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# Existence of solutions for fractional $p$ -Laplacian problems via Leray-Schauder's nonlinear alternative

Hong Qiu and Mingqi Xiang\*

\*Correspondence:  
mqxiang@cauc.edu.cn  
College of Science, Civil Aviation  
University of China, Tianjin, 300300,  
P.R. China

## Abstract

In this paper, we are concerned with the existence of solutions for a class of quasilinear elliptic problems driven by a nonlocal integro-differential operator with homogeneous Dirichlet boundary data. As a particular case, we study the following problem:

$$\begin{aligned}(-\Delta)_p^s u &= f(x, u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{aligned}$$

where  $(-\Delta)_p^s$  is the fractional  $p$ -Laplace operator,  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$  with Lipschitz boundary, and  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function. The existence of nonnegative solutions is obtained by using Leray-Schauder's nonlinear alternative.

**MSC:** 35R11; 35A15; 47G20

**Keywords:** fractional  $p$ -Laplacian equation; integro-differential operator; Leray-Schauder alternative theorem

## 1 Introduction and main results

Recently, a great deal of attention has been paid to the study of problems involving fractional and nonlocal operators, both in pure mathematical research and in real-world applications, such as optimization, finance, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces, and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes; see [1–4] and the references therein. Especially, the fractional Laplacian operators of the form  $(-\Delta)^s$  can be viewed as the infinitesimal generators of stable Lévy processes; see for instance [5]. Some interesting topics concerning the nonlocal fractional operators, such as the nonlinear fractional Schrödinger equation (see [6]), the fractional porous medium equation (see [7, 8]), and so on, have attracted considerable attention. There is no doubt that the literature on fractional and nonlocal operators is quite large; see for example [9–18]. For the basic properties of fractional Sobolev spaces and their applications to elliptic fractional problems, we refer the reader to [19, 20] and the references therein.

In this paper we are interested in the existence of solutions for the following problem:

$$\begin{cases} \mathcal{L}_K u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where  $\Omega \subset \mathbb{R}^N$  is an open bounded set with Lipschitz boundary  $\partial\Omega$ ,  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function, and  $\mathcal{L}_K$  is a nonlocal operator defined as

$$\mathcal{L}_K \varphi(x) = \lim_{\varepsilon \rightarrow 0^+} 2 \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |\varphi(x) - \varphi(y)|^{p-2} (\varphi(x) - \varphi(y)) K(x - y) dy, \quad x \in \mathbb{R}^N,$$

along any  $\varphi \in C_0^\infty(\mathbb{R}^N)$ , where  $B_\varepsilon(x)$  denotes the ball in  $\mathbb{R}^N$  of radius  $\varepsilon > 0$  at the center  $x \in \mathbb{R}^N$  and  $K : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^+$  is a measurable function with the following property:

$$\begin{cases} \gamma K \in L^1(\mathbb{R}^N), \text{ where } \gamma(x) = \min\{|x|^p, 1\}; \\ \text{there exist } s \in (0, 1) \text{ and } K_0 > 0 \\ \text{such that } K(x) \geq K_0 |x|^{-(N+ps)} \text{ for any } x \in \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{1.2}$$

Throughout the paper, without further mention, we always assume that  $0 < s < 1$  and  $1 < p < N/s$ .

A typical example for  $K$  is given by the singular kernel  $K(x) = |x|^{-(N+ps)}$ . In this case, problem (1.1) becomes

$$\begin{cases} (-\Delta)_p^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where  $(-\Delta)_p^s$  is the fractional  $p$ -Laplace operator; see [19] for more details. When  $p = 2$ , problem (1.3) reduces to the fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

A distinguishing characterization of the fractional operator  $(-\Delta)^s$  in (1.4) is in the non-locality, in the sense that this operator takes care of the behavior of the solution in the whole space. This is in contrast with the usual elliptic partial differential equations, which are governed by local differential operators like the Laplace operator. Of course, there are other explanations for this feature; see for example [21, 22]. The functional space that takes into account this boundary condition was introduced in [23]. In [22], the authors get the existence of nontrivial weak solutions of problem (1.4) by using the mountain pass theorem. See also [24] for the related discussions. In [25], the author studied the existence of solutions to a variational inequality involving nonlocal elliptic operators and obtained the existence of solutions by using Schauder’s fixed point theorem combined with adequate variational arguments and a penalization technique.

