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Existence and iterative solutions of a new kind of Sturm-Liouville-type boundary value problem with one-dimensional p-Laplacian

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Abstract
We study a kind of Sturm-Liouville-type four-point boundary value problems. The main tool is monotone iteration theory.

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1 Introduction

In this paper, we are concerned with the following Sturm-Liouville-type four-point boundary value problem with one-dimensional p-Laplacian:

$$\begin{cases} (\phi_p(x'(t)))' + h(t)f(t,x(t),x'(t)) = 0, & 0 < t < 1, \\ x'(0) - \alpha x(\xi) = 0, & x'(1) + \beta x(\eta) = 0, \end{cases} \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $0 < \alpha \leq \frac{1}{\xi}$, $0 < \beta \leq \frac{1}{1-\eta}$, $0 < \xi < \eta < 1$. By applying the monotone iterative technique, we not only prove the existence of positive solutions for the problem, but also establish iterative schemes for approximating the solutions.

We will assume throughout:

(C_1) $h(t) \in L(0,1)$ is nonnegative on $(0,1)$ and is not identically zero on any subset of $(0,1)$.
(C_2) $f \in C([0,1] \times [0,+\infty) \times R, [0,+\infty))$, $f(t,0,0) \not\equiv 0$ for $0 \leq t \leq 1$.

Boundary value problems (BVPs) have been studied for a long period. At the beginning, most researchers focused on two-point BVPs with four classical boundary conditions (BCs) of Dirichlet type $u(0) = u(1) = 0$, Neumann type $u'(0) = u'(1) = 0$, Robin type $u(0) = u'(1) = 0$ or $u'(0) = u(1) = 0$, and Sturm-Liouville type $\alpha u(0) - \beta u'(0) = 0$, $\gamma u(1) + \delta u'(1) = 0$. Later, in order to meet the requirements of various applications, some researchers began to pay their attentions on multipoint BVPs, such as three-point BC $u(0) = \alpha u(\eta)$, $u(1) = 0$ or $u'(0) = 0$, $u(1) = \alpha u(\eta)$, and so on. Although the points involved are larger than that involved in two-point BC, the difficulties remain similar. However, when we study this kind of four-point BVPs, difficulties have a qualitative leap.

Recently, some research articles on the theory of positive solutions to multipoint BVPs have appeared [1–5]. More recently, in [6–8], BVPs subject to the boundary conditions

$$\alpha x(0) - \beta x'(\xi) = 0, \qquad \gamma x(1) + \delta x'(\eta) = 0 \tag{1.2}$$

(Sturm-Liouville-type BC) were studied. Notice that BC in equation (1.1) can also be seen as a Sturm-Liouville-type BC. However, to the best knowledge of the authors, such a kind of BVPs has been rarely considered up to now. The reason is that it is not easy to convert BVP (1.1) to its equivalent integral equation. In this paper, we overcome this difficulty and also get its iterative solutions. The main tool is the monotone iterative technique. For more references, we refer the readers to [9–11].

2 Background material

In the following, there are some lemmas.

Definition 2.1 A map α is said to be a nonnegative concave continuous function if α: $P \to [0,\infty)$ is continuous and

$$\alpha\bigl(\lambda x + (1-\lambda)y\bigr) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y)$$

for all $x, y \in P$ and $0 \leq \lambda \leq 1$.

By ϕ_q we denote the inverse to ϕ_p, where $\frac{1}{p} + \frac{1}{q} = 1$. Consider the following BVP:

$$\begin{cases} (\phi_p(x'(t)))' + \nu(t) = 0, & 0 < t < 1, \\ x'(0) - \alpha x(\xi) = 0, & x'(1) + \beta x(\eta) = 0. \end{cases} \tag{2.1}$$

Let

$$B_1(t) = \frac{1}{\alpha}\phi_q\biggl(\int_0^t \nu(s)\,\mathrm{d}s\biggr) + \int_\xi^t \phi_q\biggl(\int_s^t \nu(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s,$$
$$B_2(t) = \frac{1}{\beta}\phi_q\biggl(\int_t^1 \nu(s)\,\mathrm{d}s\biggr) + \int_t^\eta \phi_q\biggl(\int_t^s \nu(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s.$$

