Xu et al. *Boundary Value Problems* (2016) 2016:73
DOI 10.1186/s13661-016-0582-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A fixed point operator for systems of vector p-Laplacian with singular weights

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Abstract
In this paper, after establishing a fixed point operator for a strongly coupled vector p-Laplacian with a singular and sign-changing weight function, which may not be integrable, we investigate the existence for the Dirichlet boundary value problems of strongly coupled vector p-Laplacian systems with a nonlinear term consisting of Hadamard product. The proofs are mainly based on topological degree arguments and the global continuation theorem.

MSC: 34B16; 34B18

Keywords: p-Laplacian system; sign-changing weight; existence; nontrivial solution

1 Introduction

We are concerned with the existence of nontrivial solutions for strongly coupled nonlinear differential systems of the form

$$(P_\lambda) \quad \begin{cases} -\Psi_p(u')' = \lambda h(t) \cdot f(u), & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $p > 1$, $\Psi_p : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $\Psi_p(x) = |x|^{p-2}x$, $\lambda > 0$ is a parameter, $h(t) = (h_1(t), \dots, h_N(t))$ with $h_i : (0,1) \to \mathbb{R}$, and $f(u) = (f_1(u), \dots, f_N(u))$ with continuous $f_i : \mathbb{R}^N \to \mathbb{R}$. Here we denote $x \cdot y = (x_1y_1, x_2y_2, \dots, x_Ny_N)$ the Hadamard product of x and y in $\mathbb{R}^N$. Thus, problem (P_λ) can be rewritten as

$$\begin{cases} -(|u'(t)|^{p-2}u_1'(t))' = \lambda h_1(t) f_1(u), \\ \vdots \\ -(|u'(t)|^{p-2}u_N'(t))' = \lambda h_N(t) f_N(u), & t \in (0,1), \\ u_i(0) = 0 = u_i(1), \quad i = 1, \dots, N. \end{cases}$$

Throughout the paper, we denote by $|\cdot|$ the absolute value on $\mathbb{R}$ or the Euclidean norm on $\mathbb{R}^N$ and by $\langle\cdot,\cdot\rangle$ the inner product on $\mathbb{R}^N$ and define $\varphi_p : \mathbb{R} \to \mathbb{R}$ by $\varphi_p(s) = |s|^{p-2}s$. For a weight function h, we assume that $h_i \in \mathcal{H}$, where

$$\mathcal{H} = \left\{ g \in L^1_{\mathrm{loc}}((0,1), \mathbb{R}) \;\middle|\; \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} |g(\tau)|\, d\tau \right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s |g(\tau)|\, d\tau \right) ds < \infty \right\}.$$

It is well known that $L^1(0,1) \subsetneqq \mathcal{H}$. Thus, a function in $\mathcal{H}$ may have stronger singularity at the boundary than a function in $L^1(0,1)$ (see examples in Section 4). If $h_i \in \mathcal{H}$ for all $i = 1, 2, \ldots, N$, then $|h| \in \mathcal{H}$. In this sense, we shall denote $h \in \mathcal{H}$ whenever $h_i \in \mathcal{H}$ for all $i = 1, 2, \ldots, N$.

Scalar equations or systems of p-Laplacian-like problem (P_λ) appear in various applications, which describe reaction-diffusion systems, nonlinear elasticity, glaciology, population biology, combustion theory, and non-Newtonian fluids (see [1–4]). The study on the existence of solutions for p-Laplacian scalar equations or systems or more generalized Laplacian systems has attracted much attention recently (see [5–18] and the references therein).

Among their general setup, a solution operator for nonlinear p-Laplacian systems was introduced in the pioneering works of Manásevich and Mawhin [19, 20]. They applied the solution operator to study the existence of solutions for systems of strongly coupled vector p-Laplacian-like operators with L^1-Carathéodory nonlinear perturbations.

We see that the L^1-Carathéodory condition in problem (P_λ) corresponds to the condition $h \in L^1((0,1), \mathbb{R}^N)$. As a generalization of the L^1-Carathéodory condition, it is interesting to consider the case $h \in \mathcal{H}$. Since our problem involves systems of strongly coupled differential operators and the weight function h may change sign, related studies are not known yet, as far as the authors know. Recently, for a scalar equation of (P_λ), Sim and Lee [21] established a new solution operator and proved an existence result by the global continuation theorem.

Thus, the goal of this paper is to get an existence result for (P_λ) where the differential operator is related to strongly coupled vector p-Laplacian and the weight function has stronger singularity at the boundary than L^1 and sign-changing. The novelty of the paper is providing a new solution operator, which is the most generalized so far.

This paper is organized as follows. In Section 2, we derive a solution operator for problem (W)+(D) with $g \in \mathcal{H}$. In Section 3, we prove the compactness of the solution operator for (P_λ) with $\lambda = 1$. In Section 4, we show the existence of solutions and give some illustrative examples, which satisfy all assumptions in the paper and are not given in other studies.

2 A fixed point operator

In this section, we construct a solution operator for a strongly coupled vector p-Laplacian. Let us consider a problem of the form

$$(W) \qquad -\Psi_p\big(w'\big)' = g(t), \qquad t \in (0,1),$$

$$(D) \qquad w(0) = 0 = w(1),$$

where $g \in \mathcal{H}$. Since g may not be in $L^1((0,1), \mathbb{R}^N)$, the solution of (W)+(D) may not be in $C^1([0,1], \mathbb{R}^N)$. For an example of a simple scalar case, take $g(t) = (p-1)t^{-1}|1 + \ln t|^{p-2}$, $p > 2$; then $g \notin L^1(0,1)$, but $g \in \mathcal{H}$, and the solution u is given by $u(t) = -t \ln t$, which is not in $C^1[0,1]$.

So by a solution to this problem we mean a function $w \in C([0,1], \mathbb{R}^N) \cap C^1((0,1), \mathbb{R}^N)$ with $\Psi_p(w')$ absolutely continuous that satisfies equations (W)+(D).

We first give some remarks for calculations later on.

Remark 2.1 From the definition of Ψ_p and φ_p we get, for any $x, y \in \mathbb{R}^N$,

$$\left|\Psi_p^{-1}(x+y)\right| \leq \varphi_p^{-1}\left(|x|+|y|\right) \leq C_p\left(\varphi_p^{-1}\left(|x|\right)+\varphi_p^{-1}\left(|y|\right)\right),$$

where

$$C_p = \begin{cases} 1, & p > 2, \\ 2^{\frac{2-p}{p-1}}, & 1 < p \leq 2. \end{cases}$$

Remark 2.2 By the homogeneity of φ_p^{-1} we can deduce that if $h \in \mathcal{H}$, then $\alpha \cdot h \in \mathcal{H}$ for all $\alpha \in C([0,1], \mathbb{R}^N)$.

