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The existence of almost periodic solution: via coincidence degree theory

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Abstract

In the present paper, a new method is developed to study the existence of an almost periodic solution for the ordinary or functional differential equations. The approaches are based on topological degree and novel estimation techniques for the *a priori* bounds of unknown solutions for $Lx = \lambda Nx$. To investigate the existence of an almost periodic solution, a few good methods have been presented in the previous literature (such as using the Lyapunov function, averaging, exponential dichotomy, stability, separate conditions, and so on). But topological degree theory was never employed to study the almost periodic differential equations. Though Mawhin's coincidence degree is employed to study the existence of periodic differential equations extensively, it cannot be applied to study the almost periodic systems immediately. Some essentially new and interesting lemmas should be proved before applying topological degree theory to almost periodic systems. To the best knowledge of the authors, it is the first time that topological degree theory is employed to study the existence of *almost periodic* solution and this method can be seen as a good supplement to the known methods. Therefore, it will be of great significance to study the almost periodic systems by using this method. The approach followed in the paper could be further generalized to investigate the existence of almost periodic oscillatory in some other nonlinear dynamical systems. It is believed that it can be applied to image patterns, digital image processing, data processing, signal sparse decomposition and information technology, etc.

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1 Introduction

The existence of almost periodic solutions of ordinary differential equations has been discussed extensively in theory and in practice (for example, see [1–48] and the references cited therein). In particular, many useful methods are developed to study the almost periodic differential equations in the classical references such as Hale [1–4], Fink [5–7], Yoshizawa [8, 9], Hino *et al.* [10], Seifert [11–14], Copple [15], Kato [16], Sell [17, 18], He [19], Favard [20], Bohr and Neugebauer [21], and Lakshmikantham and Leela [22]. We summarize their methods to study the existence of an almost periodic solution for the differential equations as six big categories:

(I) By using the semi-separated condition or the separated condition (considering the hull system), including the famous theorem (each hull system has a unique solution in S ,

then these solutions are all almost periodic, see He [19], Theorem 32, and Fink [5], Theorem 10.1).

(II) By using the Lyapunov function.

(III) By using the stability, including (weakly) quasi-uniformly asymptotic stability, total stability, stability under the perturbation of the hull (see Fink [5], Yoshizawa [8], and Kato [16]), relatively weakly uniformly asymptotic stability, and relatively total stability (see Hino *et al.* [10]).

(IV) By using the relations between asymptotically almost periodic function and the almost periodic functions.

(V) By combining the exponential dichotomy and fixed point theory including the Banach fixed point, Schauder fixed point, Lerry-Schauder fixed point approaches, and so on.

(VI) By the average method.

(VII) By using the smallest solution with respect to the norm (see Favard [20]).

(VIII) By the comparison method (see Fink [5] and Lakshmikantham and Leela [22]).

These methods have great significance in the study of the almost periodic differential equations and thus have many application in the specific systems arising from biology, neural networks, physics, chemistry, engineering, and so on (one can refer to [23–48]).

Though so many good methods were developed and applied to study the almost periodic equations, there is no paper studying the existence of *almost periodic solutions* by using topological degree theory. It is well known that topological degree theory is a powerful tool to study the *periodic* differential equations. It is usually used to study the existence of periodic solutions for the boundary value problem, the initial value problem, the two-point boundary value problem and so on (for examples, one can refer to [41, 47, 48] and references cited therein). Since topological degree theory plays such a great role in the periodic differential equations, it is natural to ask the question:

Can topological degree theory be employed to study the almost periodic differential equations? If so, how should one use topological degree theory to study the almost periodic systems?

Therefore, the present paper is devoted to giving an affirmative reply to the question. Topological degree theory can be applied to study the almost periodic nonlinear systems including the ordinary differential equations and functional differential equations. However, the method used to study the periodic systems cannot be applied to an almost periodic system directly. The method used in previous work (*e.g.* [41]) cannot be applied to the almost periodic system directly. Therefore, many definitions should be modified and many indispensable and essentially new lemmas should be proved to suit for the almost periodic systems.