Lemma 2.1 *Suppose that* $\nu \in L[0,1]$, $\nu(t) \geq 0$, *and* $\nu(t) \not\equiv 0$ *on any subinterval of* $[0,1]$. *Then BVP* (2.1) *has the unique solution*

$$x(t) = \begin{cases} \frac{1}{\alpha}\phi_q(\int_0^{\sigma_x} \nu(s)\,\mathrm{d}s) + \int_\xi^t \phi_q(\int_s^{\sigma_x} \nu(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s, & t \in [0,\sigma_x], \quad (2.2_1) \\ \frac{1}{\beta}\phi_q(\int_{\sigma_x}^1 \nu(s)\,\mathrm{d}s) + \int_t^\eta \phi_q(\int_{\sigma_x}^s \nu(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s, & t \in [\sigma_x,1], \quad (2.2_2) \end{cases} \tag{2.2}$$

where σ_x *is a solution of the equation*

$$B_1(t) - B_2(t) = 0, \quad t \in [0,1]. \tag{2.3}$$

Proof We first prove that the solution of (2.1) can be expressed as (2.2). Let x be a solution of BVP (2.1). Then $(\phi_p(x'(t)))' = -\nu(t) \leq 0$ means that $x'(t)$ is nonincreasing. We show that $x'(0) > 0 > x'(1)$, which implies that there exists a point $\sigma \in (0,1)$ such that $x'(\sigma) = 0$.

If not, then, for example, $x'(0) \leq 0$. Then $x'(t) \leq 0$ on $[0,1]$ and $x'(1) < 0$ at the same time. Considering $\xi < \eta$, we have $x(\eta) \leq 0$. Then from the boundary condition in (2.1) we have $x'(1) \geq 0$, a contradiction.

Integrating both sides of

$$-\big(\phi_p\big(x'(t)\big)\big)' = \nu(t) \tag{2.4}$$

from σ to t, we get

$$\phi_p\big(x'(t)\big) = -\int_\sigma^t \nu(s)\,\mathrm{d}s.$$

Then

$$x'(t) = -\phi_q\left(\int_\sigma^t \nu(s)\,\mathrm{d}s\right), \tag{2.5}$$

where q is given by $\frac{1}{p} + \frac{1}{q} = 1$.

Integrating both sides of (2.5) from t to 1, we have

$$x(t) = x(1) + \int_t^1 \phi_q\left(\int_\sigma^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{2.6}$$

By (2.5) and (2.6) we have

$$\begin{aligned} x'(1) &= -\phi_q\left(\int_\sigma^1 \nu(s)\,\mathrm{d}s\right), \\ x(\eta) &= x(1) + \int_\eta^1 \phi_q\left(\int_\sigma^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned}$$

Considering the BC in (2.1), we have

$$x(1) = \frac{1}{\beta}\left(\phi_q\left(\int_\sigma^1 \nu(s)\,\mathrm{d}s\right)\right) - \int_\eta^1 \phi_q\left(\int_\sigma^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{2.7}$$

Substituting (2.7) into (2.6), we obtain

$$\begin{aligned} x(t) &= \frac{1}{\beta}\left(\phi_q\left(\int_\sigma^1 \nu(s)\,\mathrm{d}s\right)\right) - \int_\eta^1 \phi_q\left(\int_\sigma^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \int_t^1 \phi_q\left(\int_\sigma^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= \frac{1}{\beta}\left(\phi_q\left(\int_\sigma^1 \nu(s)\,\mathrm{d}s\right)\right) + \int_t^\eta \phi_q\left(\int_\sigma^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0,1]. \end{aligned} \tag{2.8}$$

By a similar argument we have

$$x(t) = \frac{1}{\alpha}\left(\phi_q\left(\int_0^\sigma \nu(s)\,\mathrm{d}s\right)\right) + \int_\xi^t \phi_q\left(\int_s^\sigma \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0,1]. \tag{2.9}$$

Let $t = \sigma$ in (2.8) and (2.9). Then $B_1(\sigma) = B_2(\sigma)$, that is, σ can be determined by $B_1(t) - B_2(t) = 0$. Next, we show that such a σ is unique.

Clearly, $B_1(t) - B_2(t)$ is increasing on $t \in [0,1]$. It can be easily seen that $B_1(0) - B_2(0) < 0$ and $B_1(1) - B_2(1) > 0$. Indeed,

$$\begin{aligned} B_1(0) &= \int_\xi^0 \phi_q\left(\int_s^0 \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s = \int_0^\xi \phi_q\left(\int_0^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &< \int_0^\eta \phi_q\left(\int_0^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \le B_2(0). \end{aligned}$$

Thus, $B_1(0) - B_2(0) < 0$. Similarly, $B_1(1) - B_2(1) > 0$. Therefore, $B_1(t)$ and $B_2(t)$ must intersect at one point in $(0,1)$, which solves (2.3), that is, σ exists and is unique. This also implies that $x(t)$ defined by (2.1) is continuous at σ.

Since σ has something to do with x, we denote σ by σ_x.

Hence, for $t \in [0,1]$, the solution of (2.1) can be expressed as (2.2), which completes the proof. □

Remark 2.1 In fact, for any $t \in [0,1]$, the solution of (2.1) can be expressed both by (2.2_1) and (2.2_2), but just for convenience, we write it in two parts.