Let w be a solution of (W)+(D). Then integrating both sides of (W) on the intervals $[s, \frac{1}{2}]$ and $[\frac{1}{2}, s]$ for $s \in (0, \frac{1}{2}]$ and $s \in [\frac{1}{2}, 1)$, respectively, we find that (W)+(D) is equivalent to

$$\begin{cases} w'(s) = \Psi_p^{-1}(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau), & w(0) = 0, \quad s \in (0, \frac{1}{2}], \\ w'(s) = \Psi_p^{-1}(a - \int_{\frac{1}{2}}^s g(\tau)\,d\tau), & w(1) = 0, \quad s \in [\frac{1}{2}, 1), \end{cases} \tag{2.1}$$

where $a = \Psi_p(w'(\frac{1}{2}))$. Applying Remark 2.1 with $x = a$ and $y = \int_s^{\frac{1}{2}} g(\tau)\,d\tau$, we get

$$\begin{aligned} \left|\Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right)\right| &\leq \varphi_p^{-1}\left(|a| + \int_s^{\frac{1}{2}} \left|g(\tau)\right|\,d\tau\right) \\ &\leq C_p\varphi_p^{-1}\left(|a|\right) + C_p\varphi_p^{-1}\left(\int_s^{\frac{1}{2}} \left|g(\tau)\right|\,d\tau\right). \end{aligned}$$

Since $g \in \mathcal{H}$, we know that

$$\Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right) \in L^1\left(\left(0, \frac{1}{2}\right]\right), \qquad \Psi_p^{-1}\left(a - \int_{\frac{1}{2}}^s g(\tau)\,d\tau\right) \in L^1\left(\left[\frac{1}{2}, 1\right)\right).$$

Thus, we may integrate both sides of (2.1) on the interval $[0, t]$ for $t \in [0, \frac{1}{2}]$ and on the interval $[t, 1]$ for $t \in [\frac{1}{2}, 1]$, and we get

$$w(t) = \begin{cases} \int_0^t \Psi_p^{-1}(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau)\,ds, & t \in [0, \frac{1}{2}], \\ \int_t^1 \Psi_p^{-1}(-a + \int_{\frac{1}{2}}^s g(\tau)\,d\tau)\,ds, & t \in [\frac{1}{2}, 1]. \end{cases}$$

We need to check that $w(\frac{1}{2}^-) = w(\frac{1}{2}^+)$. For $a \in \mathbb{R}^N$, define

$$G_g(a) = \int_0^{\frac{1}{2}} \Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right) ds - \int_{\frac{1}{2}}^1 \Psi_p^{-1}\left(-a + \int_{\frac{1}{2}}^s g(\tau)\,d\tau\right) ds. \tag{2.2}$$

Then the function $G_g : \mathbb{R}^N \to \mathbb{R}^N$ is well defined. If G_g has a unique zero, then $w(\frac{1}{2}^-) = w(\frac{1}{2}^+)$. For this, we give the following lemma.

Lemma 2.3 *For given $g \in \mathcal{H}$, the function G_g defined in* (2.2) *has a unique zero $a = a(g)$ in $\mathbb{R}^N$.*

Proof I. Existence. We claim that there exists $r > 0$ such that $\langle G_g(a), a\rangle > 0$ for all $a \in \partial B_r(0) \subset \mathbb{R}^N$. If the claim is valid, then we consider the homotopy

$$h(\lambda, a) = \lambda a + (1-\lambda)G_g(a) \quad \text{for } \lambda \in [0,1].$$

By the claim,

$$\bigl\langle h(\lambda, a), a\bigr\rangle = \lambda\langle a, a\rangle + (1-\lambda)\bigl\langle G_g(a), a\bigr\rangle > 0$$

for any $a \in \partial B_r(0)$, $\lambda \in [0,1]$. Taking $\Omega = B_r(0)$, we see that the Brouwer degree $d_B(h(\lambda, a), \Omega, 0)$ is well defined, and by the homotopy invariance property we get

$$d_B\bigl(G_g(\cdot), \Omega, 0\bigr) = d_B\bigl(h(0,a), \Omega, 0\bigr) = d_B\bigl(h(1,a), \Omega, 0\bigr) = d_B(id, \Omega, 0) = 1$$

since $0 \in \Omega$. This completes the proof of the existence of a zero of G_g. We now prove the claim. For convenience, we denote

$$H_g(a) \triangleq \int_0^{\frac{1}{2}} \Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right) ds, \qquad W_g(a) \triangleq \int_{\frac{1}{2}}^{1} \Psi_p^{-1}\left(-a + \int_{\frac{1}{2}}^{s} g(\tau)\,d\tau\right) ds.$$

Then it suffices to show that there exists $r > 0$ such that $\langle H_g(a), a\rangle > 0$ and $\langle W_g(a), a\rangle < 0$ for all $a \in \partial B_r(0) \subset \mathbb{R}^N$. Indeed, we have

$$\begin{aligned}\bigl\langle H_g(a), a\bigr\rangle &= \int_0^{\frac{1}{2}} \left\langle \Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right), a\right\rangle ds \\ &= \int_0^{\delta} \left\langle \Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right), a\right\rangle ds + \int_\delta^{\frac{1}{2}} \left\langle \Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right), a\right\rangle ds,\end{aligned}$$

where $\delta \in (0, \frac{1}{2})$ will be determined later. Since $g \in \mathcal{H}$, both integrations are well defined, and we denote

$$\begin{aligned}H_{1,\delta} &\triangleq \int_0^{\delta} \left\langle \Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right), a\right\rangle ds, \\ H_{2,\delta} &\triangleq \int_\delta^{\frac{1}{2}} \left\langle \Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right), a\right\rangle ds.\end{aligned}$$

We first consider $H_{1,\delta}$. Since

$$\left|\int_s^{\frac{1}{2}} g(\tau)\,d\tau\right| \le \int_s^{\frac{1}{2}} \bigl|g(\tau)\bigr|\,d\tau,$$

applying Remark 2.1, we obtain

$$\begin{aligned}|H_{1,\delta}| &\le \int_0^{\delta} \left|\left\langle \Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right), a\right\rangle\right| ds \le \int_0^{\delta} \left|\Psi_p^{-1}\left(a + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\right)\right| |a|\,ds \\ &\le \int_0^{\delta} \varphi_p^{-1}\left(|a| + \left|\int_s^{\frac{1}{2}} g(\tau)\,d\tau\right|\right)|a|\,ds \le \int_0^{\delta} \varphi_p^{-1}\left(|a| + \int_s^{\frac{1}{2}} \bigl|g(\tau)\bigr|\,d\tau\right)|a|\,ds\end{aligned}$$

$$\begin{aligned}&\leq \int_0^{\delta} C_p\left(\varphi_p^{-1}\big(|a|\big)+\varphi_p^{-1}\left(\int_s^{\frac{1}{2}}\big|g(\tau)\big|\,d\tau\right)\right)|a|\,ds\\&= C_p\delta|a|^{p^*}+C_p\left[\int_0^{\delta}\varphi_p^{-1}\left(\int_s^{\frac{1}{2}}\big|g(\tau)\big|\,d\tau\right)ds\right]|a|,\end{aligned}$$

where $p^* = \frac{p}{p-1}$. Thus, we get

$$\begin{aligned}H_{1,\delta} &\geq -C_p\delta|a|^{p^*}-C_p\left[\int_0^{\delta}\varphi_p^{-1}\left(\int_s^{\frac{1}{2}}\big|g(\tau)\big|\,d\tau\right)ds\right]|a|\\&= |a|^{p^*}\left[-C_p\delta-C_p\left[\int_0^{\delta}\varphi_p^{-1}\left(\int_s^{\frac{1}{2}}\big|g(\tau)\big|\,d\tau\right)ds\right]\frac{1}{|a|^{p^*-1}}\right].\end{aligned}\tag{2.3}$$

Now we consider $H_{2,\delta}$. Since $\langle\Psi_p(x),x\rangle = |x|^p$, $x\in\mathbb{R}^N$, we see that

$$\big\langle\Psi_p^{-1}(x),x\big\rangle = \big|\Psi_p^{-1}(x)\big|^p = |x|^{(p^*-1)p} = |x|^{p^*}.$$

