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Asymptotic behavior of solutions to a class of semilinear parabolic equations

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Abstract

This paper concerns the asymptotic behavior of solutions to the homogeneous Neumann exterior problems of a class of semilinear parabolic equations with convection and reaction terms. The critical Fujita exponents theorems are established. It is shown that the global existence and blow-up of solutions depends on the reaction term, the convection term and the spatial dimension.

MSC: 35K65; 35B33

Keywords: convection; reaction; asymptotic behavior

1 Introduction

In this paper, we consider the asymptotic behavior of solutions to the following problem:

$$\frac{\partial u}{\partial t} = \Delta u + \lambda_1 \frac{x}{|x|^2} \cdot \nabla u + |x|^{\lambda_2} t^{\lambda_3} u^p, \quad x \in \mathbb{R}^n \setminus \overline{B}_1, t > 0, \tag{1}$$

$$\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial B_1, t > 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \setminus \overline{B}_1, \tag{3}$$

where $\lambda_1, \lambda_2, \lambda_3 \geq 0$, $p > 1$, $0 \leq u_0 \in C(\mathbb{R}^n \setminus B_1) \cap L^\infty(\mathbb{R}^n \setminus B_1)$, B_1 is the unit ball in $\mathbb{R}^n$.

The studies on asymptotic behavior of solutions to diffusion equations with nonlinear reaction was begun in 1966 by Fujita in [1], where it was proved that for the Cauchy problem to the semilinear equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^n, t > 0, \tag{4}$$

the problem does not have any nontrivial, nonnegative global solution if $1 < p < p_c = 1 + 2/n$, whereas if $p > p_c$, there exist both global (with small data) and non-global (with large initial data) solutions. This result shows that the exponent p of the nonlinear reaction affects the properties of solutions directly. We call p_c the critical Fujita exponent and such a result a blow-up theorem of Fujita type.

The elegant work of Fujita revealed a new phenomenon of nonlinear evolution equations. There have been a number of extensions of Fujita's results in several directions since then, including similar results for numerous of quasilinear parabolic equations and systems in various of geometries (whole spaces, cones, and exterior domains) with nonlinear reactions or nonhomogeneous boundary conditions, and even degenerate equations in domains with non-compact boundary [2, 3]. We refer to the survey papers [4, 5] and the references therein, and more recent work [6–20]. Qi [8] considered the Cauchy problem of (1) without convection, *i.e.* $\lambda_1 = 0$, and proved that the Fujita exponent is

$$p_c = 1 + \frac{2 + \lambda_2 + 2\lambda_3}{n}.$$

Zheng and Wang [15] studied the homogeneous Neumann exterior problem (1)-(3) with the special case $\lambda_3 = 0$, and proved that the Fujita exponent is

$$p_c = 1 + \frac{2 + \lambda_2}{n + \lambda_1}.$$

In this paper, we consider the homogeneous Neumann exterior problem (1)-(3), and prove that the Fujita exponent is

$$p_c = 1 + \frac{2 + \lambda_2 + 2\lambda_3}{n + \lambda_1},$$

which depends on $n, \lambda_1, \lambda_2, \lambda_3$. The technique used in this paper for establishing the Fujita type results for the problem (1)-(3) is mainly inspired by [8, 10, 11]. To prove the blow-up of solutions, we will determine the interactions among the diffusion terms, convection terms and reaction terms by a series of precise integral estimates instead of pointwise comparisons. Here, we also need to construct some subsolutions to get some integral estimates of the solutions. As to the existence of global solutions, we construct some global self-similar supersolutions.

This paper is arranged as follows. The main results of the paper are stated in Section 2 and their proofs are given in Section 3 subsequently.

2 Main results

For $0 \le u_0 \in C(\mathbb{R}^n \setminus B_1) \cap L^\infty(\mathbb{R}^n \setminus B_1)$, it follows from the classical theory for parabolic equations that the problem (1)-(3) admits a nonnegative solution locally in time. Moreover, the comparison principle holds for the problem (1)-(3). A solution u to the problem (1)-(3) is said to blow up in a finite time $0 < T < +\infty$, if

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n\setminus B_1)} = \sup_{x\in\mathbb{R}^n\setminus B_1} u(x,t) \to +\infty \quad \text{as } t \to T^-.$$

Otherwise, u is said to be global.

The main results of this paper are the following two theorems.