2 Preliminary

In this section, in order to obtain the existence of almost periodic solutions of the differential equations, we shall make some preparation. For convenience, we first summarize in the following a few concepts and prove some preliminary results on almost periodic functions and topological degree theory that will be basic for the next section.

Definition 2.1 $x(t) : R \rightarrow R^n$, which is continuous in t and $x(t)$ is said to almost periodic on R , if, for any $\varepsilon > 0$, it is possible to find a real number $l(\varepsilon) > 0$ such that in any interval

of length $l(\varepsilon)$ there exists $\tau = \tau(\varepsilon)$ such that the inequality

$$\|f(t + \tau) - f(t)\| < \varepsilon$$

is satisfied for all $t \in (-\infty, +\infty)$. The number τ is called a ε -translation number of $f(t)$ and $l(\varepsilon)$ is called the length of inclusion interval $T(f, \varepsilon)$.

Definition 2.2 A function $f(t, x) = (f_1(t, x), f_2(t, x), \dots, f_n(t, x))^T$, where $f(t, x)$ is an n -vector, t is a real variable, and $x = (x_1(t), x_2(t), \dots, x_n(t))^T$ is an n -vector, is said to be almost periodic in t uniformly with respect to $x \in S \subset R^n$, if $f(t, x)$ is continuous in $t \in R$ and $x \in S$, and if, for any $\varepsilon > 0$, it is possible to find a constant $l(\varepsilon, S) > 0$ such that in any interval of length $l(\varepsilon, S)$ there exists τ such that the inequality

$$\|f(t + \tau, x) - f(t, x)\| = \sum_{i=1}^n |f_i(t + \tau, x) - f_i(t, x)| < \varepsilon$$

is satisfied for all $t \in (-\infty, +\infty)$, $x \in S$. The number τ is called a ε -translation number of $f(t, x)$ and $l(\varepsilon, S)$ is called the length of inclusion interval of $T(f, \varepsilon, S)$.

Definition 2.3 Let $f \in C(R, R^n)$, if the limit $\lim_{T \rightarrow \infty} T^{-1} \int_0^T f(t) dt$ exists, then the limit is called the mean value of f . Denote $m(f(t)) = \lim_{T \rightarrow \infty} T^{-1} \int_0^T f(t) dt$.

Obviously, $m(a) = a$ and $m(m(f(t))) = m(f(t))$, where $a \in R^n$ is a constant vector.

Lemma 2.1 [5, 8, 19]

- (a) Let $f(t) \in C(R, R^n)$, $f(t)$ is almost periodic if only if $x(t)$ is bounded and uniformly continuous on R .
- (b) Let $f : R \times S \rightarrow R^n$ be almost periodic in t uniformly with respect to $x \in S \subset R^n$, where S is any compact set of R^n . Then $f(t, x)$ is bounded on $R \times S$ and uniformly continuous.
- (c) Let $f(t)$ and $g(t)$ be all almost periodic, then $f(t) + g(t)$ is almost periodic. Moreover, if $\inf_{t \in R} |g(t)| > 0$, then $f(t)/g(t)$ is also almost periodic.

Lemma 2.2 [5, 8, 19] Suppose that $x \in C(R, R^n)$ is an almost periodic function. Then

$$\Phi(t) = \int_0^t x(s) ds$$

is almost periodic, if only if $\Phi(t)$ is bounded.

Lemma 2.3 [5, 8, 19] (mean value theorem) Suppose that $f \in C(R, R^n)$ is an almost periodic function, then $m(f(t))$ exists, i.e. $\|m(f(t))\| < +\infty$. Moreover, if $f(t) \geq 0$ and $f(t) \not\equiv 0$, then $m(f(t)) > 0$.