Moreover, for $s\in[\delta,\frac{1}{2}]$, $|\int_s^{\frac{1}{2}} g(\tau)\,d\tau| \leq \int_\delta^{\frac{1}{2}} |g(\tau)|\,d\tau < \infty$; thus, denoting $\int_\delta^{\frac{1}{2}} |g(\tau)|\,d\tau \triangleq M_\delta$, we obtain

$$\begin{aligned}H_{2,\delta} &= \int_\delta^{\frac{1}{2}}\left\langle\Psi_p^{-1}\left(a+\int_s^{\frac{1}{2}}g(\tau)\,d\tau\right),a+\int_s^{\frac{1}{2}}g(\tau)\,d\tau\right\rangle ds\\&\quad-\int_\delta^{\frac{1}{2}}\left\langle\Psi_p^{-1}\left(a+\int_s^{\frac{1}{2}}g(\tau)\,d\tau\right),\int_s^{\frac{1}{2}}g(\tau)\,d\tau\right\rangle ds\\&\geq \int_\delta^{\frac{1}{2}}\left|a+\int_s^{\frac{1}{2}}g(\tau)\,d\tau\right|^{p^*}ds - M_\delta\int_\delta^{\frac{1}{2}}\left|\Psi_p^{-1}\left(a+\int_s^{\frac{1}{2}}g(\tau)\,d\tau\right)\right|ds.\end{aligned}$$

Since $p^* > 1$ and

$$\left|a+\int_s^{\frac{1}{2}}g(\tau)\,d\tau\right| \geq |a| - \left|\int_s^{\frac{1}{2}}g(\tau)\,d\tau\right| \geq |a| - M_\delta$$

for $s\in[\delta,\frac{1}{2}]$, taking $|a|$ large enough to satisfy $|a|-M_\delta>0$, we get

$$\begin{aligned}H_{2,\delta} &\geq \int_\delta^{\frac{1}{2}}\big(|a|-M_\delta\big)^{p^*}ds - M_\delta\int_\delta^{\frac{1}{2}}\big(|a|+M_\delta\big)^{p^*-1}ds\\&= \left(\frac{1}{2}-\delta\right)\big(|a|-M_\delta\big)^{p^*}-\frac{M_\delta}{2}\big(|a|+M_\delta\big)^{p^*-1}\\&= |a|^{p^*}\left[\left(\frac{1}{2}-\delta\right)\left(1-\frac{M_\delta}{|a|}\right)^{p^*}-\frac{M_\delta}{2}\left(1+\frac{M_\delta}{|a|}\right)^{p^*-1}\frac{1}{|a|}\right].\end{aligned}\tag{2.4}$$

Combining (2.3) and (2.4), we get that

$$\begin{aligned}\big\langle H_g(a),a\big\rangle \geq |a|^{p^*}&\left[\left(\frac{1}{2}-\delta\right)\left(1-\frac{M_\delta}{|a|}\right)^{p^*}-\frac{M_\delta}{2}\cdot\left(1+\frac{M_\delta}{|a|}\right)^{p^*-1}\cdot\frac{1}{|a|}\right.\\&\left.\quad-C_p\delta-C_p\int_0^{\delta}\varphi_p^{-1}\left(\int_s^{\frac{1}{2}}\big|g(\tau)\big|\,d\tau\right)ds\cdot\frac{1}{|a|^{p^*-1}}\right].\end{aligned}\tag{2.5}$$

Since $g \in \mathcal{H}$, we have that $\varphi_p^{-1}(\int_s^{\frac{1}{2}} |g(\tau)|\,d\tau) \in L^1(0,\delta]$. Choosing $\delta > 0$ sufficiently small and $|a| = r$ sufficiently large, we can make the right-hand side of (2.5) strictly greater than 0. This implies that there exists $r > 0$ such that $\langle H_g(a), a\rangle > 0$ for all $a \in \partial B_r(0)$. Applying a similar argument, we can show that $\langle W_g(a), -a\rangle > 0$ for all $a \in \partial B_r(0)$. Therefore, we conclude that there exists $r > 0$ such that $\langle G_g(a), a\rangle > 0$ for all $a \in \partial B_r(0)$, and the claim is proved.

II. Uniqueness. Suppose that a_1 and a_2 are two distinct zeros of G_g. Then

$$\bigl\langle G_g(a_1) - G_g(a_2), a_1 - a_2\bigr\rangle = 0.$$

On the contrary,

$$\begin{aligned}
&\bigl\langle G_g(a_1) - G_g(a_2), a_1 - a_2\bigr\rangle \\
&\quad = \bigl\langle H_g(a_1) - H_g(a_2), a_1 - a_2\bigr\rangle + \bigl\langle W(a_2) - W(a_1), a_1 - a_2\bigr\rangle \\
&\quad = \int_0^{\frac{1}{2}} \biggl\langle \Psi_p^{-1}\biggl(a_1 + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\biggr) - \Psi_p^{-1}\biggl(a_2 + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\biggr), a_1 - a_2\biggr\rangle\,ds \\
&\qquad + \int_{\frac{1}{2}}^1 \biggl\langle \Psi_p^{-1}\biggl(-a_2 + \int_{\frac{1}{2}}^s g(\tau)\,d\tau\biggr) - \Psi_p^{-1}\biggl(-a_1 + \int_{\frac{1}{2}}^s g(\tau)\,d\tau\biggr), a_1 - a_2\biggr\rangle\,ds.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\bigl\langle G_g(a_1) - G_g(a_2), a_1 - a_2\bigr\rangle \\
&\quad = \int_0^{\frac{1}{2}} \biggl\langle \Psi_p^{-1}\biggl(a_1 + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\biggr) - \Psi_p^{-1}\biggl(a_2 + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\biggr), \\
&\qquad \biggl(a_1 + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\biggr) - \biggl(a_2 + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\biggr)\biggr\rangle\,ds \\
&\qquad + \int_{\frac{1}{2}}^1 \biggl\langle \Psi_p^{-1}\biggl(-a_2 + \int_{\frac{1}{2}}^s g(\tau)\,d\tau\biggr) - \Psi_p^{-1}\biggl(-a_1 + \int_{\frac{1}{2}}^s g(\tau)\,d\tau\biggr), \\
&\qquad \biggl(-a_2 + \int_{\frac{1}{2}}^s g(\tau)\,d\tau\biggr) - \biggl(-a_1 + \int_{\frac{1}{2}}^s g(\tau)\,d\tau\biggr)\biggr\rangle\,ds > 0
\end{aligned}$$

since $\langle \Psi_p^{-1}(x) - \Psi_p^{-1}(y), x - y\rangle > 0$ for all $x, y \in \mathbb{R}^N$, $x \neq y$. This contradiction completes the proof of uniqueness. □

Lemma 2.3 implies that if $g \in \mathcal{H}$, then the solution w of (W)+(D) can be represented by

$$w(t) = \begin{cases} \int_0^t \Psi_p^{-1}(a(g) + \int_s^{\frac{1}{2}} g(\tau)\,d\tau)\,ds, & t \in [0, \frac{1}{2}], \\ \int_t^1 \Psi_p^{-1}(-a(g) + \int_{\frac{1}{2}}^s g(\tau)\,d\tau)\,ds, & t \in [\frac{1}{2}, 1], \end{cases} \tag{2.6}$$

where $a(g) \in \mathbb{R}^N$ satisfies

$$\int_0^{\frac{1}{2}} \Psi_p^{-1}\biggl(a(g) + \int_s^{\frac{1}{2}} g(\tau)\,d\tau\biggr)\,ds = \int_{\frac{1}{2}}^1 \Psi_p^{-1}\biggl(-a(g) + \int_{\frac{1}{2}}^s g(\tau)\,d\tau\biggr)\,ds. \tag{2.7}$$

We note that $a(g)$ is determined uniquely up to g, and from this uniqueness property the following corollary is obvious.

Corollary 2.4 *Let $g \in \mathcal{H}$, Then, as a function of g, a is homogeneous, that is,*

$$a(\lambda g) = \lambda a(g) \quad \text{for all } \lambda \in \mathbb{R}.$$

On the other hand, it is not hard to see that the function w defined in (2.6) satisfies $w \in C([0,1],\mathbb{R}^N) \cap C^1((0,1),\mathbb{R}^N)$, $\Psi_p(w')$ is absolutely continuous on $(0,1)$, and w satisfies (W)+(D). Therefore, we conclude that if $g \in \mathcal{H}$, then w is a solution of (W)+(D) if and only if w satisfies (2.6).