Theorem 2.1 *Assume that* $1 < p < p_c = 1 + (2 + \lambda_2 + 2\lambda_3)/(n + \lambda_1)$. *Then each nontrivial nonnegative solution to the problem* (1)-(3) *blows up in a finite time.*

Theorem 2.2 *Assume that* $p > p_c = 1 + (2 + \lambda_2 + 2\lambda_3)/(n + \lambda_1)$. *Then there exist both nontrivial global and blow-up nonnegative solutions to the problem* (1)-(3).

3 Proof of main results

First we prove Theorem 2.1 by determining the interactions among the diffusion terms, convection terms and reaction terms by a series of precise integral estimates. Moreover, we need to construct some subsolutions to get some integral estimates of the solutions.

Proof of Theorem 2.1 We prove the theorem by a contradiction argument. Assume that the problem (1)-(3) admits a nontrivial nonnegative global solution u. For each $l > 1$, set

$$\psi_l(r) = \begin{cases} r^{\lambda_1}, & 1 \le r \le l, \\ \frac{1}{2} r^{\lambda_1}\left(1 + \cos \frac{(r-l)\pi}{(\delta-1)l}\right), & l < r < \delta l, \\ 0, & r \ge \delta l, \end{cases}$$

where $\delta = \pi/(n + \lambda_1 - 1) + 1$. Similar to the proof of Lemma 2.1 in [15], one can show that for each $l > 1$,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l\big(|x|\big)\,dx \\ &\quad\ge -C_0 l^{-2}\int_{B_{\delta l}\setminus B_l} u(x,t)\psi_l\big(|x|\big)\,dx \\ &\qquad + t^{\lambda_3}\int_{\mathbb{R}^n\setminus B_1} |x|^{\lambda_2} u^p(x,t)\psi_l\big(|x|\big)\,dx, \quad t > 0, \end{aligned} \tag{5}$$

where $C_0 = (\delta - 1)^{-2}\pi^2$, B_l is a ball centered at the origin and with radius l in $\mathbb{R}^n$. It follows from the Hölder inequality that

$$\begin{aligned} &C_0\int_{B_{\delta l}\setminus B_l} u(x,t)\psi_l\big(|x|\big)\,dx \\ &\quad\le C_0\left(\int_{B_{\delta l}\setminus B_l} |x|^{-\lambda_2/(p-1)}\psi_l\big(|x|\big)\,dx\right)^{(p-1)/p} \\ &\qquad\times\left(\int_{B_{\delta l}\setminus B_l} |x|^{\lambda_2} u^p(x,t)\psi_l\big(|x|\big)\,dx\right)^{1/p} \\ &\quad\le C_1 l^{n+\lambda_1-(n+\lambda_1+\lambda_2)/p}\left(\int_{\mathbb{R}^n\setminus B_1} |x|^{\lambda_2} u^p(x,t)\psi_l\big(|x|\big)\,dx\right)^{1/p}, \quad t > 0, \end{aligned} \tag{6}$$

with some constant $C_1 > 0$ depending only on n, λ_1, λ_2, and p. Substitute (6) into (5) to get

$$\begin{aligned} \frac{d}{dt}w_l(t) \ge{}& \left(\int_{\mathbb{R}^n\setminus B_1} |x|^{\lambda_2} u^p(x,t)\psi_l\big(|x|\big)\,dx\right)^{1/p}\bigg\{-C_1 l^{n+\lambda_1-2-(n+\lambda_1+\lambda_2)/p} \\ &+ t^{\lambda_3}\left(\int_{\mathbb{R}^n\setminus B_1} |x|^{\lambda_2} u^p(x,t)\psi_l\big(|x|\big)\,dx\right)^{(p-1)/p}\bigg\}, \quad t > 0, \end{aligned} \tag{7}$$

where

$$w_l(t) = \int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l\big(|x|\big)\,dx, \quad t \ge 0.$$

The Hölder inequality shows

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l(|x|)\,dx \\
&\quad\le \left(\int_{B_{\delta l}\setminus B_1} |x|^{-\lambda_2/(p-1)}\psi_l(|x|)\,dx\right)^{(p-1)/p} \\
&\qquad\times \left(\int_{\mathbb{R}^n\setminus B_1} |x|^{\lambda_2}u^p(x,t)\psi_l(|x|)\,dx\right)^{1/p} \\
&\quad\le C_2 l^{(n+\lambda_1)-(n+\lambda_1+\lambda_2)/p}\left(\int_{\mathbb{R}^n\setminus B_1} |x|^{\lambda_2}u^p(x,t)\psi_l(|x|)\,dx\right)^{1/p}, \quad t>0,
\end{aligned}$$

