\bigr)-\sum_{j=k+1}^{p}I_j\bigl(y(t_j)\bigr)\Biggr]\Biggr| \\
&\leq\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&+\frac{1}{\Gamma(q)}\int_0^1(1-s)^{q-1}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\,ds \\
&+\frac{1}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{j=1}^{p}\bigl|J_j\bigl(x(t_j)\bigr)-J_j\bigl(y(t_j)\bigr)\bigr|+2\sum_{j=1}^{p}\bigl|I_j\bigl(x(t_j)\bigr)-I_j\bigl(y(t_j)\bigr)\bigr| \\
&+\sum_{j=k+1}^{p}\bigl|J_j\bigl(x(t_j)\bigr)-J_j\bigl(y(t_j)\bigr)\bigr|+\sum_{j=k+1}^{p}\bigl|I_j\bigl(x(t_j)\bigr)-I_j\bigl(y(t_j)\bigr)\bigr| \\
\le{}& \frac{L_1}{\Gamma(q+1)}\|x-y\|_{PC}+\frac{L_1}{\Gamma(q+1)}\|x-y\|_{PC}+\frac{L_1}{\Gamma(q)}\|x-y\|_{PC} \\
&+2\sum_{j=1}^{p}L_3\|x-y\|_{PC}+2\sum_{j=1}^{p}L_2\|x-y\|_{PC} \\
&+\sum_{j=k+1}^{p}L_3\|x-y\|_{PC}+\sum_{k=j+1}^{p}L_2\|x-y\|_{PC} \\
\le{}& \frac{2L_1}{\Gamma(q+1)}\|x-y\|_{PC}+\frac{L_1}{\Gamma(q)}\|x-y\|_{PC}+2pL_3\|x-y\|_{PC} \\
&+2pL_2\|x-y\|_{PC}+pL_3\|x-y\|_{PC}+pL_2\|x-y\|_{PC} \\
={}& \left[L_1\left(\frac{2}{\Gamma(q+1)}+\frac{1}{\Gamma(q)}\right)+3p(L_2+L_3)\right]\|x-y\|_{PC}.
\end{aligned}
$$

Since

$$L_1\left(\frac{2}{\Gamma(q+1)}+\frac{1}{\Gamma(q)}\right)+3p(L_2+L_3)<1,$$

T is a contraction mapping. Thus, the conclusion follows by the contraction mapping principle. □

Theorem 3.2 *Assume that* $|f(t,u)|\le\mu(t)$ *for all* $(t,u)\in J\times R$ *where* $\mu\in L^{1/\sigma}(J,R)$ *and* $\sigma\in(0,q-1)$, *furthermore, there exist positive constants* L_2, L_3, M_2, M_3 *such that* $|I_k(x)-I_k(y)|\le L_2|x-y|$, $|J_k(x)-J_k(y)|\le L_3|x-y|$, $|I_k(x)|\le M_2$, $|J_k(x)|\le M_3$, $x,y\in R$, $k=1,2,\dots,p$, *with* $3p(L_2+L_3)<1$. *Then problem* (1.1) *has at least one solution on* J.

Proof Choose

$$r\ge\|\mu\|_{L^{\frac{1}{\sigma}}(J)}\left[\frac{2}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}}+\frac{1}{\Gamma(q-1)(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}}\right]+p(M_2+3M_3)$$

and denote

$$B_r=\bigl\{u\in PC(J,R)\mid\|u\|_{PC}\le r\bigr\}.$$

Define the operators P and Q on B_r as

$$
\begin{aligned}
(Pu)(t)={}&\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f\bigl(s,u(s)\bigr)\,ds+\frac{1-t}{\Gamma(q)}\int_0^1(1-s)^{q-1}f\bigl(s,u(s)\bigr)\,ds \\
&+\frac{1-t}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}f\bigl(s,u(s)\bigr)\,ds,
\end{aligned}
$$

$$\begin{aligned}(Qu)(t) &= (2-t)\sum_{k=1}^{p} J_k\bigl(u(t_k)\bigr)(1-t_k) + (2-t)\sum_{k=1}^{p} I_k\bigl(u(t_k)\bigr)\\ &\quad - (t-t_k)\sum_{k=j+1}^{p} J_k\bigl(u(t_k)\bigr) - \sum_{k=j+1}^{p} I_k\bigl(u(t_k)\bigr).\end{aligned}$$