Lemma 2.4 [47] (continuation theorem) Let $\Omega \subset X$ be an open and bounded set. Let L be a Fredholm mapping of index zero and N be L -compact on $\overline{\Omega}$. Assume

- (a) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \text{Dom } L$, $Lx \neq \lambda Nx$;
- (b) for each $x \in \partial\Omega \cap \text{Ker } L$, $QNx \neq 0$;

(c) $\text{deg}\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \text{Dom } L$.

In the following we shall introduce some function spaces and their norms, which are valid throughout this paper. Denote

$$\begin{aligned} AP(R, R^n) &= \{x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \mid x(t) \in C(R, R^n), x(t) \text{ is almost periodic}\}, \\ X &= \{x(t) \in C(R, R^n) \mid x(t) \in AP(R, R^n), \dot{x}(t) \in AP(R, R^n)\}, \\ Z &= \{x(t) \in C(R, R^n) \mid x(t) \in AP(R, R^n)\}. \end{aligned}$$

The norms are given by

$$\begin{aligned} |x_i(t)|_0 &= \sup_{t \in R} |x_i(t)|, & |x_i(t)|_1 &= |x_i(t)|_0 + |\dot{x}_i(t)|_0, \quad i = 1, 2, \dots, n, \\ \|x(t)\|_0 &= \max_{1 \leq i \leq n} \{|x_i(t)|_0\}, & \|x(t)\|_1 &= \|x(t)\|_0 + \|\dot{x}(t)\|_0 = \max_{1 \leq i \leq n} \{|x_i(t)|_1\}. \end{aligned}$$

Obviously, X and Z , respectively, endowed with the norms $\|\cdot\|_1$ and $\|\cdot\|_0$ are Banach spaces.

To proceed our study of the existence of almost periodic solutions, we need to prove the following important lemmas.

Lemma 2.5 *If $x(t) \in AP(R, R^n)$, then there exists $t_0 \in R$ such that $x(t_0) = \sup_{t \in R} x(t)$.*

Proof In view of Lemma 2.1(a), $x(t)$ is bounded. Let $M = \sup_{t \in R} x(t)$. By way of contradiction, suppose that $x(t) < M$ for all $t \in R$. Obviously, $M - x(t) > 0$ is also a continuously almost periodic function on R . Note that $\inf_{t \in R} (M - x(t)) > 0$. Hence, it follows from Lemma 2.1(c) that

$$\frac{1}{M - x(t)} \text{ is also an almost periodic function.}$$

By Lemma 2.1(a) again, $\frac{1}{M - x(t)}$ is bounded. That is, there exists a constant $C_0 > 0$ such that

$$\frac{1}{M - x(t)} \leq C_0 \quad \text{or} \quad x(t) \leq M - \frac{1}{C_0}, \quad \text{for all } t \in R.$$

This implies that $\sup_{t \in R} x(t) \leq M - \frac{1}{C_0} < M$, which contradicts $M = \sup_{t \in R} x(t)$. Therefore, there exists $t_0 \in R$ such that $x(t_0) = M = \sup_{t \in R} x(t)$. □

Lemma 2.6 *Supposing that $x(t) \in AP(R, R^n)$, $m(x(t)) = 0$, then there exists $\eta_0 > 0$ such that, for every fixed $\eta_0 \leq \eta < +\infty$, $\Phi(t) = \int_0^t e^{-\eta(t-s)} x(s) ds$ is an almost periodic function and $\dot{\Phi}(t)$ is also an almost periodic function satisfying $\dot{\Phi}(t) = x(t)$.*

Proof To $\Phi(t)$ is an almost periodic function, it suffices to show that $\Phi(t)$ is bounded. To this end, we just need show that $\int_{-\infty}^t e^{-\eta(t-s)} x(s) ds$ is bounded.