which leads to

$$\int_{\mathbb{R}^n\setminus B_1} |x|^{\lambda_2}u^p(x,t)\psi_l(|x|)\,dx \ge C_3 l^{(n+\lambda_1+\lambda_2)-p(n+\lambda_1)} w_l^p(t), \quad t>0, \tag{8}$$

where $C_2, C_3 > 0$ are constants depending only on n, λ_1, λ_2, and p. It follows from (8) and (7) that

$$\begin{aligned}
\frac{d}{dt}w_l(t) \ge{}& \left(C_3 l^{n+\lambda_1+\lambda_2-p(n+\lambda_1)}\right)^{1/p} w_l(t)\bigl\{-C_1 l^{n+\lambda_1-2-(n+\lambda_1+\lambda_2)/p} \\
&+ C_3^{(p-1)/p} l^{(p-1)(n+\lambda_1+\lambda_2)/p-(p-1)(n+\lambda_1)} t^{\lambda_3} w_l^{p-1}(t)\bigr\}, \quad t>0.
\end{aligned}\tag{9}$$

It is noted that the restriction on p in (9) is $p>1$ instead of $1<p<p_c$.

Next we prove that there exists a constant $C_4>0$ depending only on u_0, n, and λ_1, such that

$$w_l(l^2) = \int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l(|x|)\,dx \ge C_4, \quad l\ge 2. \tag{10}$$

Let v be the solution to the following problem:

$$\begin{aligned}
&\frac{\partial v}{\partial t} = \Delta v + \lambda_1\frac{x}{|x|^2}\cdot\nabla v, \quad x\in\mathbb{R}^n\setminus\overline{B}_1, t>0, \\
&\frac{\partial v}{\partial \nu} = 0, \quad x\in\partial B_1, t>0, \\
&v(x,0) = u_0(x), \quad x\in\mathbb{R}^n\setminus\overline{B}_1.
\end{aligned}$$

Then the comparison principle gives

$$u(x,t)\ge v(x,t), \quad x\in\mathbb{R}^n\setminus B_1, t\ge 0. \tag{11}$$

Since $0\le u_0\in C(\mathbb{R}^n\setminus B_1)\cap L^\infty(\mathbb{R}^n\setminus B_1)$ is nontrivial, one can prove by the Green function method that

$$v(x,1)\ge C_5 C_6^{-(n+\lambda_1)/2}\exp\left\{-\frac{(|x|-1)^2}{4C_6}\right\}, \quad x\in\mathbb{R}^n\setminus\overline{B}_1, \tag{12}$$

with some constants $C_5, C_6 > 0$ depending only on u_0, n, and λ_1. Set

$$\underline{z}(x,t) = C_5(t-1+C_6)^{-(n+\lambda_1)/2}\exp\left\{-\frac{(|x|-1)^2}{4(t-1+C_6)}\right\}, \quad x\in\mathbb{R}^n\setminus B_1, t\geq 1. \tag{13}$$

It is easy to verify that $\underline{z}$ is a subsolution to the problem

$$\begin{aligned}
&\frac{\partial z}{\partial t} = \Delta z + \lambda_1\frac{x}{|x|^2}\cdot\nabla z, \quad x\in\mathbb{R}^n\setminus\overline{B}_1, t>1,\\
&\frac{\partial z}{\partial \nu} = 0, \quad x\in\partial B_1, t>1,\\
&z(x,1) = C_5C_6^{-(n+\lambda_1)/2}\exp\left\{-\frac{(|x|-1)^2}{4C_6}\right\}, \quad x\in\mathbb{R}^n\setminus\overline{B}_1.
\end{aligned}$$

The comparison principle, together with (12) and (13), shows that

$$\begin{aligned}
v(x,t) &\geq \underline{z}(x,t)\\
&= C_5(t-1+C_6)^{-(n+\lambda_1)/2}\exp\left\{-\frac{(|x|-1)^2}{4(t-1+C_6)}\right\}, \quad x\in\mathbb{R}^n\setminus B_1, t\geq 1.
\end{aligned}\tag{14}$$

It follows from (11) and (14) that

$$u(x,t) \geq C_5(t-1+C_6)^{-(n+\lambda_1)/2}\exp\left\{-\frac{(|x|-1)^2}{4(t-1+C_6)}\right\}, \quad x\in\mathbb{R}^n\setminus B_1, t\geq 1,$$

which implies (10).