3 Compactness of the fixed point operator

Consider a nonlinear problem of the form

$$(P) \quad \begin{cases} -\Psi_p(u')' = h(t) \cdot f(u), & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases}$$

where $h \in \mathcal{H}$ and $f \in C(\mathbb{R}^N,\mathbb{R}^N)$. We note that, by Remark 2.2, $h \cdot f(u) \in \mathcal{H}$. Let us apply the solution representation for (W)+(D) given in (2.6) replacing g with $h \cdot f(u)$. Then we may rewrite problem (P) equivalently as

$$u = T(u),$$

where $T : C([0,1],\mathbb{R}^N) \to C([0,1],\mathbb{R}^N)$ is defined by

$$T(u)(t) = \begin{cases} \int_0^t \Psi_p^{-1}(a(h \cdot f(u)) + \int_s^{\frac{1}{2}} h(\tau) \cdot f(u(\tau))\,d\tau)\,ds, & t \in [0,\frac{1}{2}], \\ \int_t^1 \Psi_p^{-1}(-a(h \cdot f(u)) + \int_{\frac{1}{2}}^s h(\tau) \cdot f(u(\tau))\,d\tau)\,ds, & t \in [\frac{1}{2},1]. \end{cases}$$

In this section, we prove that the solution operator T is completely continuous. For this, we need two lemmas about the properties of $a(h \cdot f(u))$. Since h and f are fixed, we regard $a(h \cdot f(u))$ as a function of $u \in C([0,1],\mathbb{R}^N)$.

Lemma 3.1 *The function a sends bounded sets in $C([0,1],\mathbb{R}^N)$ into bounded sets in $\mathbb{R}^N$.*

Proof Assume that a sequence $\{u_n\}$ is bounded in $C([0,1],\mathbb{R}^N)$. Let us denote $a_n \triangleq a(h \cdot f(u_n))$ and $G_n \triangleq G_{h\cdot f(u_n)}$. Suppose that $\{a_n\}$ is unbounded in $\mathbb{R}^N$. Then there exists a subsequence $\{a_{n_k}\}$ such that $|a_{n_k}| \to \infty$ as $k \to \infty$. Since each a_{n_k} is a zero of G_{n_k}, we see that $\langle G_{n_k}(a_{n_k}), a_{n_k}\rangle = 0$ for all k. On the other hand, by the same calculation as in the proof of Lemma 2.3 we obtain

$$\begin{aligned} \langle H_{n_k}(a_{n_k}), a_{n_k}\rangle \geq |a_{n_k}|^{p^*}\Biggl[&\left(\frac{1}{2} - \delta\right)\left(1 - \frac{MH_\delta}{|a_{n_k}|}\right)^{p^*} - \frac{MH_\delta}{2} \cdot \left(1 + \frac{MH_\delta}{|a_{n_k}|}\right)^{p^*-1} \cdot \frac{1}{|a_{n_k}|} \\ &- C_p\delta - C_p\varphi_p^{-1}(M)\int_0^\delta \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} \bigl|h(\tau)\bigr|\,d\tau\right)ds \cdot \frac{1}{|a_{n_k}|^{p^*-1}}\Biggr], \end{aligned}$$

where $M = \sup_{k\in\mathbb{N}} \|f(u_{n_k})\|_\infty$ and $H_\delta = \int_\delta^{\frac{1}{2}} |h(\tau)|\,d\tau$. Since $|a_{n_k}| \to \infty$ as $k \to \infty$, we may choose sufficiently large k and then $\delta > 0$ small enough to satisfy $\langle H_{n_k}(a_{n_k}), a_{n_k}\rangle > 0$. Applying a similar argument for W_{n_k}, we conclude that $\langle G_{n_k}(a_{n_k}), a_{n_k}\rangle > 0$ for sufficiently large k, and this contradiction completes the proof. □

Remark 3.2 If B is a bounded set in $C([0,1],\mathbb{R}^N)$, then $\{a(h\cdot v)|v\in B\}$ is also bounded in $\mathbb{R}^N$. The proof is similar to that of Lemma 3.1 by replacing M with $\sup_{v\in B}\|v\|_\infty$.

Lemma 3.3 *The function* $a : C([0,1],\mathbb{R}^N)\to\mathbb{R}^N$ *is continuous.*

Proof Assume that $u_n\to u$ in $C([0,1],\mathbb{R}^N)$. Then for the continuity of a, we need to show that $a(h\cdot f(u_n))\to a(h\cdot f(u))$ in $\mathbb{R}^N$ as $n\to\infty$. Denote again $a_n\triangleq a(h\cdot f(u_n))$. We know that $\{a_n\}$ is bounded in $\mathbb{R}^N$ by Lemma 3.1; thus, it has a convergent subsequence $\{a_{n_k}\}$, which converges to, say, $\hat a\in\mathbb{R}^N$. We first claim that

$$\begin{aligned}&\int_0^{\frac12}\Psi_p^{-1}\left(\hat a+\int_s^{\frac12}h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds\\&\quad=\int_{\frac12}^1\Psi_p^{-1}\left(-\hat a+\int_{\frac12}^s h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds.\end{aligned}\tag{3.1}$$

Indeed, let us take $K=\sup_{n\in\mathbb{N}}|a_n|$, $M=\sup_{n\in\mathbb{N}}\|f(u_n)\|_\infty$ and fix $s\in(0,\frac12]$. Then we get

$$\bigl|h(\tau)\cdot f\bigl(u_{n_k}(\tau)\bigr)\bigr|\le M\bigl|h(\tau)\bigr|$$

for all $\tau\in[s,\frac12]$. Moreover, $h_i\in L^1_{\mathrm{loc}}(0,1)$ implies $|h|\in L^1[s,\frac12]$. Thus, by the continuity of Ψ_p^{-1} and applying the Lebesgue dominated convergence theorem componentwise, we get

$$\lim_{k\to\infty}\Psi_p^{-1}\left(a_{n_k}+\int_s^{\frac12}h(\tau)\cdot f\bigl(u_{n_k}(\tau)\bigr)\,d\tau\right)=\Psi_p^{-1}\left(\hat a+\int_s^{\frac12}h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right).$$

Similarly, for $k\in\mathbb{N}$,

$$\left|\Psi_p^{-1}\left(a_{n_k}+\int_s^{\frac12}h(\tau)\cdot f\bigl(u_{n_k}(\tau)\bigr)\,d\tau\right)\right|\le A+B\varphi_p^{-1}\left(\int_s^{\frac12}\bigl|h(\tau)\bigr|\,d\tau\right),$$

where $A=C_p\varphi_p^{-1}(K)$ and $B=C_p\varphi_p^{-1}(M)$. Since $h\in\mathcal{H}$, the right-hand side of the last inequality is in $L^1(0,\frac12]$. Thus, applying the Lebesgue dominated convergence theorem componentwise again, we have

$$\begin{aligned}&\lim_{k\to\infty}\int_0^{\frac12}\Psi_p^{-1}\left(a_{n_k}+\int_s^{\frac12}h(\tau)\cdot f\bigl(u_{n_k}(\tau)\bigr)\,d\tau\right)ds\\&\quad=\int_0^{\frac12}\Psi_p^{-1}\left(\hat a+\int_s^{\frac12}h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds.\end{aligned}\tag{3.2}$$

By the same argument, for fixed $s\in[\frac12,1)$, we also get

$$\begin{aligned}&\lim_{k\to\infty}\int_{\frac12}^1\Psi_p^{-1}\left(-a_{n_k}+\int_{\frac12}^s h(\tau)\cdot f\bigl(u_{n_k}(\tau)\bigr)\,d\tau\right)ds\\&\quad=\int_{\frac12}^1\Psi_p^{-1}\left(-\hat a+\int_{\frac12}^s h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds.\end{aligned}\tag{3.3}$$