Lemma 2.2 *Let $\nu(t)$ satisfy all the conditions in Lemma* 2.1. *Then the solution $x(t)$ of BVP* (2.1) *is concave on $t \in [0,1]$. Moreover, $x(t) \ge 0$.*

Proof Since $(\phi_p(x'(t)))' = -\nu(t) \le 0$, we have $x''(t) \le 0$, so $x(t)$ is concave on $t \in [0,1]$.

Next, we prove that $x(t) \ge 0$. By Lemma 2.1 we know that $x(t)$ can be expressed as (2.2). When $t \in [\sigma_x, 1]$, since $0 < \beta \le \frac{1}{1-\eta}$, that is, $\frac{1}{\beta} \ge 1 - \eta$, we have

$$\begin{aligned} (2.2_2) &= \frac{1}{\beta}\phi_q\left(\int_{\sigma_x}^1 \nu(s)\,\mathrm{d}s\right) + \int_t^\eta \phi_q\left(\int_{\sigma_x}^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= \frac{1}{\beta}\phi_q\left(\int_{\sigma_x}^1 \nu(s)\,\mathrm{d}s\right) - \int_\eta^1 \phi_q\left(\int_{\sigma_x}^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \int_t^1 \phi_q\left(\int_{\sigma_x}^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\ge \int_\eta^1 \phi_q\left(\int_{\sigma_x}^1 \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \int_\eta^1 \phi_q\left(\int_{\sigma_x}^1 \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \int_t^1 \phi_q\left(\int_{\sigma_x}^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\ge \int_t^1 \phi_q\left(\int_{\sigma_x}^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \ge 0. \end{aligned}$$

Similarly, when $t \in [0, \sigma_x]$ and $0 < \alpha \le \frac{1}{\xi}$, we get $(2.2_1) \ge 0$. Thus, $x(t) \ge 0$ for all $t \in [0,1]$. The proof is complete. □

Let $X = C^1[0,1]$ be endowed with the maximum norm, $\|x\| = \max\{\|x\|_1, \|x'\|_1\}$, where $\|x\|_1 = \max_{0\le t\le 1} |x(t)|$. Define the cone $P \subset X$ as

$$\begin{aligned} P = \bigl\{&x \in X : x \text{ is concave on } t \in [0,1], \\ &\text{and there exists one point } \sigma_x \in (0,1) \text{ such that } x'(\sigma_x) = 0\bigr\}. \end{aligned}$$

For $x, y \in P$, by $x \le y$ we mean that $x(t) \le y(t)$ and $|x'(t)| \le |y'(t)|$ for $t \in [0,1]$.

Define $T : P \to X$ as follows:

$$(Tx)(t) = \begin{cases} \frac{1}{\alpha}\phi_q(\int_0^{\sigma_x} h(s)f(s,x(s),x'(s))\,\mathrm{d}s) \\ \quad + \int_\xi^t \phi_q(\int_s^{\sigma_x} h(\tau)f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, & t \in [0,\sigma_x], \\ \frac{1}{\beta}\phi_q(\int_{\sigma_x}^1 h(s)f(s,x(s),x'(s))\,\mathrm{d}s) \\ \quad + \int_t^\eta \phi_q(\int_{\sigma_x}^s h(\tau)f(\tau,x(\tau),x'(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, & t \in [\sigma_x,1]. \end{cases} \tag{2.10}$$

Lemma 2.3 *For* $x \in P$, $x(t) \ge \min\{t, 1-t\} \max_{0\le t\le 1} |x(t)|$.

Lemma 2.4 *Suppose that* (C_1) *and* (C_2) *hold. Then* $T : P \to P$ *is completely continuous.*

Proof We divide the proof into three steps.

Step 1. We first show that $T : P \to P$ is well defined. Let $x \in P$. Then Tx is concave on $t \in [0,1]$. Indeed, by (2.4),

$$(Tx)'(t) = \begin{cases} \phi_q(\int_t^{\sigma_x} h(s)f(s,x(s),x'(s))\,\mathrm{d}s), & t \in [0,\sigma_x], \\ -\phi_q(\int_{\sigma_x}^t h(s)f(s,x(s),x'(s))\,\mathrm{d}s), & t \in [\sigma_x,1]. \end{cases} \tag{2.11}$$

Obviously, $(Tx)''(t) \le 0$, that is, Tx is concave on $t \in [0,1]$. Further, $(Tx)'(t) \ge 0$ on $t \in [0,\sigma_x]$, $(Tx)'(t) \le 0$ on $t \in [\sigma_x,1]$, and $(Tx)'(\sigma_x) = 0$. Thus, $T : P \to P$ is well defined.