Now, taking $t\geq l^2$ in (9), one gets

$$\begin{aligned}
\frac{d}{dt}w_l(t) \geq{}& \left(C_3 l^{n+\lambda_1+\lambda_2-p(n+\lambda_1)}\right)^{1/p} w_l(t)\left\{-C_1 l^{n+\lambda_1-2-(n+\lambda_1+\lambda_2)/p}\right.\\
&\left.+ C_3^{(p-1)/p} l^{(p-1)(n+\lambda_1+\lambda_2)/p-(p-1)(n+\lambda_1)+2\lambda_3} w_l^{p-1}(t)\right\}, \quad t\geq l^2.
\end{aligned}\tag{15}$$

It follows from $p<p_c$ that

$$n+\lambda_1-2-(n+\lambda_1+\lambda_2)/p < (p-1)(n+\lambda_1+\lambda_2)/p-(p-1)(n+\lambda_1)+2\lambda_3. \tag{16}$$

In addition, (10) yields

$$\inf\left\{w_l\left(l^2\right) : l\in(2,+\infty)\right\} \geq C_4. \tag{17}$$

Owing to (16) and (17), one sees from (15) that there exist two constants $L>2$ and $\gamma>0$ depending only on u_0, n, λ_1, λ_2, and λ_3 such that, for each $l>L$,

$$\begin{aligned}
\frac{d}{dt}w_l(t) &\geq \left(C_3 l^{n+\lambda_1+\lambda_2-p(n+\lambda_1)}\right)^{1/p}\\
&\quad\times w_l(t)\left(\frac{1}{2}C_3^{(p-1)/p} l^{(p-1)(n+\lambda_1+\lambda_2)/p-(p-1)(n+\lambda_1)+2\lambda_3} w_l^{p-1}(t)\right)\\
&\geq \gamma w_l^p(t), \quad t\geq l^2.
\end{aligned}$$

Thus there exists some $0 < T < +\infty$ such that

$$w_l(t) = \int_{\mathbb{R}^n\setminus B_1} u(x,t)\psi_l(|x|)\,dx \to +\infty \quad \text{as } t \to T^-.$$

Since $\operatorname{supp}\psi_l(|x|) = \overline{B}_{\delta l} \setminus B_1$, one further gets

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n\setminus B_1)} \to +\infty \quad \text{as } t \to T^-.$$

That is to say, u blows up in a finite time. □

Remark 3.1 Similar to the proof of critical case in [15] and the proof in Theorem 2.1, one can show that the critical case $p = p_c$ is also blow-up case.

We turn to the proof of Theorem 2.2. We use the integral estimate (9) to show the existence of nontrivial blow-up nonnegative solutions, while we construct some global self-similar supersolutions to show the existence of nontrivial global nonnegative solutions.

Proof of Theorem 2.2 First consider the blow-up case. As mentioned in the proof of Theorem 2.1, (9) holds for each $p > 1$, especially for $p > p_c$. In particular, for $l = 2$,

$$\begin{aligned}\frac{d}{dt}w_2(t) \geq{}& \bigl(C_3 2^{n+\lambda_1+\lambda_2-p(n+\lambda_1)}\bigr)^{1/p} w_2(t)\bigl\{-C_1 2^{n+\lambda_1-2-(n+\lambda_1+\lambda_2)/p} \\ &+ C_3^{(p-1)/p} 2^{(p-1)(n+\lambda_1+\lambda_2)/p-(p-1)(n+\lambda_1)} t^{\lambda_3} w_2^{p-1}(t)\bigr\}, \quad t > 0,\end{aligned} \tag{18}$$

where w_2, C_1, and C_3 are given in the proof of Theorem 2.1. Let z be the solution to the following problem:

$$\frac{\partial z}{\partial t} = \Delta z + \lambda_1 \frac{x}{|x|^2}\cdot\nabla z, \quad x \in \mathbb{R}^n \setminus \overline{B}_1, t > 0, \tag{19}$$

$$\frac{\partial z}{\partial \nu} = 0, \quad x \in \partial B_1, t > 0, \tag{20}$$

$$z(x,0) = C_7 \exp\left\{-\frac{(|x|-1)^2}{4}\right\}, \quad x \in \mathbb{R}^n \setminus \overline{B}_1, \tag{21}$$

where $C_7 > 0$ is a constant to be determined. If

$$u_0(x) \geq C_7 \exp\left\{-\frac{(|x|-1)^2}{4}\right\}, \quad x \in \mathbb{R}^n \setminus \overline{B}_1, \tag{22}$$

then it follows from the comparison principle that

$$u(x,t) \geq z(x,t), \quad x \in \mathbb{R}^n \setminus B_1, t \geq 0. \tag{23}$$

Set

$$\underline{z}(x,t) = C_7(t+1)^{-(n+\lambda_1)/2} \exp\left\{-\frac{(|x|-1)^2}{4(t+1)}\right\}, \quad x \in \mathbb{R}^n \setminus B_1, t \geq 0. \tag{24}$$