Moreover, by the definition of a_{n_k} given in (2.7), we know that

$$\begin{aligned}&\int_0^{\frac{1}{2}} \Psi_p^{-1}\left(a_{n_k} + \int_s^{\frac{1}{2}} h(\tau)\cdot f\bigl(u_{n_k}(\tau)\bigr)\,d\tau\right)ds\\&\quad= \int_{\frac{1}{2}}^1 \Psi_p^{-1}\left(-a_{n_k} + \int_{\frac{1}{2}}^s h(\tau)\cdot f\bigl(u_{n_k}(\tau)\bigr)\,d\tau\right)ds.\end{aligned}$$

This implies that both limits in (3.2) and (3.3) are the same, and thus (3.1) is valid. Equation (3.1) implies that $\hat{a} = a(h\cdot f(u))$ by the uniqueness of $\hat{a}$. So we conclude that $\lim_{k\to\infty} a_{n_k}(= a(h\cdot f(u_{n_k}))) = a(h\cdot f(u))$ in $\mathbb{R}^N$. It is not hard to see by the standard subsequence argument that $\lim_{n\to\infty} a_n(= a(h\cdot f(u_n))) = a(h\cdot f(u))$, and the proof is done. □

Remark 3.4 If $v_n \in C([0,1],\mathbb{R}^N)$ with $v_n \to v$ as $n\to\infty$, then $a(h\cdot v_n) \to a(h\cdot v)$ as $n\to\infty$. In particular, if $v = 0$, then $a(h\cdot v_n)\to 0$ as $n\to\infty$. The proof is similar to that of Lemma 3.3 by replacing M with $\sup_{v\in B}\|v\|_\infty$.

Lemma 3.5 *The operator* $T : C([0,1],\mathbb{R}^N) \to C([0,1],\mathbb{R}^N)$ *is completely continuous.*

Proof The continuity of T is easily verified mainly by Lemma 3.1 and the Lebesgue dominated convergence theorem. Let B be a bounded subset of $C([0,1],\mathbb{R}^N)$. Then by the Arzelà-Ascoli theorem, it suffices to show that $T(B)$ is uniformly bounded and equicontinuous. Take $M_B = \sup_{u\in B}\|f(u)\|_\infty$, $K_B = \sup_{u\in B}|a(h\cdot f(u))|$, and denote $a_u \triangleq a(h\cdot f(u))$. Then, for $t\in(0,\frac{1}{2}]$,

$$\begin{aligned}\bigl|T(u)(t)\bigr| &\le \int_0^t \left|\Psi_p^{-1}\left(a_u + \int_s^{\frac{1}{2}} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)\right|ds\\&\le \int_0^t \varphi_p^{-1}\left(K_B + M_B\int_s^{\frac{1}{2}}\bigl|h(\tau)\bigr|\,d\tau\right)ds\\&\le \frac{1}{2}C_p\varphi_p^{-1}(K_B) + C_p\varphi_p^{-1}(M_B)\int_0^{\frac{1}{2}}\varphi_p^{-1}\left(\int_s^{\frac{1}{2}}\bigl|h(\tau)\bigr|\,d\tau\right)ds.\end{aligned}$$

Since $h\in\mathcal{H}$, we see that the last bound is independent of $u\in B$ and $t\in(0,\frac{1}{2}]$. The bound on the interval $[\frac{1}{2},1)$ can be obtained similarly, and thus $T(B)$ is uniformly bounded.

To show the equicontinuity of $T(B)$, let $t_1,t_2\in[0,1]$ with $t_1<t_2$.

Case 1. $t_1,t_2\in[0,\frac{1}{2}]$ or $t_1,t_2\in[\frac{1}{2},1]$. We have

$$\begin{aligned}&\bigl|T(u)(t_1) - T(u)(t_2)\bigr|\\&\quad\le \int_{t_1}^{t_2}\left|\Psi_p^{-1}\left(a_u + \int_s^{\frac{1}{2}} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)\right|ds\\&\quad\le C_p\varphi_p^{-1}(K_B)(t_2-t_1) + C_p\varphi_p^{-1}(M_B)\int_{t_1}^{t_2}\varphi_p^{-1}\left(\int_s^{\frac{1}{2}}\bigl|h(\tau)\bigr|\,d\tau\right)ds.\end{aligned}$$

The bound is independent of $u\in B$ and $\varphi_p^{-1}(\int_s^{\frac{1}{2}}|h(\tau)|\,d\tau)\in L^1(0,\frac{1}{2}]$ since $h\in\mathcal{H}$; thus, we see that the bound converges to 0 as $|t_1-t_2|\to 0$. The case of $t_1,t_2\in[\frac{1}{2},1]$ can be similarly proved.

Case 2. $0 < t_1 \leq \frac{1}{2} < t_2 < 1$. Since t_1 and t_2 can be considered sufficiently close, without loss of generality, we assume that $\frac{1}{4} \leq t_1 \leq \frac{1}{2} < t_2 \leq \frac{3}{4}$. Then, by the definition of T,

$$\begin{aligned} T(u)(t_1) &= \int_0^{t_1} \Psi_p^{-1}\left(a_u + \int_s^{\frac{1}{2}} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds \\ &= \int_0^{\frac{1}{2}} \Psi_p^{-1}\left(a_u + \int_s^{\frac{1}{2}} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds \\ &\quad - \int_{t_1}^{\frac{1}{2}} \Psi_p^{-1}\left(a_u + \int_s^{\frac{1}{2}} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds \end{aligned}$$

and

$$\begin{aligned} T(u)(t_2) &= \int_{t_2}^{1} \Psi_p^{-1}\left(-a_u + \int_{\frac{1}{2}}^{s} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds \\ &= \int_{\frac{1}{2}}^{1} \Psi_p^{-1}\left(-a_u + \int_{\frac{1}{2}}^{s} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds \\ &\quad - \int_{\frac{1}{2}}^{t_2} \Psi_p^{-1}\left(-a_u + \int_{\frac{1}{2}}^{s} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds. \end{aligned}$$

Since, by the definition of a_u,

$$\begin{aligned} &\int_0^{\frac{1}{2}} \Psi_p^{-1}\left(a_u + \int_s^{\frac{1}{2}} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds \\ &\quad = \int_{\frac{1}{2}}^{1} \Psi_p^{-1}\left(-a_u + \int_{\frac{1}{2}}^{s} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds, \end{aligned}$$

we get

$$\begin{aligned} &\bigl|T(u)(t_1) - T(u)(t_2)\bigr| \\ &\quad = \left|\int_{\frac{1}{2}}^{t_2} \Psi_p^{-1}\left(-a_u + \int_{\frac{1}{2}}^{s} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds \right. \\ &\qquad \left. - \int_{t_1}^{\frac{1}{2}} \Psi_p^{-1}\left(a_u + \int_s^{\frac{1}{2}} h(\tau)\cdot f\bigl(u(\tau)\bigr)\,d\tau\right)ds\right| \\ &\quad \leq \int_{\frac{1}{2}}^{t_2} \varphi_p^{-1}\left(K_B + M_B \int_{\frac{1}{2}}^{s} \bigl|h(\tau)\bigr|\,d\tau\right)ds + \int_{t_1}^{\frac{1}{2}} \varphi_p^{-1}\left(K_B + M_B \int_s^{\frac{1}{2}} \bigl|h(\tau)\bigr|\,d\tau\right)ds \\ &\quad \leq \int_{\frac{1}{2}}^{t_2} \varphi_p^{-1}\left(K_B + M_B \int_{\frac{1}{2}}^{\frac{3}{4}} \bigl|h(\tau)\bigr|\,d\tau\right)ds + \int_{t_1}^{\frac{1}{2}} \varphi_p^{-1}\left(K_B + M_B \int_{\frac{1}{4}}^{\frac{1}{2}} \bigl|h(\tau)\bigr|\,d\tau\right)ds. \end{aligned}$$