In the present paper, motivated by the above papers, we study the existence of weak solutions for problem (1.1) involving nonlocal fractional operator. It is worth noticing that

there are a few authors addressing the fractional  $p$ -Laplacian problems by using the non-linear alternative of Leray-Schauder.

In this spirit, we suppose that  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function satisfying:

(f<sub>1</sub>) there exist  $q \in (1, p)$  and  $C > 0$  such that

$$|f(x, t)| \leq a(x) + C|t|^{q/q'} \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where  $q'$  satisfies  $\frac{1}{q} + \frac{1}{q'} = 1$ , and  $a \in L^{q'}(\Omega)$ , with  $a \geq 0$  a.e. in  $\Omega$ .

A simple example of  $f$  is given by  $f(x, t) = |t|^{q-2}t + \ln(1 + |t|)$  for all  $(x, t) \in \Omega \times \mathbb{R}$ .

Now we are in a position to give the main result as follows.

**Theorem 1.1** *Let  $K : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^+$  be a function satisfying (1.2). Suppose that  $f$  satisfies (f<sub>1</sub>). If  $1 < q < p$ , then there exists  $R > 0$  such that problem (1.1) has a solution  $u \in W_0$  with  $\|u\|_{W_0} \leq R$ , where the fractional Sobolev space  $W_0$  will be introduced in Section 2.*

This paper is organized as follows. In Section 2, we recall some necessary definitions and properties of the fractional Sobolev space  $W_0$ . In Section 3, applying the properties of the operator  $\mathcal{L}_K$  and the nonlinear alternative of Leray-Schauder, we obtain the existence of solutions for problem (1.1).

## 2 Preliminaries

In this section, we first recall some basic results, which will be used in the next section. Let  $0 < s < 1 < p < \infty$  be real numbers and the fractional critical exponent  $p_s^*$  be defined as

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ \infty & \text{if } sp \geq N. \end{cases}$$

In the following, we denote  $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$ , where

$$\mathcal{O} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N},$$

and  $\mathcal{C}(\Omega) = \mathbb{R}^N \setminus \Omega$ .  $W$  is a linear space of Lebesgue measurable functions from  $\mathbb{R}^N$  to  $\mathbb{R}$  such that the restriction to  $\Omega$  of any function  $u$  in  $W$  belongs to  $L^p(\Omega)$  and

$$\iint_Q |u(x) - u(y)|^p K(x - y) \, dx \, dy < \infty.$$

The space  $W$  is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \left( \iint_Q |u(x) - u(y)|^p K(x - y) \, dx \, dy \right)^{1/p}.$$

It is easy to prove that  $\|\cdot\|_W$  is a norm on  $W$ . We shall work in the closed linear subspace

$$W_0 = \{u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

endowed with the norm

$$\|u\|_{W_0} := [u]_{s,p,K} = \left( \iint_Q |u(x) - u(y)|^p K(x - y) \, dx \, dy \right)^{1/p}.$$

Then  $(W_0, \|\cdot\|_{W_0})$  is a uniformly convex Banach space; see [26]. Moreover,  $C_0^\infty(\Omega)$  is dense in  $W_0$ ; see [27].

**Lemma 2.1** (see Lemma 2.5 of [26]) *Let  $K : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^+$  satisfy (1.2) and let  $v_j$  be a bounded sequence in  $W_0$ . Then there exists  $v \in L^v(\mathbb{R}^N)$  with  $v = 0$  a.e. in  $\mathbb{R}^N \setminus \Omega$  such that up to a subsequence,*

$$v_j \rightarrow v \text{ strongly in } L^v(\Omega) \text{ as } j \rightarrow \infty$$

for any  $v \in [1, p_s^*)$ .

### 3 Proof of the main result

In this section we prove the existence of solutions to problem (1.1). Our method is based on the properties of the operator  $\mathcal{L}_K$  and again on the nonlinear alternative of Leray-Schauder.