Since $m(x(t)) = 0$, there exists a continuous scalar function $\epsilon(T)$ defined on $0 < T < +\infty$ which satisfies $\epsilon(T) \rightarrow 0$ as $T \rightarrow +\infty$, so that $|\frac{1}{T} \int_t^{t+T} x(s) ds| \leq \epsilon(T)$ and, for any $T > 0$, we

have

$$\int_{-\infty}^t e^{-\eta(t-s)} x(s) \, ds = \int_0^\infty e^{-\eta s} x(t-s) \, ds = \sum_{k=0}^\infty e^{-\eta kT} \int_{kT}^{(k+1)T} e^{-\eta(s-kT)} x(t-s) \, ds.$$

Let $M = \sup_{t \in R} x(t)$, then

$$\begin{aligned} \left| \int_{-\infty}^t e^{-\eta(t-s)} x(s) \, ds \right| &\leq \sum_{k=0}^\infty \left[\left| \int_{kT}^{(k+1)T} x(t-s) \, ds \right| + M \int_{kT}^{(k+1)T} (1 - e^{-\eta(s-kT)}) \, ds \right] \\ &\leq \sum_{k=0}^\infty \left[T\epsilon(T) + M \int_0^T (1 - e^{-\eta s}) \, ds \right] \\ &\leq \frac{T\epsilon(T)}{1 - e^{-\eta T}} + MT \\ &\leq T(\vartheta(T) + M), \end{aligned}$$

where $\vartheta(T) = \frac{\epsilon(T)}{1 - e^{-\eta T}}$ is a strictly decreasing function with respect to T so that $\vartheta(0^+) > 1$ and $\vartheta(+\infty) = 0$. Thus, there is a unique $T(\eta)$ such that $\vartheta(T(\eta)) = 1$. Moreover, $\frac{dT(\eta)}{d\eta} < 0$. In fact, $\vartheta(T(\eta)) = 1$ implies that

$$\epsilon(T(\eta)) - 1 + e^{-\eta T(\eta)} = 0. \tag{1}$$

Taking the derivative of the above inequality with respect to η leads to

$$\frac{d\epsilon(T(\eta))}{dT} \frac{dT(\eta)}{d\eta} - e^{-\eta T(\eta)} \left[T(\eta) + \eta \frac{dT(\eta)}{d\eta} \right] = 0.$$

Note that $\vartheta(T)$ is strictly decreasing, it follows that

$$\frac{dT(\eta)}{d\eta} = \frac{e^{-\eta T(\eta)} T(\eta)}{\frac{d\epsilon(T(\eta))}{dT} - e^{-\eta T(\eta)}} < 0.$$

Thus, $T(\eta)$ is decreasing with respect to η .

First of all, we claim that $T(0^+) = \infty$. If this is not true, suppose that $T(0^+) \leq \gamma < +\infty$. Then, for any $\eta > 0$, we have

$$1 - e^{-\eta T(\eta)} = \epsilon(T(\eta)) \geq \epsilon(\gamma) > 0.$$

Letting $\eta \rightarrow 0^+$, the above inequality leads to $0 > 0$. This is a contradiction. Thus, $T(0^+) = \infty$.

Second, we claim that there exists $\delta > 0$ such that $T(+\infty) = \delta < +\infty$. If this is not true, suppose that $T(+\infty) = +\infty$. Then, for any $\eta > 0$, we have

$$1 - e^{-\eta T(\eta)} = \epsilon(T(\eta)).$$

Letting $\eta \rightarrow +\infty$, the above inequality leads to $1 = 1 - e^{-\infty} = \epsilon(+\infty) = 0$. This is a contradiction. Thus, there exists $\delta > 0$ such that $T(+\infty) = \delta < +\infty$.