For any $u, v \in B_r$ and $t \in J$, using the condition that $|f(t,u)| \le \mu(t)$ and the Hölder inequality,

$$\begin{aligned}&\int_0^t \bigl|(t-s)^{q-1} f\bigl(s,u(s)\bigr)\bigr|\,ds\\ &\quad\le \left(\int_0^t (t-s)^{\frac{q-1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^t \bigl(\mu(s)\bigr)^{\frac{1}{\sigma}}\,ds\right)^{\sigma} \le \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{(\frac{q-\sigma}{1-\sigma})^{1-\sigma}},\\ &\int_0^t \bigl|(1-s)^{q-1} f\bigl(s,u(s)\bigr)\bigr|\,ds\\ &\quad\le \left(\int_0^t (1-s)^{\frac{q-1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^t \bigl(\mu(s)\bigr)^{\frac{1}{\sigma}}\,ds\right)^{\sigma} \le \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{(\frac{q-\sigma}{1-\sigma})^{1-\sigma}},\\ &\int_0^t \bigl|(1-s)^{q-2} f\bigl(s,u(s)\bigr)\bigr|\,ds\\ &\quad\le \left(\int_0^t (1-s)^{\frac{q-2}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^t \bigl(\mu(s)\bigr)^{\frac{1}{\sigma}}\,ds\right)^{\sigma} \le \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|Pu+Qv\|_{PC}\\ &\quad\le \frac{2\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} + \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q-1)(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}}\\ &\qquad + 2pM_3 + 2pM_2 + pM_3 + pM_2\\ &\quad= \|\mu\|_{L^{\frac{1}{\sigma}}(J)}\left(\frac{2}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} + \frac{1}{\Gamma(q-1)(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}}\right) + 3p(M_2+M_3).\end{aligned}$$

Thus $Pu + Qv \in B_r$. It is obvious that Q is a contraction mapping (the proof is just similar to Theorem 3.1). On the other hand, the continuity of f implies that the operator P is continuous. Also, P is uniformly bounded on B_r since

$$\|Pu\|_{PC} \le \frac{2\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q)(\frac{q-\sigma}{1-\sigma})^{1-\sigma}} + \frac{\|\mu\|_{L^{\frac{1}{\sigma}}(J)}}{\Gamma(q-1)(\frac{q-\sigma-1}{1-\sigma})^{1-\sigma}} \le r.$$

Now we prove the quasi-equicontinuity of the operator P.

Let $\Omega = J \times B_r$, $f_{\max} = \sup_{(t,u)\in\Omega} |f(t,u)|$. For any $t_k < \tau_2 < \tau_1 \le t_{k+1}$, we have

$$\begin{aligned}&\bigl|(Pu)(\tau_2) - (Pu)(\tau_1)\bigr|\\ &\quad= \left|\frac{1}{\Gamma(q)}\int_0^{\tau_2} (\tau_2-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds + \frac{1-\tau_2}{\Gamma(q)}\int_0^1 (1-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\tau_2}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}f\big(s,u(s)\big)\,ds-\frac{1}{\Gamma(q)}\int_0^{\tau_1}(\tau_1-s)^{q-1}f\big(s,u(s)\big)\,ds\\
&-\frac{1-\tau_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}f\big(s,u(s)\big)\,ds-\frac{1-\tau_1}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}f\big(s,u(s)\big)\,ds\bigg|\\
\le{}&\frac{f_{\max}}{\Gamma(q)}\bigg|\int_0^{\tau_2}\big[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}\big]\,ds+\int_{\tau_2}^{\tau_1}(\tau_1-s)^{q-1}\,ds\bigg|\\
&+\bigg|\frac{(\tau_1-\tau_2)f_{\max}}{\Gamma(q)}\int_0^1(1-s)^{q-1}\,ds\bigg|+\bigg|\frac{(\tau_1-\tau_2)f_{\max}}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}\,ds\bigg|\\
\le{}& f_{\max}\bigg[\frac{2(\tau_1-\tau_2)^q+\tau_1^q-\tau_2^q+\tau_1-\tau_2}{\Gamma(q+1)}+\frac{\tau_1-\tau_2}{\Gamma(q)}\bigg],
\end{aligned}
$$