Thus, using Remark 2.1, we obtain

$$\begin{aligned} &\bigl|T(u)(t_1) - T(u)(t_2)\bigr| \\ &\quad \leq C_p \int_{\frac{1}{2}}^{t_2} \varphi_p^{-1}(K_B)\,ds + C_p \int_{\frac{1}{2}}^{t_2} \varphi_p^{-1}(M_B)\varphi_p^{-1}\left(\int_{\frac{1}{2}}^{\frac{3}{4}} \bigl|h(\tau)\bigr|\,d\tau\right)ds \end{aligned}$$

$$+ C_p \int_{t_1}^{\frac{1}{2}} \varphi_p^{-1}(K_B)\,ds + C_p \int_{t_1}^{\frac{1}{2}} \varphi_p^{-1}(M_B)\varphi_p^{-1}\left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left|h(\tau)\right|d\tau\right)ds$$

$$\leq \left[C_p\varphi_p^{-1}(K_B) + C_p\varphi_p^{-1}(M_B)\varphi_p^{-1}\left(\int_{\frac{1}{4}}^{\frac{3}{4}} \left|h(\tau)\right|d\tau\right)\right](t_2 - t_1).$$

Since the coefficient at $t_2 - t_1$ is a constant independent on $u \in B$, the proof of the equicontinuity of $T(B)$ is complete. □

4 Applications

In this section, we apply the solution operator obtained in Section 2 and use the compactness of the operator in Section 3 to show the existence of nontrivial solutions for the problem

$$(P_\lambda)\qquad \begin{cases} -\Psi_p(u')' = \lambda h(t)\cdot f(u), & t \in (0,1), \\ u(0) = 0 = u(1). \end{cases}$$

For this, we first give one assumption on f.

(F) $f_i(0,\ldots,0) > 0$ and $\lim_{|s|\to\infty} f_i(s)/|s|^{p-1} = 0$ for $s \in \mathbb{R}^N$, $i = 1,\ldots,N$.

Let X be a Banach space, and $G : \mathbb{R}\times X \to X$ be completely continuous with $G(0,u) = 0$. Consider

$$u = G(\lambda, u). \tag{4.1}$$

Denote by $\mathcal{S}$ the set of solutions of (4.1), $\mathbb{R}_+ = [0,\infty)$, and $\mathbb{R}_- = (-\infty, 0]$. As the basic tool for the proof of our main theorem, we introduce the following theorem known as the global continuation theorem.

Theorem 4.1 ([22]) *Let X be a Banach space, and $G : \mathbb{R}\times X \to X$ be continuous and compact with $G(0,u) = 0$. Then $\mathcal{S}$ contains a pair of unbounded components $\mathcal{C}^+$ and $\mathcal{C}^-$ in $\mathbb{R}_+ \times X$ and $\mathbb{R}_- \times X$, respectively, and $\mathcal{C}^+ \cap \mathcal{C}^- = \{(0,0)\}$.*

For our fitting, let us take $X = C([0,1],\mathbb{R}^N)$. Then the usual norm for X to be a Banach space is defined by $\|u\|_\infty = \sum_{i=1}^N \|u_i\|_\infty$. In this paper, for the convenience of computation, we establish an equivalent norm, which is defined by

$$\|u\|_X = \max_{0\leq t\leq 1}\left|\left(u_1(t),\ldots,u_N(t)\right)\right| = \max_{0\leq t\leq 1}\left(u_1^2(t) + \cdots + u_N^2(t)\right)^{1/2}.$$

Indeed, it is easy to see that

$$\|u\|_X \leq \|u\|_\infty \leq N\|u\|_X.$$

We are ready to state our main existence theorem.

Theorem 4.2 *Assume that $h \in \mathcal{H}$ and that (F) holds. Then (P_λ) has at least one nontrivial solution for all $\lambda > 0$.*

We know that to solve (P_λ) is equivalent to solve

$$u = G(\lambda, u),$$

where $G : (0,\infty) \times X \to X$ is defined by

$$G(\lambda,u)(t) = \begin{cases} \int_0^t \Psi_p^{-1}(a(\lambda h \cdot f(u)) + \int_s^{\frac{1}{2}} \lambda h(\tau) \cdot f(u(\tau))\,d\tau)\,ds, & t \in [0,\frac{1}{2}], \\ \int_t^1 \Psi_p^{-1}(-a(\lambda h \cdot f(u)) + \int_{\frac{1}{2}}^s \lambda h(\tau) \cdot f(u(\tau))\,d\tau)\,ds, & t \in [\frac{1}{2},1]. \end{cases}$$

By Remark 2.2 and Lemma 3.5 we can easily show that G is continuous and compact with $G(0,u) = 0$. Since Theorem 4.1 guarantees an unbounded continuum $\mathcal{C}^+$, if we provide the a priori boundedness of solutions for (P_λ), then the unbounded continuum allows the existence of solutions for all $\lambda > 0$.

Lemma 4.3 *Assume that $h \in \mathcal{H}$ and that f satisfies* (F). *Let any $\Lambda > 0$ be given, and let (λ,u) be a solution for (P_λ) with $\lambda \in (0,\Lambda]$. Then there exists a constant $C(\Lambda) > 0$, depending only on Λ, such that $\|u\|_X \le C(\Lambda)$.*

Proof Assume that there exists a sequence $(\lambda_n, u_n) \in (0,\Lambda] \times X$ such that, for any $n \in \mathbb{N}$,

$$u_n = G(\lambda_n, u_n)$$

with $\|u_n\|_X \to \infty$ as $n \to \infty$.

By using Remark 2.1 with $x = a(\lambda_n h \cdot f(u_n))$, $y = \int_s^{\frac{1}{2}} \lambda_n h(\tau) \cdot f(u_n(\tau))\,d\tau$ and the homogeneity of φ_p^{-1} and a we can estimate the solution u_n as follows:

$$\begin{aligned} |u_n(t)| &= \left| \int_0^t \Psi_p^{-1}\left(a(\lambda_n h \cdot f(u_n)) + \int_s^{\frac{1}{2}} \lambda_n h(\tau) \cdot f(u_n(\tau))\,d\tau \right) ds \right| \\ &\le \int_0^t \left| \Psi_p^{-1}\left(a(\lambda_n h \cdot f(u_n)) + \int_s^{\frac{1}{2}} \lambda_n h(\tau) \cdot f(u_n(\tau))\,d\tau \right) \right| ds \\ &\le \int_0^t \varphi_p^{-1}\left(\left|a(\lambda_n h \cdot f(u_n))\right| + \left| \int_s^{\frac{1}{2}} \lambda_n h(\tau) \cdot f(u_n(\tau))\,d\tau \right| \right) ds \\ &\le \varphi_p^{-1}(\lambda_n) \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\left|a(h \cdot f(u_n))\right| + \left| \int_s^{\frac{1}{2}} h(\tau) \cdot f(u_n(\tau))\,d\tau \right| \right) ds \\ &\le \varphi_p^{-1}(\Lambda) \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\frac{|a(h \cdot f(u_n))|}{\|u_n\|_X^{p-1}} + \frac{|\int_s^{\frac{1}{2}} h(\tau) \cdot f(u_n(\tau))\,d\tau|}{\|u_n\|_X^{p-1}} \right) ds \|u_n\|_X \end{aligned}$$

for all $t \in [0,\frac{1}{2}]$. By the homogeneity of a again, we get

$$|u_n(t)| \le \varphi_p^{-1}(\Lambda) \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\left| a\left(h \cdot \frac{f(u_n)}{\|u_n\|_X^{p-1}} \right) \right| + \int_s^{\frac{1}{2}} |h(\tau)| \frac{|f(u_n(\tau))|}{\|u_n\|_X^{p-1}}\,d\tau \right) ds \|u_n\|_X.$$