The fact $T(\eta)$ is decreasing with respect to η , together with $T(0^+) = \infty$ and $T(+\infty) = \delta < +\infty$, leads to the conclusion that there exists $\eta_0 > 0$ such that, for every fixed $0 < \eta_0 \leq \eta < +\infty$, $\delta < T(\eta) \leq T(\eta_0) < T(0^+) = +\infty$. Thus, we have

$$|\Phi(t)| \leq \left| \int_{-\infty}^t e^{-\eta(t-s)} x(s) ds \right| \leq T(\vartheta(T) + M) = T(\eta)(1 + M) \leq T(\eta_0)(1 + M),$$

for $t \in R$.

The above inequality implies that, for any fixed $0 < \eta_0 \leq \eta < \infty$, $\Phi(t)$ is bounded. Thus, $\Phi(t)$ is an almost periodic function. Since $\dot{\Phi}(t) = x(t)$ and $x(t)$ is almost periodic, $\dot{\Phi}(t)$ is also an almost periodic function. The proof is complete. \square

3 Existence of an almost periodic solution in general case

In this section, we consider the almost periodic system

$$\frac{dx(t)}{dt} = f(t, x(t), x(t - \tau)), \tag{2}$$

where $f : R \times S \rightarrow R^n$ is an almost periodic function in t uniformly with respect to $x \in S \subset R^n$.

In order to apply Lemma 2.4 to study the existence of the almost periodic solutions, we define the operators $L : \text{Dom } L \subset X \rightarrow Z$ and $N : X \rightarrow Z$ as follows:

$$\begin{aligned} X \ni x(t) &\rightarrow (Lx)(t) = \frac{dx(t)}{dt} \in Z, \\ X \ni x(t) &\rightarrow (Nx)(t) = ((Nx)_1(t), (Nx)_2(t), \dots, (Nx)_n(t))^T \in Z, \end{aligned}$$

where

$$(Nx)_i(t) = f_i(t, x(t), x(t - \tau)), \quad i = 1, 2, \dots, n.$$

Define, respectively, the projectors $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ by

$$Px(t) = m(x(t)), \quad Qz(t) = m(z(t)), \quad x \in X, z \in Z.$$

From the above definitions, it is obvious that the domain of L in X is actually the whole space, and

$$\begin{aligned} \text{Ker } L &= \{x(t) \in X \mid Lx(t) = 0, \text{ i.e. } \dot{x}(t) = 0\} = R^n, \\ \text{Im } L &= \{Lx(t) \mid x(t) \in X\} \text{ is closed in } Z, \\ \text{Ker } Q &= \{z(t) \in Z \mid m(z(t)) = 0\}. \end{aligned}$$

Then we have the following results.

Lemma 3.1 *Suppose that the operators L, P, Q are as defined above. Then the following hold:*

- (i) $\text{Im } P = R^n = \text{Ker } L$;

- (ii) $\text{Im } L = \text{Ker } Q = \text{Im}(I - Q)$;
- (iii) $\dim \text{Ker } L = \text{codim } \text{Im } L = n < +\infty$.

Proof By the definition of P and $\text{Ker } L$, (i) follows immediately. Noting that $\dim \text{Ker } L + \dim \text{Im } L = n$, it is easy to check that (iii) holds. Now we devote ourselves to proving (ii).

First we show $\text{Im } L = \text{Ker } Q$. To this end, we proceed in two steps.

Step 1. $\forall \Phi(t) \in \text{Im } L, x(t) \in X$ (i.e. $x \in AP(R, R^n)$ and $\dot{x} \in AP(R, R^n)$) such that $\Phi(t) = Lx(t) = \dot{x}(t)$. Then $\Phi(t)$ is almost periodic. On the other hand, it follows from Lemma 2.1 that $x(t)$ is bounded so that $x(T) - x(0) = \text{const}$. It is easy to check that

$$m(\Phi(t)) = m(\dot{x}(t)) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \dot{x}(t) dt = \lim_{T \rightarrow +\infty} \frac{1}{T} (x(T) - x(0)) = 0.$$

Thus, $\Phi(t) \in \text{Ker } Q$, which implies that $\text{Im } L \subset \text{Ker } Q$.

