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Global existence and uniqueness of solutions to the three-dimensional Boussinesq equations

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Abstract

In this paper, we study the three-dimensional Boussinesq equations and obtain the global existence and uniqueness of a suitable weak solution in unbounded exterior domain.

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1 Introduction

Let $D \subset \mathbf{R}^3$ be a bounded domain with a connected C^2-smooth boundary S, and $D' := \mathbf{R}^3 \backslash D$ be the unbounded exterior domain. We consider following three-dimensional Boussinesq equations:

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = \theta e_3, \quad x \in D', t \geq 0, \tag{1.1}$$

$$\theta_t + u \cdot \nabla \theta - \mu \Delta \theta = 0, \tag{1.2}$$

$$\nabla \cdot u = 0, \tag{1.3}$$

$$u(0,x) = u_0(x), \qquad \theta(0,x) = \theta_0(x), \tag{1.4}$$

$$u|_S = 0, \qquad \theta|_S = 0, \tag{1.5}$$

where $p = p(x,t)$ is the scalar pressure, $u = u(x,t)$ is the velocity vector field, and $\theta = \theta(x,t)$ is a scalar quantity such as the concentration of a chemical substance or the temperature variation in a gravity field, in which case θe_3 represents the buoyancy force. The nonnegative parameters μ and ν denote the molecular diffusion and the viscosity, respectively; $u \cdot \nabla u := u_a \partial_a u$, $\partial_a u := \frac{\partial u}{\partial x_a} := u_{;a}$, $u \cdot \nabla \theta := u_a \partial_a \theta$, $\nabla \cdot u := u_{a;a} = 0$, $\frac{\partial u^2}{\partial x_a} := (u^2)_{;a}$, $u^2 = u_b u_b$. Over the repeated indices a and b summation is understood, $1 \leq a, b \leq 3$.

The Boussinesq system is one of the most commonly used fluid models since it has a vortex stretching effect similar to that in the 3D incompressible flow. The Boussinesq system has an important roles in the atmospheric sciences [1] and is a model in many geophysical applications [2]. For this reason, this system is studied systematically by scientists from different domains.

Before starting and proving our main results, let us first briefly recall the related results in the literature. For the three-dimensional case, the progress on the local well-posedness and regularity criteria on the inviscid Boussinesq equations has been made (see, *e.g.*, [3–11]). Since there is no global well-posedness for the standard high-dimensional Boussinesq equations for large initial data, it is natural to consider the blow-up (or regularity) criterion. Various criteria for the Boussinesq equations have been obtained by considerable works (see, *e.g.*, [3–5, 7–9, 12–17] and the references therein).

In this paper, we establish the global existence and uniqueness of a suitable weak solution to the three-dimensional Boussinesq equations in unbounded exterior domains.

We assume that $u \in G$ and $\theta \in J$, where

$$G := \{f | f \in L^2(0,T;H_0^1(D')) \cap f_t \in L^2(D' \times [0,T]); \nabla \cdot f = 0\},$$
$$J := \{g | g \in L^2(0,T;H_0^1(D')) \cap g_t \in L^2(D' \times [0,T])\},$$

with arbitrary $T > 0$.

In this paper, $(f,g) := \int_{D'} f_a g_a \, dx$ denotes the inner product in $L^2(D')$, $\|f\| := (f,f)^{\frac{1}{2}}$, f_{ja} denotes the ath component of the vector function f_j, and $f_{ja;b}$ is the derivative $\frac{\partial f_{ja}}{\partial x_b}$.

Let us define a suitable weak solution to problem (1.1)-(1.5) that satisfies the identity

$$\begin{aligned}&(u_t, v) + (\theta_t, w) + (\nabla u, \nabla v) + (\nabla\theta, \nabla w) + (u_a u_{b;a}, v_b) + (u_a \theta_{b;a}, w_b)\\&\quad = (\theta e_3, v), \quad \forall v \in G, w \in J.\end{aligned} \tag{1.6}$$

Here we took into account that $-(\Delta u, v) = (\nabla u, \nabla v)$, $-(\Delta\theta, w) = (\nabla\theta, \nabla w)$, and $(\nabla p, v) = -(p, v_{a;a}) = 0$, if $v \in G$, $w \in J$.

Equation (1.6) is equivalent to the integrated equation

$$\begin{aligned}&\int_0^t \bigl[(u_s, v) + (\theta_s, w) + (\nabla u, \nabla v) + (\nabla\theta, \nabla w) + (u_a u_{b;a}, v_b) + (u_a \theta_{b;a}, w_b)\bigr] \, ds\\&\quad = \int_0^t (\theta e_3, v) \, ds, \quad \forall v \in G, w \in J.\end{aligned} \tag{1.7}$$

We also assume that

$$(u_0, u_0) \le C, \qquad (\theta_0, \theta_0) \le C. \tag{1.8}$$

Now we are in position to state our main results.