By (F), for any $\varepsilon > 0$, there exists $l_\epsilon > 0$ such that for all $s \in \mathbb{R}^N$ with $|s| \ge l_\epsilon$,

$$|f_i(s)| \le \varepsilon |s|^{p-1} \quad \text{for } i = 1,\ldots,N.$$

Since f_i is continuous on $\{s \in \mathbb{R}^N \mid |s| \leq l_\epsilon\}$, there exists a constant $M_\epsilon > 0$ such that

$$\left|f_i(s)\right| \leq M_\epsilon$$

on $\{s \in \mathbb{R}^N \mid |s| \leq l_\epsilon\}$ for $i = 1, \dots, N$. Thus, we have

$$\left|f_i(s)\right| \leq \varepsilon |s|^{p-1} + M_\epsilon \quad \text{for all } s \in \mathbb{R}^N, i = 1, \dots, N. \tag{4.2}$$

Since $\|u_n\|_X \to \infty$ as $n \to \infty$, there exists $n_\epsilon \in \mathbb{N}$ such that for any $n \geq n_\epsilon$, we have

$$\|u_n\|_X \geq \left(\frac{M_\epsilon}{\epsilon}\right)^{\frac{1}{p-1}},$$

that is,

$$\frac{1}{\|u_n\|_X^{p-1}} \leq \frac{\epsilon}{M_\epsilon}.$$

Using (4.2), we get that, for any $n \geq n_\epsilon$ and $t \in [0, 1/2]$,

$$\frac{|f_i(u_n(t))|}{\|u_n\|_X^{p-1}} \leq \epsilon \cdot \frac{|u_n(t)|^{p-1}}{\|u_n\|_X^{p-1}} + \frac{M_\epsilon}{\|u_n\|_X^{p-1}} \leq \epsilon + M_\epsilon \cdot \frac{\epsilon}{M_\epsilon} = 2\epsilon$$

and

$$\frac{\|f(u_n)\|_X}{\|u_n\|_X^{p-1}} \leq \frac{\|f(u_n)\|_\infty}{\|u_n\|_X^{p-1}} = \frac{\sum_{i=1}^N \|f_i(u_n)\|_\infty}{\|u_n\|_X^{p-1}} \leq N \cdot 2\epsilon = 2\epsilon N. \tag{4.3}$$

Take

$$B = \left\{\frac{f(u_n)}{\|u_n\|_X^{p-1}}\right\}_{n \geq n_\epsilon}.$$

Then B is a bounded subset in X. Thus, by Remark 3.2 we see that the set $\{a(h \cdot v) \mid v \in B\}$ is bounded in $\mathbb{R}^N$. Moreover, by (4.3) and Remark 3.4 we may choose a constant $C_\epsilon = C_\epsilon(\epsilon N) > 0$ satisfying $C_\epsilon \to 0$ as $\epsilon \to 0$ such that

$$\left|a\left(h \cdot \frac{f(u_n)}{\|u_n\|_X^{p-1}}\right)\right| \leq C_\epsilon \quad \text{for any } n \geq n_\epsilon.$$

Therefore, for $t \in [0, \frac{1}{2}]$, we obtain

$$\begin{aligned} \left|u_n(t)\right| &\leq \left[\varphi_p^{-1}(\Lambda) \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(C_\epsilon + 2\epsilon \int_s^{\frac{1}{2}} \left|h(\tau)\right| d\tau\right) ds\right] \|u_n\|_X \\ &\leq \left[\frac{1}{2} \varphi_p^{-1}(\Lambda) C_p \varphi_p^{-1}(C_\epsilon) \right. \\ &\quad \left. + \varphi_p^{-1}(\Lambda) C_p \varphi_p^{-1}(2\epsilon) \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} \left|h(\tau)\right| d\tau\right) ds\right] \|u_n\|_X. \end{aligned} \tag{4.4}$$

By similar arguments, for $t \in [\frac{1}{2}, 1]$, we obtain

$$\begin{aligned}
\left|u_n(t)\right| &\leq \left[\varphi_p^{-1}(\Lambda) \int_{\frac{1}{2}}^{1} \varphi_p^{-1}\left(C_\epsilon + 2\epsilon \int_{\frac{1}{2}}^{s} \left|h(\tau)\right| d\tau\right) ds\right] \|u_n\|_X \\
&\leq \left[\frac{1}{2}\varphi_p^{-1}(\Lambda)C_p\varphi_p^{-1}(C_\epsilon) \right. \\
&\quad \left. + \varphi_p^{-1}(\Lambda)C_p\varphi_p^{-1}(2\epsilon) \int_{\frac{1}{2}}^{1} \varphi_p^{-1}\left(\int_{\frac{1}{2}}^{s} \left|h(\tau)\right| d\tau\right) ds\right] \|u_n\|_X. \qquad (4.5)
\end{aligned}$$

Denoting $C_h \triangleq \max\{\int_0^{\frac{1}{2}} \varphi_p^{-1}(\int_s^{\frac{1}{2}} |h(\tau)|\, d\tau)\, ds, \int_{\frac{1}{2}}^1 \varphi_p^{-1}(\int_{\frac{1}{2}}^s |h(\tau)|\, d\tau)\, ds\}$, we can choose $\epsilon > 0$ small enough such that

$$\frac{1}{2}\varphi_p^{-1}(\Lambda)C_p\varphi_p^{-1}(C_\epsilon) + \varphi_p^{-1}(\Lambda)C_p\varphi_p^{-1}(2\epsilon)C_h \leq \frac{1}{2}.$$

Consequently, combining (4.4) and (4.5), we obtain, for $t \in [0, 1]$,

$$\left|u_n(t)\right| \leq \frac{1}{2}\|u_n\|_X.$$

This implies that

$$\|u_n\|_X \leq 0 \quad \text{for } n \geq n_\epsilon,$$

which contradicts

$$\|u_n\|_X \geq \left(\frac{M_\epsilon}{\epsilon}\right)^{\frac{1}{p-1}} > 0 \quad \text{for } n \geq n_\epsilon$$

and this completes the proof. □

Example 1 Consider the following p-Laplacian system:

$$(E_1) \quad \begin{cases} -(|\mathbf{u}|^{p-2}u')' = \lambda h_1(t)[(u^2 + v^2)^{\frac{p-1}{4}} + 1], \\ -(|\mathbf{u}|^{p-2}v')' = \lambda h_2(t)e^{-v^2}[1 + (u^2)^{\frac{p-1}{3}}], \quad t \in (0, 1), \\ u(0) = v(0) = 0 = u(1) = v(1), \end{cases}$$

where $\mathbf{u} = (u, v)$, $\lambda > 0$ is a parameter, and $h(t) = (h_1(t), h_2(t))$ is given by

$$h_1(t) = \begin{cases} t^{-\alpha}, & t \in (0, \frac{1}{2}], \\ -1, & t \in (\frac{1}{2}, 1), 1 < \alpha < p, \end{cases} \qquad h_2(t) = -1, \quad t \in (0, 1).$$

We note that $h \in L^1_{\text{loc}}$ but $h_1 \notin L^1$. We now show that $h \in \mathcal{H}$. Indeed,

$$\begin{aligned}
\int_s^{\frac{1}{2}} \tau^{-\alpha}\, d\tau &= -\frac{1}{\alpha - 1}\tau^{-(\alpha-1)}\Big|_s^{\frac{1}{2}} = -\frac{1}{\alpha - 1}\left[\left(\frac{1}{2}\right)^{-(\alpha-1)} - s^{-(\alpha-1)}\right] \\
&= \frac{1}{\alpha - 1}\left[s^{-(\alpha-1)} - 2^{\alpha-1}\right] \leq \frac{1}{\alpha - 1}s^{-(\alpha-1)}.
\end{aligned}$$