Step 2. T is continuous. Let $x_n \to x_0$ in P. Similarly to Lemma 2.1, there exists a unique σ_{x_n} such that $W_{1,n}(\sigma_{x_n}) = W_{2,n}(\sigma_{x_n})$, where

$$W_{1,n}(t) = \frac{1}{\alpha}\phi_q\left(\int_0^{\sigma_{x_n}} \nu(s)\,\mathrm{d}s\right) + \int_\xi^t \phi_q\left(\int_s^{\sigma_{x_n}} \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,$$
$$W_{2,n}(t) = \frac{1}{\beta}\phi_q\left(\int_{\sigma_{x_n}}^1 \nu(s)\,\mathrm{d}s\right) + \int_{\sigma_{x_n}}^\eta \phi_q\left(\int_t^s \nu(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Meanwhile, we can obtain that $\sigma_{x_n} \to \sigma_{x_0}$ $(n \to +\infty)$, $W_{i,n} \to W_{i,0}$ $(n \to +\infty)$, $i = 1,2$. Let $\underline{\sigma}_n = \min\{\sigma_{x_n}, \sigma_{x_0}\}$ and $\overline{\sigma}_n = \max\{\sigma_{x_n}, \sigma_{x_0}\}$, $n = 1,2,\ldots$. Obviously, when $t \in \Delta_n = [\underline{\sigma}_n, \overline{\sigma}_n]$, $t - \sigma_{x_0} \to 0$ as $n \to +\infty$. Noticing that

$$\begin{aligned} \max_{t\in\Delta_n}\bigl|W_{i,n}(t) - W_{j,0}(t)\bigr| &\le \max_{t\in\Delta_n}\bigl|W_{i,n}(t) - W_{i,n}(\sigma_{x_n})\bigr| + \bigl|W_{j,n}(\sigma_{x_n}) - W_{i,n}(\sigma_{x_n})\bigr| \\ &\quad + \max_{t\in\Delta_n}\bigl|W_{j,0}(\sigma_{x_0}) - W_{j,0}(t)\bigr| \quad \text{as } n \to +\infty, i,j = 1,2, i \ne j, \end{aligned}$$

we have

$$\begin{aligned} &\max_{t\in[0,1]} |Tx_n - Tx_0| \\ &\quad = \max\bigl\{|W_{1,n} - W_{1,0}|_{[0,\underline{\sigma}_n]}, |W_{2,n} - W_{1,0}|_{\Delta_n}, |W_{1,n} - W_{2,0}|_{\Delta_n}, |W_{2,n} - W_{2,0}|_{[\overline{\sigma}_n,1]}\bigr\} \\ &\quad \to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

Similarly, by (2.11) and the continuity of ϕ_q we can prove that

$$\max_{t\in[0,1]}\bigl|(Tx_n)' - (Tx_0)'\bigr| \to 0 \quad \text{as } n \to +\infty.$$

Thus, T is continuous.

It is easy to prove that $T(D)$ is bounded and equicontinuous, where $D \subset P$ is a bounded set. By the Arzelà-Ascoli theorem, $T(D)$ is relatively compact. So $T: P \to P$ is completely continuous. □

Lemma 2.5 *Suppose that* (C_1) *and* (C_2) *hold. Then* T *is increasing with respect to* $x \in P$.

Proof Suppose $x_1, x_2 \in P$, $x_1 \leq x_2$. Then $x_1(t) \leq x_2(t)$ and $|x_1'(t)| \leq |x_2'(t)|$. Let us prove that $Tx_1 \leq Tx_2$. According to the definition of P, we know that there exists $\sigma_{x_2} \in (0,1)$ such that $x_2'(\sigma_{x_2}) = 0$, and considering $|x_1'(t)| \leq |x_2'(t)|$, we have $x_1'(\sigma_{x_2}) = 0$, which means that $\sigma_{x_1} = \sigma_{x_2}$. In what follows, we try to prove that $Tx_1 \leq Tx_2$.