Step 2. $\forall z(t) \in \text{Ker } Q$, that is, $z(t) \in AP(R, R^n)$ and $m(z(t)) = 0$. $z(t) \in AP(R, R^n)$ implies $z(t) \in C(R, R^n)$. Thus, there exists an original function of $z(t)$, denoted by $\Phi(t)$ such that $\dot{\Phi}(t) = z(t)$. On the other hand, it is easy to check that, for any fixed $0 < \eta_0 \leq \eta < \infty$, $\Phi_0(t) = \int_0^t e^{-\eta(t-s)} z(s) ds$ is also an original function of $z(t)$ such that $\dot{\Phi}_0(t) = z(t)$. In fact, $\Phi(t) = \Phi_0(t) + C$, where C is an arbitrary constant.

Since $m(z(t)) = 0$, by Lemma 2.6, $\Phi_0(t) = \int_0^t e^{-\eta(t-s)} z(s) ds$ is an almost periodic function and $\dot{\Phi}_0(t)$ is also an almost periodic function satisfying $\dot{\Phi}_0(t) = z(t)$. Therefore, $\Phi(t) = \Phi_0(t) + C$ is almost periodic and $\dot{\Phi}(t) = z(t) = L\Phi(t)$ is also almost periodic. This implies that $z(t) = L\Phi(t) \in \text{Im } L$. Hence, $\text{Ker } Q \subset \text{Im } L$.

It follows from the above steps that $\text{Im } L = \text{Ker } Q$.

Now we show that $\text{Ker } Q = \text{Im}(I - Q) = \{z(t) - m(z(t)) | z(t) \in Z\}$. In fact, $\forall z(t) \in \text{Ker } Q$, then $m(z(t)) = 0$. It follows that

$$z(t) = z(t) - m(z(t)) \in \text{Im}(I - Q),$$

which implies that $\text{Ker } Q \subset \text{Im}(I - Q)$.

Conversely, $\forall F(t) \in \text{Im}(I - Q)$, then there exists $z(t) \in Z$ such that $F(t) = z(t) - m(z(t))$. It follows that

$$m(F(t)) = m(z(t) - m(z(t))) = m(z(t)) - m(z(t)) = 0,$$

which implies that $\text{Im}(I - Q) \subset \text{Ker } Q$. Therefore, which implies that $\text{Ker } Q = \text{Im}(I - Q)$.

In a word, $\text{Im } L = \text{Ker } Q = \text{Im}(I - Q)$. That is, (ii) holds. Thus, the proof of Lemma 3.1 is complete. □

It immediately follows that L is a Fredholm mapping of index zero and $L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the inverse of that map by K_P , it is easy to see that the map $K_P : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$ is given by

$$K_P y(t) = \int_0^t y(s) ds - m\left(\int_0^t y(s) ds\right), \tag{3}$$

or

$$(K_P y)_i(t) = \int_0^t y_i(s) ds - m\left(\int_0^t y_i(s) ds\right).$$

In fact, for any $y(t) \in \text{Im } L$, there exists $x(t) \in X$ such that

$$y(t) = \frac{dx(t)}{dt}.$$

An integration of the above inequality over $[0, t]$ leads to

$$x(t) = x(0) + \int_0^t y(s) ds.$$

Let $K_P y(t) = x(t)$. In view of $x(t) \in X$, $K_P y(t) \in \text{Dom } L = X$. We ensure that $K_P y(t) \in \text{Ker } P$, $m(K_P y(t)) = 0$. Hence, we have