Theorem 1.1 *If assumption* (1.8) *holds,* $u_0 \in H_0^1(D)$ *satisfies* (1.3), *and* $\theta_0 \in H_0^1(D)$, *then for all* $t \ge 0$, *there exist weak solutions* $u \in G$, $\theta \in J$ *to* (1.6), *and the solutions are unique, provided that* $|\theta(\cdot,t)|$, $\|\nabla u\|^4 \in L^1_{\mathrm{loc}}(0,\infty)$.

Remark 1.1 The proof follows the method of [16, 18], and [19].

2 Proof of Theorem 1.1

In this section, we prove the global existence and uniqueness of suitable weak solutions to problem (1.1)-(1.5). We begin with the following lemma.

Lemma 2.1 *Let $\theta_0 \in L^1 \cap L^2$ and $u_0 \in L^2$. Then the weak solutions u, θ of problem* (1.1)-(1.5) *satisfy*

$$\|\theta(t)\| \le \|\theta_0\|_1 (1 + Ct)^{-\frac{3}{4}}, \qquad \|u(t)\| \le \|u_0\| + C\|\theta_0\|_1 t^{\frac{1}{4}}. \tag{2.1}$$

Proof See, *e.g.*, [20]. □

Lemma 2.2 *Under assumption* (1.8), *the following estimate holds*:

$$\sup_{t\ge 0}\|u(t)\|^2 + \|\theta(t)\|^2 + 2\int_0^t \big[\|\nabla u(s)\|^2 + \|\nabla\theta(s)\|^2\big]\,ds \le C, \tag{2.2}$$

where $C > 0$ is a constant.

Proof Take $v = u$ and $w = \theta$ in (1.6). Then

$$(u_a u_{b;a}, u_b) = -(u_a u_b, u_{b;a}) = -\frac{1}{2}\big(u_a, (u^2)_{;a}\big) = \frac{1}{2}\big(u_{a;a}, u^2\big) = 0,$$
$$(u_a \theta_{b;a}, \theta_b) = -\frac{1}{2}\big(u_a, (\theta^2)_{;a}\big) = \frac{1}{2}\big(u_{a;a}, \theta^2\big) = 0.$$

Thus, equation (1.6) with $v = u$ and $w = \theta$ implies

$$\frac{1}{2}\partial_t\big[(u,u) + (\theta,\theta)\big] + (\nabla u, \nabla u) + (\nabla\theta, \nabla\theta) = (\theta e_3, u). \tag{2.3}$$

Integrating (2.3) over $[0, t]$, by virtue of (1.8), we get the following estimate:

$$\begin{aligned}
&\|u(t)\|^2 + \|\theta(t)\|^2 + 2\int_0^t \big[\|\nabla u(s)\|^2 + \|\nabla\theta(s)\|^2\big]\,ds \\
&\quad\le C + \int_0^t (\theta e_3, u)\,ds \le C + \int_0^t \|\theta\|\|u\|\,ds \\
&\quad\le C + C\sup_{s\in[0,t]}\|u(s)\|\int_0^t (1+s)^{-\frac{5}{4}}\,ds \le C + C\sup_{s\in[0,t]}\|u(s)\|.
\end{aligned} \tag{2.4}$$

Denote $\sup_{s\in[0,t]}\|u(s)\| = \alpha(t)$. Then (2.4) implies

$$\alpha^2(t) \le C + C\alpha(t). \tag{2.5}$$

Since $\alpha(t) \ge 0$, inequality (2.5) yields

$$\sup_{t\ge 0}\alpha(t) \le C,$$

which, together with (2.4), gives (2.2). □

Remark 2.3 *A priori* estimate (2.2) implies that, for every $T \in [0, \infty)$, $u \in L^\infty(0, T; L^2(D')) \cap L^2(0, T; H_0^1(D'))$, $\theta \in L^2(0, T; H_0^1(D'))$. This and equation (1.6) also imply that $u_t, \theta_t \in L^2(D' \times [0, T])$.

Lemma 2.4 *Under the assumptions of Theorem* 1.1, *there exist suitable weak solutions* $u \in G$, $\theta \in J$ *to problem* (1.1)-(1.5).

Proof The idea of the proof is to reduce the problem to the existence of the solution to a Cauchy problem for ordinary differential equations (ODE) of finite order and then to use an *a priori* estimate to establish the convergence of these solutions of ODE to the solutions of equations (1.6) and (1.7).