For convenience, we give the notation

$$F_i(t) = h(t)f\bigl(t, x_i(t), x_i'(t)\bigr), \quad i = 1, 2.$$

If $t \in [0, \sigma_{x_1}(\sigma_{x_2})]$, then, in view of (C_2), we have

$$\begin{aligned}
(Tx_2)(t) - (Tx_1)(t) &= \frac{1}{\alpha}\left(\phi_q\left(\int_0^{\sigma_{x_2}} F_2(s)\,\mathrm{d}s\right) - \phi_q\left(\int_0^{\sigma_{x_2}} F_1(s)\,\mathrm{d}s\right)\right) \\
&\quad + \int_\xi^t \left(\phi_q\left(\int_s^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \phi_q\left(\int_s^{\sigma_{x_1}} F_1(\tau)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \\
&= \frac{1}{\alpha}\left(\phi_q\left(\int_0^{\sigma_{x_2}} F_2(s)\,\mathrm{d}s\right) - \phi_q\left(\int_0^{\sigma_{x_2}} F_1(s)\,\mathrm{d}s\right)\right) \\
&\quad + \int_\xi^t \left(\phi_q\left(\int_s^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \phi_q\left(\int_s^{\sigma_{x_2}} F_1(\tau)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \\
&= \frac{1}{\alpha}\left(\phi_q\left(\int_0^{\sigma_{x_2}} F_2(s)\,\mathrm{d}s\right) - \phi_q\left(\int_0^{\sigma_{x_2}} F_1(s)\,\mathrm{d}s\right)\right) \\
&\quad - \int_0^\xi \left(\phi_q\left(\int_s^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \phi_q\left(\int_s^{\sigma_{x_2}} F_1(\tau)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \\
&\quad + \int_0^t \left(\phi_q\left(\int_s^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \phi_q\left(\int_s^{\sigma_{x_2}} F_1(\tau)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \\
&\geq \int_0^\xi \left(\phi_q\left(\int_0^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau\right) - \phi_q\left(\int_0^{\sigma_{x_2}} F_1(\tau)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \\
&\quad - \int_0^\xi \left(\phi_q\left(\int_0^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \phi_q\left(\int_0^{\sigma_{x_2}} F_1(\tau)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \\
&\quad + \int_0^t \left(\phi_q\left(\int_s^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \phi_q\left(\int_s^{\sigma_{x_2}} F_1(\tau)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \\
&= \int_0^t \left(\phi_q\left(\int_s^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s - \phi_q\left(\int_s^{\sigma_{x_2}} F_1(\tau)\,\mathrm{d}\tau\right)\right)\mathrm{d}s \geq 0, \\
(Tx_2)'(t) - (Tx_1)'(t) &= \phi_q\left(\int_t^{\sigma_{x_2}} F_2(s)\,\mathrm{d}s\right) - \phi_q\left(\int_t^{\sigma_{x_1}} F_1(s)\,\mathrm{d}s\right) \\
&= \phi_q\left(\int_t^{\sigma_{x_2}} F_2(s)\,\mathrm{d}s\right) - \phi_q\left(\int_t^{\sigma_{x_2}} F_1(s)\,\mathrm{d}s\right) \geq 0.
\end{aligned}$$

If $t \in [\sigma_{x_1}(\sigma_{x_2}), 1]$, then we can similarly prove that $(Tx_2)(t) - (Tx_1)(t) \geq 0$ and $(Tx_2)'(t) - (Tx_1)'(t) \leq 0$.

To sum up, we have $Tx_1 \leq Tx_2$, which is the desired result. The proof is complete. □

Remark 2.2 We can easily verify that $\phi_q(\int_s^{\sigma_{x_2}} F_2(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s - \phi_q(\int_s^{\sigma_{x_2}} F_1(\tau)\,\mathrm{d}\tau)$ is nonincreasing with respect to $s \in [0, \sigma_{x_2}]$ by calculating its derivative.

3 The existence of positive solutions

Let

$$\begin{aligned}\lambda = \max\biggl\{&\frac{1}{\alpha}\phi_q\biggl(\int_0^{\frac{1}{2}} h(s)\,\mathrm{d}s\biggr) + \int_\xi^{\frac{1}{2}} \phi_q\biggl(\int_s^{\frac{1}{2}} h(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s, \frac{1}{\beta}\phi_q\biggl(\int_{\frac{1}{2}}^1 h(s)\,\mathrm{d}s\biggr) \\ &+ \int_{\frac{1}{2}}^{\eta} \phi_q\biggl(\int_{\frac{1}{2}}^{s} h(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s, \phi_q\biggl(\int_0^1 h(s)\,\mathrm{d}s\biggr)\biggr\} \cdot \biggl(\max\biggl\{\frac{1}{\alpha}, \frac{1}{\beta}\biggr\} + \eta + \frac{5}{4}\biggr).\end{aligned}$$

Theorem 3.1 *Assume that* (C_1) *and* (C_2) *hold. Further, suppose that there exists* $r > 0$ *such that*:

(C_3) $f(t, u_1, v_1) \leq f(t, u_2, v_2)$ *for any* $0 \leq t \leq 1$, $0 \leq u_1 \leq u_2 \leq r$, $0 \leq |v_1| \leq |v_2| \leq r$;
(C_4) $\max_{t\in[0,1]} f(t, r, r) \leq \phi_p(\frac{r}{\lambda})$.