$$0 = m(K_P y(t)) = m(x(0)) + m\left(\int_0^t y(s) ds\right),$$

which implies

$$x(0) = -\left(\int_0^t y(s) ds\right).$$

Therefore,

$$K_P y(t) = \int_0^t y(s) ds - m\left(\int_0^t y(s) ds\right).$$

Lemma 3.2 *For any open bounded subset of X , denoted by*

$$\Omega = \{x(t) \in X \mid \|x(t)\|_1 = \|x(t)\|_0 + \|\dot{x}(t)\|_0 < h\},$$

if there exists $\tilde{M} > 0$ such that

$$\|f(t, x_1, x_2) - f(t, y_1, y_2)\|_0 \leq \tilde{M}[\|x_1 - y_1\|_0 + \|x_2 - y_2\|_0], \quad x_1, x_2, y_1, y_2 \in R^n, t \in R,$$

then the mapping N is L -compact on $\overline{\Omega}$. Here, the constant h is independent of the choice of $x(t)$.

Proof It suffices to show that $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \rightarrow X$ is compact. We first arrive at the result that, for the constant $M = \|f(t, x(t), x(t - \tau))\|_0 > 0$,

$$\|QNx\|_0 = \|m(f(t, x(t), x(t - \tau)))\|_0 \leq m(M) = M, \quad \text{for all } x \in \overline{\Omega},$$

which implies that $QN(\overline{\Omega})$ is bounded in the space $(Z, \|\cdot\|_0)$. Second, by (3),

$$\begin{aligned} & (K_P(I - Q)Nx)(t) \\ &= \int_0^t [f(s, x(s), x(s - \tau)) - m(f(s, x(s), x(s - \tau)))] ds \\ & \quad - m\left(\int_0^t [f(s, x(s), x(s - \tau)) - m(f(s, x(s), x(s - \tau)))] ds\right). \end{aligned} \tag{4}$$

We shall show that $(K_P(I - Q)Nx)(\overline{\Omega})$ is relatively compact in the space $(X, \| \cdot \|_1)$. To this end, we proceed in two steps.

Step 1. We claim that, for any $x(t) \in \overline{\Omega}$, $(K_P(I - Q)Nx)(t)$ is bounded. In fact, for any $x(t) \in \overline{\Omega}$, then $f(t, x(t)) \in N(\overline{\Omega})$ due to the definition of the operator N . This implies that $f(t, x(t)) - m(f(t, x(t))) \in (I - Q)N(\overline{\Omega}) \subset \text{Im}(I - Q)N$. From (ii) of Lemma 3.1, we know that $\text{Im}(I - Q)N = \text{Im}LN$. Thus, $f(t, x(t)) - m(f(t, x(t))) \in \text{Im}LN = \{Lr(t) | r(t) \in NX \subset Z, \text{ i.e. } r(t) \in AP(R, R^n)\}$, which implies that there exists $r(t) \in AP(R, R^n)$ such that

$$Lr(t) = f(t, x(t), x(t - \tau)) - m(f(t, x(t), x(t - \tau))).$$

This, together with (4), leads to

$$\begin{aligned} (K_P(I - Q)Nx)(t) &= \int_0^t Lr(s) ds - m\left(\int_0^t Lr(s) ds\right) \\ &= \int_0^t \dot{r}(s) ds - m\left(\int_0^t \dot{r}(s) ds\right) \\ &= r(t) - r(0) - m(r(t) - r(0)) \\ &= r(t) - m(r(t)). \end{aligned} \tag{5}$$

Since $r(t) \in AP(R, R^n)$, by Lemma 2.1 and Lemma 2.3, $r(t)$ is bounded and $\|m(r(t))\|_0 < +\infty$. Therefore, it follows from (5) that, for any $x(t) \in \overline{\Omega}$, there exists a constant $M_0 > 0$ such that $\|(K_P(I - Q)Nx)(t)\|_0 \leq M_0$.

On the other hand, we have

$$(K_P(I - Q)Nx)'(t) = \dot{r}(t) = Lr(t) = f(t, x(t), x(t - \tau)) - m(f(t, x(t), x(t - \tau))). \tag{6}$$

In view of the almost periodicity of $f(t, x(t))$, $(K_P(I - Q)Nx)'(t)$ is bounded for any $x(t) \in \overline{\Omega}$. That is, there exists $M_1 > 0$ such that $(K_P(I - Q)Nx)'(t) \leq M_0$.