Let us look for the solutions to equation (1.6) of the form $u^n := \sum_{j=1}^n c_j^n(t)\phi_j(x)$, $\theta^n := \sum_{k=1}^n d_k^n(t)\phi_k(x)$, where $\{\phi_i(x)\}_{i=1}^\infty$ is an orthonormal basis of the space $L^2(D')$ of divergence-free vector functions belonging to $H_0^1(D')$, and in the expression u^n, θ^n, the upper index n is not a power. If we substitute u^n, θ^n into (1.6), take $v = \phi_v$, $w = \phi_w$, and use the orthonormality of the system $\{\phi_i(x)\}_{i=1}^\infty$ and relation $(\nabla\phi_j, \nabla\phi_v) = \lambda_v\delta_{jv}$, $(\nabla\phi_k, \nabla\phi_w) = \lambda_w\delta_{kw}$, where λ_v, λ_w are the eigenvalues of the vector Dirichlet Laplacian in D on the divergence-free vector fields, then we obtain a system of ODE for the unknown coefficients c_v^n, d_w^n:

$$\partial_t c_v^n + \lambda_v c_v^n + \sum_{i,j=1}^n (\phi_{ia}\phi_{jb;a}, \phi_{vb})c_i^n c_j^n = \theta_m, \quad c_v^n(0) = (u_0, \phi_v), \tag{2.6}$$

$$\partial_t d_w^n + \lambda_w d_w^n + \sum_{i,k=1}^n (\phi_{ia}\phi_{kb;a}, \phi_{wb})c_i^n d_k^n = 0, \quad d_w^n(0) = (\theta_0, \phi_w). \tag{2.7}$$

Problem (2.4)-(2.5) has a global solution because of an *a priori* estimate that follows from (2.2) and Parseval's relations:

$$\big(u^n(t), u^n(t)\big) = \sum_{j=1}^n \big[c_j^n(t)\big]^2 \le C, \qquad \big(\theta^n(t), \theta^n(t)\big) = \sum_{k=1}^n \big[d_k^n(t)\big]^2 \le C. \tag{2.8}$$

Consider the set $\{u^n = u^n(t)\}_{n=1}^\infty$, $\{\theta^n = \theta^n(t)\}_{n=1}^\infty$. Inequalities (2.2) and (2.8) for $u = u^n$, $\theta = \theta^n$ imply the existence of the weak limits $u^n \rightharpoonup u$, $\theta^n \rightharpoonup \theta$ in $L^2(0, T; H_0^1(D'))$. This allows us to pass to the limit in equation (1.7) in all the terms except the term $\int_0^t [(u_s, v) + (\theta_s, w)]\,ds$. The weak limits of the terms $(u_a^n u_{b;a}^n, v_b)$, $(u_a^n \theta_{b;a}^n, w_b)$ exist and are equal to $(u_a u_{b;a}, v_b)$, $(u_a \theta_{b;a}, w_b)$, respectively, because

$$\begin{aligned}
&\big(u_a^n u_{b;a}^n, v_b\big) = -\big(u_a^n u_b^n, v_{b;a}\big) \to -(u_a u_b, v_{b;a}) = (u_a u_{b;a}, v_b), \\
&\big(u_a^n \theta_{b;a}^n, w_b\big) = -\big(u_a^n \theta_b^n, w_{b;a}\big) \to -(u_a \theta_b, w_{b;a}) = (u_a \theta_{b;a}, w_b).
\end{aligned}$$

Note that $v_{b;a}, w_{b;a} \in L^2(D')$ and $u_a^n u_b^n, u_a^n \theta_b^n \in L^4(D')$. Applying the interpolation inequality and the Schwarz inequality, we get

$$\|u\|_{L^4(D')}^2 \le \sqrt{2}\|u\|^{\frac{1}{4}}\|\nabla u\|^{\frac{3}{4}} \le \epsilon\|\nabla u\|^2 + \frac{27}{16\epsilon^3}\|u\|^2, \quad u \in H_0^1\big(D'\big), \tag{2.9}$$

$$\|\theta\|_{L^4(D')}^2 \le \sqrt{2}\|\theta\|^{\frac{1}{4}}\|\nabla \theta\|^{\frac{3}{4}} \le \epsilon\|\nabla \theta\|^2 + \frac{27}{16\epsilon^3}\|\theta\|^2, \quad \theta \in H_0^1\big(D'\big), \tag{2.10}$$

where $\epsilon > 0$ is an arbitrary small number. We have $u_a^n u_b^n \rightharpoonup u_a u_b$ in $L^2(D')$ as $n \to \infty$ because bounded sets in a reflexive Banach space $L^4(D')$ are weakly compact. Consequently, $(u_a^n u_{b;a}^n, v_b) \to (u_a u_{b;a}, v_b)$ as $n \to \infty$. Similarly, we have $(u_a^n \theta_{b;a}^n, w_b) \to (u_a \theta_{b;a}, w_b)$ as $n \to \infty$.