Then the boundary value problem (1.1) *has at least two positive solutions* w^* *and* v^* *in* P *such that*

$$0 < w^* \leq r, \qquad 0 < \bigl|\bigl(w^*\bigr)'\bigr| \leq r,$$

and

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*, \qquad \lim_{n\to\infty} (w_n)' = \lim_{n\to\infty} \bigl(T^n w_0\bigr)' = \bigl(w^*\bigr)',$$

where

$$w_0(t) = \frac{r}{\lambda}\biggl(\max\biggl\{\frac{1}{\alpha}, \frac{1}{\beta}\biggr\} + \eta + t + 1 - t^2\biggr) \cdot \phi_q\biggl(\int_0^1 h(s)\,\mathrm{d}s\biggr)$$

and

$$0 < v^* \leq r, \qquad 0 < \bigl|\bigl(v^*\bigr)'\bigr| \leq r,$$

and

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^*, \qquad \lim_{n\to\infty} (v_n)' = \lim_{n\to\infty} \bigl(T^n v_0\bigr)' = \bigl(v^*\bigr)',$$

where $v_0(t) = 0$, $0 \leq t \leq 1$.

Proof Let $\overline{P}_r = \{u \in P \mid \|u\| \leq r\}$. First, we prove that $T : \overline{P}_r \to \overline{P}_r$. For any $u \in \overline{P}_r$, $\|u\| \leq r$, we have

$$0 \leq u(t) \leq \max_{0\leq t\leq 1} \bigl|u(t)\bigr| \leq \|u\| \leq r, \qquad \bigl|u'(t)\bigr| \leq \max_{0\leq t\leq 1} \bigl|u'(t)\bigr| \leq \|u\| \leq r.$$

Then considering (C_1)-(C_4), we get

$$0 \le f\big(t,u(t),u'(t)\big) \le f(t,r,r) \le \max_{0\le t\le 1} f(t,r,r) \le \phi_p\left(\frac{r}{\lambda}\right).$$

By (2.4) and (2.5) we obtain

$$\begin{aligned}
(Tu)(\sigma_x) &= \frac{1}{\alpha}\phi_q\left(\int_0^{\sigma_x} h(s)f\big(s,x(s),x'(s)\big)\,\mathrm{d}s\right) \\
&\quad + \int_\xi^{\sigma_x} \phi_q\left(\int_s^{\sigma_x} h(\tau)f\big(\tau,x(\tau),x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\le \max\left\{\frac{1}{\alpha}\phi_q\left(\int_0^{\frac{1}{2}} h(s)f\big(s,x(s),x'(s)\big)\,\mathrm{d}s\right)\right. \\
&\quad + \int_\xi^{\frac{1}{2}} \phi_q\left(\int_s^{\frac{1}{2}} h(\tau)f\big(\tau,x(\tau),x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s, \frac{1}{\beta}\phi_q\left(\int_{\frac{1}{2}}^1 h(s)f\big(s,x(s),x'(s)\big)\,\mathrm{d}s\right) \\
&\quad \left. + \int_{\frac{1}{2}}^{\eta} \phi_q\left(\int_{\frac{1}{2}}^{s} h(\tau)f\big(\tau,x(\tau),x'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right\} \\
&\le \frac{r}{\lambda}\cdot\lambda = r,
\end{aligned}$$

and

$$\begin{aligned}
(Tu)'(0) &= \phi_q\left(\int_0^{\sigma_x} h(s)f\big(s,x(s),x'(s)\big)\,\mathrm{d}s\right) \\
&\le \phi_q\left(\int_0^{1} h(s)f\big(s,x(s),x'(s)\big)\,\mathrm{d}s\right) \le \frac{r}{\lambda}\cdot\lambda = r, \\
-(Tu)'(1) &= \phi_q\left(\int_{\sigma_x}^{1} h(s)f\big(s,x(s),x'(s)\big)\,\mathrm{d}s\right) \\
&\le \phi_q\left(\int_0^{1} h(s)f\big(s,x(s),x'(s)\big)\,\mathrm{d}s\right) \le \frac{r}{\lambda}\cdot\lambda = r.
\end{aligned}$$

Thus, we obtain that $\|Tu\| \le r$. So, we have shown that $T:\overline{P}_r \to \overline{P}_r$.

Second, we will establish iterative schemes for approximating the solutions. Let

$$w_0(t) = \frac{r}{\lambda}\big(-t^2+t+c\big)\cdot\phi_q\left(\int_0^1 h(s)\,\mathrm{d}s\right),$$

where $c = \frac{1}{\alpha} + \frac{1}{\alpha} + 1 + \frac{1}{\beta}$. Obviously, $w_0(t) \in P$ and $w_0'(\frac{1}{2}) = 0$. Let $w_1(t) = Tw_0(t)$. Then we have $w_1 \in \overline{P}_r$. We denote $w_{n+1} = Tw_n = T^n w_0$, $n = 1,2,\ldots$. Then we have $w_n \in \overline{P}_r$. Since T is completely continuous, $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set. We have