Therefore, for any $x(t) \in \overline{\Omega}$,

$$\|(K_P(I - Q)Nx)(t)\|_1 \leq M_0 + M_1.$$

Step 2. By similar arguments to [41], we can prove that $(K_P(I - Q)Nx)(\overline{\Omega})$ is equicontinuous.

Therefore, by generalizing the famous Arzela-Ascoli theorem, $(K_P(I - Q)Nx)(\overline{\Omega})$ is relatively compact in the space $(X, \| \cdot \|_1)$. The proof of this lemma is complete. \square

Then by Lemma 3.1, Lemma 3.2, and Lemma 2.4, we have the following results on the existence of almost periodic solutions.

Theorem 3.1 *Let the operators L, N be defined as above. Suppose that $f(t, x)$ is almost periodic in t uniformly with respect to $x \in S$ and there exists $\tilde{M} > 0$ such that*

$$\|f(t, x_1, x_2) - f(t, y_1, y_2)\|_0 \leq \tilde{M}[\|x_1 - y_1\|_0 + \|x_2 - y_2\|_0], \quad x_1, x_2, y_1, y_2 \in R^n, t \in R.$$

If there exists a constant h (h is independent of the choice of $x(t)$) such that the open bounded set $\Omega = \{x(t) \in X | \|x(t)\|_1 < h\}$ satisfies the following conditions:

- (i) for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \text{Dom } L$, $Lx \neq \lambda Nx$;
 - (ii) for each $x \in \partial\Omega \cap \text{Ker } L$, $QNx \neq 0$;
 - (iii) $\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$.
- Then system (2) has at least one almost periodic solution in $\overline{\Omega} \cap \text{Dom } L$.

4 Conclusion

In the present paper, a new method is developed to study the existence of an almost periodic solution for the ordinary or functional differential equations. The approach is based on topological degree and novel estimation techniques for the *a priori* bounds of unknown solutions for $Lx = \lambda Nx$. This new method can be seen as a supplement of the other classical methods. The approach performed in the paper could be further generalized to investigate the existence of almost periodic oscillatory cases in some other nonlinear dynamical systems. It is believed that it can be applied to the image pattern, digital image processing, data processing, signal sparse decomposition and information technology, *etc.*

It should be noted that there are particular differences between this paper and previous work [41].

1. This paper considered the *almost periodic solution*. However, the authors in [41] studied the *periodic solutions*. In fact, *periodic solutions* are a special case of *almost periodic solutions*. For a periodic function, it is defined by $f(t + T) = f(t)$ for some T . However, for the almost periodic function, it is defined in a more complicated way. For the detailed definition of an almost periodic function, one can refer to Definition 2.1.

2. Due to the big difference between the concepts of *almost periodic* and *periodic*, the method used in [41] cannot be applied to the *almost periodic* case directly. So we need to prove a lot of preliminary results in Section 2. From Lemma 2.1 to Lemma 2.6, all these results are very original, they have never been studied. This is the big contribution of this paper. In fact, these results are very interesting. For *periodic solutions*, it is not necessary to prove these results. It is obvious for the *periodic case*.

3. In Section 3, it seems that the method is similar to [41]. But please note that my aim of this paper is to generalize the *periodic results* in [41] to the *almost periodic case*. In fact, the detailed proof is different. Thus, it seems similar, but it is a very different problem. For example, for the *periodic case*, the operators P and Q are defined by $Px(t) = \frac{1}{T} \int_0^T x(t) dt$, however, for the *almost periodic case*, we have $Px(t) = m(x(t))$.

Competing interests

The author declares that there is no conflict of interests regarding the publication of this article.

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