The weak limit of the $\int_0^t [(\nabla u, \nabla v) + (\nabla\theta, \nabla w)]\,ds$ exists because of the *a priori* estimate (2.2) and the weak compactness of the bounded sets in a Hilbert space. Since equation (1.7) holds and the limits of all its terms, except $\int_0^t [(u_s^n, v) + (\theta_s^n, w)]\,ds$, do exist, there exists the limit $\int_0^t [(u_s^n, v) + (\theta_s^n, w)]\,ds \to \int_0^t [(u_s, v) + (\theta_s, w)]\,ds$ for all $v, w \in G$. By passing to the limit as $n \to \infty$ we prove that the limits u, θ satisfy (1.7). Differentiating (1.7) with respect to t yields (1.6) almost everywhere. □

Lemma 2.5 *The global weak solutions u, θ from Lemma* 2.4 *are unique*.

Proof Suppose that there are two solutions to (1.6), $u \in G$, $\theta \in J$ and $u' \in G$, $\theta' \in J$, and let $U = u - u'$, $\Theta = \theta - \theta'$. Then

$$\begin{aligned}&(U_t, v) + (\Theta_t, w) + (\nabla U, \nabla v) + (\nabla\Theta, \nabla w) + \bigl(u_a u_{b;a} - u'_a u'_{b;a}, v_b\bigr)\\&\quad + \bigl(u_a \theta_{b;a} - u'_a \theta'_{b;a}, w_b\bigr) = 0.\end{aligned} \tag{2.11}$$

Since $U \in G$, $\Theta \in J$, we may set $v = U$, $w = \Theta$ in (2.11) and have

$$\partial_t\bigl(\|U\|^2 + \|\Theta\|^2\bigr) + 2\bigl(\|\nabla U\|^2 + \|\nabla\Theta\|^2\bigr) + 2(U_a u_{b;a}, U_b) + 2\bigl(U_a \theta'_{b;a}, \Theta_b\bigr) = 0, \tag{2.12}$$

where we have used the following conclusions:

$$\begin{aligned}&\bigl(u_a u_{b;a} - u'_a u'_{b;a}, U_b\bigr) = (U_a u_{b;a}, U_b) + \bigl(u'_a U_{b;a}, U_b\bigr),\\&\bigl(u'_a U_{b;a}, U_b\bigr) = -\frac{1}{2}\bigl(u'_{a;a}, \bigl(U^2\bigr)_b\bigr) = 0,\\&\bigl(u_a \theta_{b;a} - u'_a \theta'_{b;a}, \Theta_b\bigr) = (U_a \theta_{b;a}, \Theta_b) + \bigl(u'_a \Theta_{b;a}, \Theta_b\bigr),\\&\bigl(u'_a \Theta_{b;a}, \Theta_b\bigr) = -\frac{1}{2}\bigl(u'_{a;a}, \bigl(\Theta^2\bigr)_b\bigr) = 0.\end{aligned}$$

Thus, (2.12) implies

$$\partial_t\bigl(\|U\|^2 + \|\Theta\|^2\bigr) + 2\bigl(\|\nabla U\|^2 + \|\nabla\Theta\|^2\bigr) \le 2\bigl[\bigl|(U_a u_{b;a}, h_b)\bigr| + \bigl|(U_a \theta_{b;a}, \Theta_b)\bigr|\bigr]. \tag{2.13}$$

From Young's inequality we get the following estimates:

$$\bigl|(U_a u_{b;a}, U_b)\bigr| \le \|U\|^2_{L^4(D')} \|\nabla u\| \le \|\nabla u\| \biggl(\epsilon\|\nabla U\|^2 + \frac{27}{16\epsilon^3}\|U\|^2\biggr), \tag{2.14}$$

$$\bigl|(U_a \theta_{b;a}, \Theta_b)\bigr| = \bigl|(U_a \theta_b, \Theta_{b;a})\bigr| \le 9\|\nabla\Theta\| \bigl\| |U||\theta| \bigr\| \le \frac{1}{2}\|\nabla\Theta\|^2 + 18\bigl\| |U||\theta| \bigr\|^2. \tag{2.15}$$

Denoting $z := \|U\|^2 + \|\Theta\|^2$ and choosing $\epsilon = \frac{1}{2\|\nabla u\|}$ in inequalities (2.14), we obtain

$$\partial_t z \le \biggl(\frac{27}{2}\|\nabla u\|^4 + 18\bigl|\theta(\cdot, t)\bigr|\biggr) z, \quad z|_{t=0} = 0. \tag{2.16}$$

Assume that $|\theta(\cdot, t)|$, $\|\nabla u\|^4 \in L^1_{\mathrm{loc}}(0, \infty)$. Then we get $z = 0$ for all $t \ge 0$. □

The proof of Theorem 1.1 is complete.

Competing interests
The author declares that they have no competing interests.

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