Then $\underline{z}$ is a subsolution to the problem (19)-(21) and thus the comparison principle gives

$$z(x,t) \geq \underline{z}(x,t), \quad x \in \mathbb{R}^n \setminus B_1, t \geq 0. \tag{25}$$

It follows from (23)-(25) that

$$u(x,t) \geq C_7(t+1)^{-(n+\lambda_1)/2} \exp\left\{-\frac{(|x|-1)^2}{4(t+1)}\right\}, \quad x \in \mathbb{R}^n \setminus B_1, t \geq 0.$$

Thus, there exists sufficiently large C_7 such that

$$C_3^{(p-1)/p} 2^{(p-1)(n+\lambda_1+\lambda_2)/p-(p-1)(n+\lambda_1)} w_2^{p-1}(1) \geq 2C_1 2^{n+\lambda_1-2-(n+\lambda_1+\lambda_2)/p},$$

which, together with (18), leads to

$$\frac{d}{dt} w_2(t) \geq \frac{1}{2}\left(C_3 2^{n+\lambda_1+\lambda_2-p(n+\lambda_1)}\right)^{1/p} C_3^{(p-1)/p} 2^{(p-1)(n+\lambda_1+\lambda_2)/p-(p-1)(n+\lambda_1)} w_2^p(t), \quad t \geq 1.$$

Then, similar to the discussion in the proof of Theorem 2.1, one sees that if u_0 satisfies (22), then u blows up in a finite time.

We turn to the global existence case. We prove that the problem (1)-(3) admits a self-similar supersolution of the form

$$\overline{u}(x,t) = (t+1)^{-\alpha} v\left((t+1)^{-\beta}|x|\right), \quad x \in \mathbb{R}^n \setminus B_1, t \geq 0 \tag{26}$$

if

$$u_0(x) \leq \overline{u}(x,0), \quad x \in \mathbb{R}^n \setminus \overline{B}_1, \tag{27}$$

where

$$\alpha = \frac{2+\lambda_2+2\lambda_3}{2(p-1)}, \qquad \beta = \frac{1}{2}.$$

Direct calculations show that if v is a supersolution to the following ordinary differential equation:

$$v''(r) + \frac{n+\lambda_1-1}{r} v'(r) + \beta r v'(r) + \alpha v(r) + r^{\lambda_2} v^p(r) = 0, \quad r > 0, \tag{28}$$

then $\overline{u}$ defined by (26) is a supersolution to equation (1). It follows from $p > p_c$ that $0 < \frac{2+\lambda_2+2\lambda_3}{4(p-1)(n+\lambda_1)} < \frac{1}{4}$, which ensure that we can choose $A > 0$ such that

$$\frac{\alpha}{2(n+\lambda_1)} = \frac{2+\lambda_2+2\lambda_3}{4(p-1)(n+\lambda_1)} < A < \frac{1}{4} = \frac{\beta}{2}.$$

Set

$$v(r) = \eta e^{-Ar^2}, \quad r > 0 \tag{29}$$

with $\eta > 0$ to be determined. We verify that for sufficiently small $\eta > 0$, v is a supersolution to (28). That is to say,

$$v''(r) + \frac{n + \lambda_1 - 1}{r} v'(r) + \beta r^{\lambda_1+1} v'(r) + \alpha r^{\lambda_1} v(r) + r^{\lambda_2} v^p(r) \le 0, \quad r > 0,$$

which is equivalent to

$$4A^2 r^2 - 2A - 2(n + \lambda_1 - 1)A - 2\beta A r^2 + \alpha + r^{\lambda_2} \eta^{p-1} \mathrm{e}^{-A(p-1)r^2} \le 0, \quad r > 0$$

or

$$2(2A - \beta)Ar^2 + 2\big(\alpha - (n + \lambda_1)A\big) + r^{\lambda_2} \eta^{p-1} \mathrm{e}^{-A(p-1)r^{\lambda_1+2}} \le 0, \quad r > 0. \tag{30}$$

The choice of A shows that (30) holds for sufficiently small $\eta > 0$. Therefore, for sufficiently small $\eta > 0$, $\overline{u}$ defined by (26) and (29) is a supersolution to equation (1). Noting

$$v'(r) \le 0, \quad r > 0,$$

one further sees that $\overline{u}$ is a supersolution to the problem (1)-(3) if u_0 satisfies (27). Then the comparison principle shows that if u_0 satisfies (27), the solution to the problem (1)-(3) exists globally. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed to each part of this study equally and approved the final version of the manuscript.

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