which tends to zero as $\tau_2\to\tau_1$. This shows that P is quasi-equicontinuous on the interval $(t_k,t_{k+1}]$. It is obvious that P is compact by Lemma 2.2, so P is relatively compact on B_r.

Thus all the assumptions of Lemma 2.1 are satisfied and problem (1.1) has at least one solution on J. □

4 Example

Example 4.1 Consider the following impulsive fractional boundary value problem:

$$
\begin{cases}
{}^CD_{0+}^{\frac{3}{2}}u(t)=\frac{1}{(t+3)^2}\frac{\sin^5 u(t)}{1+u^4(t)}, & t\in[0,1], t\neq\frac{1}{4},\\
\Delta u(\frac{1}{4})=\frac{|u(\frac{1}{4})|}{15+|u(\frac{1}{4})|}, & \Delta u'(\frac{1}{4})=\frac{|u(\frac{1}{4})|}{17+|u(\frac{1}{4})|}\\
u(0)+u'(0)=0, & u(1)+u'(1)=0.
\end{cases}
\tag{4.1}
$$

Obviously, $L_1=1/9$, $L_2=1/15$, $L_3=1/17$, $M_2=1/15$, $M_3=1/17$, $p=1$,

$$
\begin{aligned}
&\frac{\Gamma(q+1)}{2(2+q)}=\frac{3\sqrt{\pi}}{28},\qquad L_1<\frac{\Gamma(q+1)}{2(2+q)},\\
&L_1\bigg(\frac{2}{\Gamma(q+1)}+\frac{1}{\Gamma(q)}\bigg)+3p(L_2+L_3)=\frac{14}{27\sqrt{\pi}}+\frac{96}{255}<1.
\end{aligned}
$$

Thus, all the assumptions in Theorem 3.1 are satisfied. Hence, the impulsive fractional boundary value problem (4.1) has a unique solution on $[0,1]$.

Example 4.2 Consider the following impulsive fractional boundary value problem:

$$
\begin{cases}
{}^CD_{0+}^{\frac{3}{2}}u(t)=\frac{e^t}{(t+1)^2}\frac{|u(t)|}{1+|u(t)|}, & t\in[0,1], t\neq\frac{1}{4},\\
\Delta u(\frac{1}{4})=\frac{3+2|u(\frac{1}{4})|}{1+|u(\frac{1}{4})|}, & \Delta u'(\frac{1}{4})=\frac{5|u(\frac{1}{4})|}{1+|u(\frac{1}{4})|}\\
u(0)+u'(0)=0, & u(1)+u'(1)=0.
\end{cases}
\tag{4.2}
$$

Set

$$
f(t,u)=\frac{e^t}{(t+1)^2}\frac{|u|}{1+|u|},\quad (t,u)\in[0,1]\times[0,\infty).
$$

Obviously,

$$
\big|f(t,u)\big|\le\frac{e^t}{(t+1)^2}.
$$

Set

$$L_2 = L_3 = 1, \qquad M_2 = 3, \qquad M_3 = 5 \quad \text{and} \quad \mu(t) = \frac{e^t}{(t+1)^2} \in L^4\bigl([0,1],R\bigr).$$

Thus, all the assumptions in Theorem 3.2 are satisfied. Hence, the impulsive fractional boundary value problem (4.2) has at least one solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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