$$\begin{aligned}
w_1(t) &= Tw_0(t) \\
&= \begin{cases}
\frac{1}{\alpha}\phi_q(\int_0^{\sigma_x} h(s)f(s,w_0(s),w_0'(s))\,\mathrm{d}s) \\
\quad + \int_\xi^t \phi_q(\int_s^{\sigma_x} h(\tau)f(\tau,w_0(\tau),w_0'(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, & t\in[0,\sigma_x], \\
\frac{1}{\beta}\phi_q(\int_{\sigma_x}^1 h(s)f(s,w_0(s),w_0'(s))\,\mathrm{d}s) \\
\quad + \int_t^{\eta} \phi_q(\int_{\sigma_x}^s h(\tau)f(\tau,w_0(\tau),w_0'(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s, & t\in[\sigma_x,1],
\end{cases}
\end{aligned}$$

$$
\begin{aligned}
&\le \begin{cases} \frac{r}{\lambda}(\frac{1}{\alpha}+\xi-t)\cdot\phi_q(\int_0^1 h(s)\,\mathrm{d}s), & 0\le t\le \min\{\xi,\sigma_x\}\le 1, \\ \frac{r}{\lambda}(\frac{1}{\alpha}+t-\xi)\cdot\phi_q(\int_0^1 h(s)\,\mathrm{d}s), & 0\le \xi\le t\le \sigma_x\le 1, \\ \frac{r}{\lambda}(\frac{1}{\beta}+\eta-t)\cdot\phi_q(\int_0^1 h(s)\,\mathrm{d}s), & 0\le \sigma_x\le t\le \eta\le 1, \\ \frac{r}{\lambda}(\frac{1}{\beta}+t-\eta)\cdot\phi_q(\int_0^1 h(s)\,\mathrm{d}s), & 0\le \max\{\eta,\sigma_x\}\le t\le 1 \end{cases} \\
&\le \frac{r}{\lambda}\left(-t^2+t+c\right)\cdot\phi_q\left(\int_0^1 h(s)\,\mathrm{d}s\right) \\
&= w_0(t), \quad 0\le t\le 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|w_1'(t)\right| &= \left|(Tw_0)'(t)\right| \\
&= \begin{cases} |\phi_q(\int_t^{\sigma_{w_0}} h(s)f(s,w_0(s),w_0'(s))\,\mathrm{d}s)|, & t\in[0,\sigma_{w_0}], \\ |-\phi_q(\int_{\sigma_{w_0}}^t h(s)f(s,w_0(s),w_0'(s))\,\mathrm{d}s)|, & t\in[\sigma_{w_0},1], \end{cases} \\
&\le \begin{cases} \frac{r}{\lambda}|\phi_q(\int_t^{\frac{1}{2}} h(s)\,\mathrm{d}s)|, & t\in[0,\frac{1}{2}], \\ \frac{r}{\lambda}|-\phi_q(\int_{\frac{1}{2}}^t h(s)\,\mathrm{d}s)|, & t\in[\frac{1}{2},1], \end{cases} \\
&\le \frac{r}{\lambda}|b-at|\phi_q\left(\int_0^1 h(s)\,\mathrm{d}s\right), \quad t\in[0,1], \\
&= \left|w_0'(t)\right|, \quad 0\le t\le 1.
\end{aligned}
$$

Then we obtain that

$$
w_1(t)\le w_0(t), \qquad \left|w_1'(t)\right|\le\left|w_0'(t)\right|, \quad 0\le t\le 1.
$$

Hence, by Lemma 2.5 we have

$$
\begin{aligned}
&w_2(t)=(Tw_1)(t)\le(Tw_0)(t)=w_1(t), \quad 0\le t\le 1, \\
&\left|w_2'(t)\right|=\left|(Tw_1)'(t)\right|\le\left|(Tw_0)'(t)\right|=\left|w_1'(t)\right|, \quad 0\le t\le 1.
\end{aligned}
$$

Thus, by induction we get

$$
w_{n+1}(t)\le w_n(t), \qquad \left|w_{n+1}'(t)\right|\le\left|w_n'(t)\right|, \quad 0\le t\le 1, n=1,2,\ldots.
$$

So, there exists $w^*\in\overline{P}_r$ such that $w_n\to w^*$. Considering that T is completely continuous and $w_{n+1}=Tw_n$, we have $Tw^*=w^*$.