Since $1 < \alpha < p$, we have $\frac{1}{\alpha-1}s^{-(\alpha-1)} > 0$ for $s \in (0,1)$ and

$$\begin{aligned}\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} \tau^{-\alpha}\,d\tau\right) ds &\le \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\frac{1}{\alpha-1}s^{-(\alpha-1)}\right) ds = \int_0^{\frac{1}{2}} \left(\frac{s^{-(\alpha-1)}}{\alpha-1}\right)^{\frac{1}{p-1}} ds \\ &= \frac{p-1}{(\alpha-1)^{\frac{1}{p-1}}(p-\alpha)} s^{\frac{p-\alpha}{p-1}}\Big|_0^{\frac{1}{2}} < \infty.\end{aligned}$$

In addition, since h_1 and h_2 are constants on $(\frac{1}{2},1)$ and $(0,1)$, respectively, by Remark 2.1 we get $h \in \mathcal{H}$.

Next, we need to check that both $f_1(u,v) = (u^2+v^2)^{\frac{p-1}{4}} + 1$ and $f_2(u,v) = e^{-v^2}[1+(u^2)^{\frac{p-1}{3}}]$ satisfy assumption (F). In fact, $f_1(0,0) = f_2(0,0) = 1 > 0$, and

$$\begin{aligned}\lim_{|(u,v)|\to\infty} \frac{f_1(u,v)}{|(u,v)|^{p-1}} &= \lim_{|(u,v)|\to\infty} \frac{(u^2+v^2)^{\frac{p-1}{4}}+1}{(u^2+v^2)^{\frac{p-1}{2}}} \\ &= \lim_{|(u,v)|\to\infty}\left(\frac{1}{(u^2+v^2)^{\frac{p-1}{4}}} + \frac{1}{(u^2+v^2)^{\frac{p-1}{2}}}\right) = 0,\\ 0 \le \lim_{|(u,v)|\to\infty} \frac{f_2(u,v)}{|(u,v)|^{p-1}} &= \lim_{|(u,v)|\to\infty} \frac{e^{-v^2}[1+(u^2)^{\frac{p-1}{3}}]}{(u^2+v^2)^{\frac{p-1}{2}}} \\ &\le \lim_{|(u,v)|\to\infty}\left(\frac{1}{e^{v^2}(u^2+v^2)^{\frac{p-1}{2}}} + \frac{1}{e^{v^2}(u^2+v^2)^{\frac{p-1}{6}}}\right) = 0.\end{aligned}$$

that is, $\lim_{|(u,v)|\to\infty} \frac{f_2(u,v)}{|(u,v)|^{p-1}} = 0$. Consequently, by Theorem 4.2 we see that problem (E_1) has at least one nontrivial solution for all $\lambda > 0$.

Example 2 Consider the following p-Laplacian system with $p = 6$:

$$(E_2)\qquad \begin{cases} -(|\mathbf{u}|^4u')' = \lambda h_1(t)[1-(u^2+v^2)^{\frac{5}{3}}], \\ -(|\mathbf{u}|^4v')' = \lambda h_2(t)[2-e^{-(u^2+v^4)}], \quad t \in (0,1), \\ u(0) = v(0) = 0 = u(1) = v(1), \end{cases}$$

where $\mathbf{u} = (u,v)$, $\lambda > 0$ is a parameter, and $h(t) = (h_1(t), h_2(t))$ is given by

$$h_1(t) = \begin{cases} t^{-2}, & t \in (0,\frac{1}{2}], \\ -1, & t \in (\frac{1}{2},1), \end{cases}$$

and

$$h_2(t) = \begin{cases} t^{-4}, & t \in (0,\frac{1}{2}], \\ 1, & t \in (\frac{1}{2},1). \end{cases}$$

By similar arguments as in Example 1, we can easily check that $h \in \mathcal{H}$ and f_1, f_2 satisfy assumption (F). Consequently, by Theorem 4.2 we see that problem (E_2) has at least one nontrivial solution for all $\lambda > 0$.

Example 3 Consider the following p-Laplacian system:

$$(E_3)\qquad \begin{cases} -(|\mathbf{u}|^{p-2}u_1')' = \lambda h_1(t)\ln((u_1^2+\cdots+u_N^2)^{\frac{1}{2}}+2), \\ \vdots \\ -(|\mathbf{u}|^{p-2}u_N')' = \lambda h_N(t)\ln((u_1^2+\cdots+u_N^2)^{\frac{1}{2}}+N+1), \quad t\in(0,1), \\ u_i(0)=0=u_i(1), \quad i=1,\dots,N, \end{cases}$$

where $\mathbf{u}=(u_1,\dots,u_N)$, $\lambda>0$ is a parameter, $h(t)=(h_1(t),\dots,h_N(t))$ is defined by

$$h_i(t) = \frac{1}{t^\alpha(1-t)^\alpha} - 4^p, \quad t\in(0,1), 1<\alpha<p, i=1,\dots,N,$$

and

$$f_i(u_1,\dots,u_N) = \ln\bigl((u_1^2+\cdots+u_N^2)^{\frac{1}{2}}+i+1\bigr), \quad i=1,\dots,N.$$

We note that each h_i is not in $L^1(0,1)$, $h_i(\frac{1}{2})=4^\alpha-4^p<0$ for $1<\alpha<p$, and $h:(0,1)\to\mathbb{R}^N$ is locally integrable. By similar arguments as in Example 1, we can easily check that $h\in\mathcal{H}$.

Next, let us check (F) for $f_i(u_1,\dots,u_N)=\ln((u_1^2+\cdots+u_N^2)^{\frac{1}{2}}+i+1)$. In fact, $f_i(0,\dots,0)=\ln(i+1)>0$, and setting $x:=(u_1^2+\cdots+u_N^2)^{\frac{1}{2}}$, we have

$$\begin{aligned} 0 &\le \lim_{|(u_1,\dots,u_N)|\to\infty} \frac{f_i(u_1,\dots,u_N)}{|(u_1,\dots,u_N)|^{p-1}} = \lim_{|(u_1,\dots,u_N)|\to\infty} \frac{\ln((u_1^2+\cdots+u_N^2)^{\frac{1}{2}}+i+1)}{(u_1^2+\cdots+u_N^2)^{\frac{p-1}{2}}} \\ &= \lim_{x\to+\infty} \frac{\ln(x+i+1)}{x^{p-1}} \\ &= \lim_{x\to+\infty} \frac{1}{x+i+1}\cdot\frac{1}{(p-1)x^{p-2}} \\ &\le \lim_{x\to+\infty} \frac{1}{(p-1)x^{p-1}} = 0, \end{aligned}$$

that is, $\lim_{|(u_1,\dots,u_N)|\to\infty} \frac{f_i(u_1,\dots,u_N)}{|(u_1,\dots,u_N)|^{p-1}} = 0$ for $i=1,\dots,N$. Consequently, by Theorem 4.2 we see that problem (E_3) has at least one nontrivial solution for all $\lambda>0$.

Competing interests
The authors declare that they have no competing interests for this paper.

Authors' contributions
All authors have equally contributed in obtaining new results in this article and also read and approved the final manuscript.

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Acknowledgements
The second author was supported by the National Research Foundation of Korea, Grant funded by the Korea Government (MEST) (NRF2012R1A1A2000739). The third author was supported by the National Research Foundation of Korea, Grant funded by the Korea Government (MEST) (NRF2014R1A1A2056339).

Received: 11 January 2016 Accepted: 30 March 2016 Published online: 05 April 2016

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