Let $v_0(t)=0$, $0\le t\le 1$. Then $v_0(t)\in\overline{P}_r$. Let $v_1=Tv_0$; then $v_1\in\overline{P}_r$. We denote $v_{n+1}=Tv_n=T^n v_0$, $n=1,2,\ldots$. Since $T:\overline{P}_r\to\overline{P}_r$, we get $v_n\in T\overline{P}_r\subseteq\overline{P}_r$, $n=1,2,\ldots$. Since T is completely continuous, $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set. We have

$$
\begin{aligned}
&v_1(t)=(Tv_0)(t)=(T0)(t)\ge 0, \quad 0\le t\le 1, \\
&\left|v_1'(t)\right|=\left|(Tv_0)'(t)\right|=\left|(T0)'(t)\right|\ge 0, \quad 0\le t\le 1.
\end{aligned}
$$

Thus,

$$v_2(t) = (Tv_1)(t) \geq (Tv_0)(t) = v_1(t), \quad 0 \leq t \leq 1,$$
$$\left|v_2'(t)\right| = \left|(Tv_1)'(t)\right| \geq \left|(T0)'(t)\right| = \left|v_1'(t)\right|, \quad 0 \leq t \leq 1.$$

Similarly, by induction we obtain

$$v_{n+1}(t) \geq v_n(t), \qquad \left|v_{n+1}'(t)\right| \geq \left|v_n'(t)\right|, \quad 0 \leq t \leq 1, n = 1, 2, \ldots.$$

So, there exists $v^* \in \overline{P}_r$ such that $v_n \to v^*$. Considering that T is completely continuous and $v_{n+1} = Tv_n$, we have $Tv^* = v^*$.

Since $f(t,0,0) \not\equiv 0$ for $0 \leq t \leq 1$, the zero function is not the solution of (1.1). Hence, since $\max|v^*(t)| > 0$, we have $v^*(t) \geq \min\{t, 1-t\} \max_{0\leq t\leq 1}|v^*(t)|$, $0 \leq t \leq 1$.

As we all know, the fixed point of T is a solution of BVP (1.1). Hence, we have shown that w^*, v^* are two positive solutions of problem (1.1).

The proof is complete. □

Remark 3.1 We can see that w^* and v^* may be the same solution of BVP (1.1), but for convenience, we say that there exist at least two solutions.

Corollary 3.2 *Assume that* (C_1) *and* (C_2) *hold. Further, suppose that there exists* $r > 0$ *such that*:

(C_5) $\lim_{l\to+\infty} \max_{0\leq t\leq 1} \frac{f(t,l,r)}{l^{p-1}} \leq \phi_p(\frac{1}{\lambda})$ (*particularly*, $\lim_{l\to+\infty} \max_{0\leq t\leq 1} \frac{f(t,l,r)}{l^{p-1}} = 0$).

Then problem (1.1) *has two positive solutions in* P.

At the end of this paper, we give an example to illustrate our main result.

Consider the following four-point boundary value problem.

Example 1

$$\begin{cases} (\phi_p(x'))' + tf(t, x(t), x'(t)) = 0, & 0 < t < 1, \\ x'(0) - 3x(1/4) = 0, & x'(1) + x(2/3) = 0, \end{cases} \tag{3.1}$$

where

$$f(t,u,v) = t^2 + \frac{u}{6} + \frac{v^2}{100}.$$

We can see that $h(t) = t$, $\xi = \frac{1}{4}$, $\eta = \frac{2}{3}$, $\alpha = 3$, $\beta = 1$. Let $p = \frac{3}{2}$, $r = 14$. By direct calculation we obtain $q = 3$, $\lambda = \frac{35}{36}$. Then the conditions of Theorem 3.1 are all satisfied. So BVP (3.1) has at least two positive solutions w^*, v^*, and there exists $\sigma_x \in (0,1)$ such that $(w^*)'(\sigma_x) = 0$, $(v^*)'(\sigma_x) = 0$. Further,

$$0 \leq w^* \leq 14, \qquad 0 \leq \left|\left(w^*\right)'\right| \leq 14,$$

and

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*, \qquad \lim_{n\to\infty} (w_n)' = \lim_{n\to\infty} \bigl(T^n w_0\bigr)' = \bigl(w^*\bigr)',$$

where

$$w_0(t) = \begin{cases} 8(\frac{5}{3} - t), & 0 \le t \le \frac{11}{24}, \\ 8(t + \frac{3}{4}), & \frac{11}{24} \le t \le 1. \end{cases}$$

At the same time, we have

$$0 < v^* \le 14, \qquad 0 < \bigl|\bigl(v^*\bigr)'\bigr| \le 14,$$

and

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^*, \qquad \lim_{n\to\infty} (v_n)' = \lim_{n\to\infty} \bigl(T^n v_0\bigr)' = \bigl(v^*\bigr)',$$

where $v_0(t) = 0$, $0 \le t \le 1$, and T is as defined in (2.4).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JZ and BS conceived of the study and participated in its coordination. JZ drafted the manuscript, and YW proofread the manuscript. All authors read and approved the final manuscript.

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