Let  $W_0^*$  denote the dual space of  $W_0$ . We first give some properties to operator  $\mathcal{L}_K$ .

**Lemma 3.1**

- (1)  $\mathcal{L}_K : W_0 \rightarrow W_0^*$  is a continuous, bounded and strictly monotone operator;
- (2)  $\mathcal{L}_K$  is a mapping of type  $(S_+)$ , i.e. if  $u_n \rightharpoonup u$  in  $W_0$  and

$$\limsup_{n \rightarrow \infty} \langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u \rangle \leq 0,$$

then  $u_n \rightarrow u$  in  $W_0$ ;

- (3)  $\mathcal{L} : W_0 \rightarrow W_0^*$  is a homomorphism.

*Proof* (1) Note that, by the Hölder inequality, one has

$$\begin{aligned} |\langle \mathcal{L}_K u, v \rangle| &= \left| \iint_Q |u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y)) K(x - y) \, dx \, dy \right| \\ &\leq \|u\|_{W_0}^{p-1} \|v\|_{W_0} \end{aligned}$$

for all  $u, v \in W_0$ . Following this inequality, we easily see that  $\mathcal{L}_K$  is continuous and bounded.

Let us now recall the well-known Simon inequality (see [26] and [28]): for all  $\xi, \eta \in \mathbb{R}$ , there exists  $C_p > 0$  such that

$$C_p (|\xi|^{p-2} \xi - |\eta|^{p-2} \eta) \cdot (\xi - \eta) \geq \begin{cases} |\xi - \eta|^p & \text{if } p \geq 2, \\ |\xi - \eta|^2 (|\xi|^p + |\eta|^p)^{(p-2)/p} & \text{if } 1 < p < 2. \end{cases} \tag{3.1}$$

It follows from (3.1) that the operator  $\mathcal{L}_K$  is strictly monotone.

(2) By (1), if  $u_n \rightharpoonup u$  and  $\limsup_{n \rightarrow \infty} \langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u \rangle \leq 0$ , then  $\lim_{n \rightarrow \infty} \langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u \rangle = 0$ . Using the well-known vector inequalities (3.1), we obtain, for  $p > 2$ ,

$$\|u_n - u\|_{W_0}^p \leq C_p \langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u \rangle = o(1), \tag{3.2}$$

and, for  $1 < p < 2$ ,

$$\begin{aligned} \|u_n - u\|_{W_0}^p &\leq C_p^{p/2} [\langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u \rangle]^{p/2} (\|u_n\|_{W_0}^p + \|u\|_{W_0}^p)^{(2-p)/2} \\ &\leq C_p^{p/2} [\langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u \rangle]^{p/2} (\|u_n\|_{W_0}^{p(2-p)/2} + \|u\|_{W_0}^{p(2-p)/2}) \\ &\leq C [\langle \mathcal{L}_K u_n - \mathcal{L}_K u, u_n - u \rangle]^{p/2} = o(1), \end{aligned} \tag{3.3}$$

where  $C > 0$  is a constant. Combining (3.2) with (3.3), we obtain  $u_n \rightarrow u$  in  $W_0$  as  $n \rightarrow \infty$ , i.e.  $\mathcal{L}_K$  is of type  $(S_+)$ .

(3) By the strictly monotonicity,  $\mathcal{L}_K$  is an injection. Since

$$\begin{aligned} \lim_{\|u\|_{W_0} \rightarrow \infty} \frac{\langle \mathcal{L}_K u, u \rangle}{\|u\|_{W_0}} &= \lim_{\|u\|_{W_0} \rightarrow \infty} \frac{\int \int_Q |u(x) - u(y)|^p K(x - y) \, dx \, dy}{\|u\|_{W_0}} \\ &= \lim_{\|u\|_{W_0} \rightarrow \infty} \|u\|_{W_0}^{p-1} = \infty, \end{aligned}$$

thanks to  $1 < p < \frac{N}{s}$ . Hence  $\mathcal{L}_K$  is coercive on  $W_0$ . Furthermore, by the Minty-Browder theorem (see Theorem 26A of [29]), we know  $\mathcal{L}_K$  is a surjection. Thus  $\mathcal{L}_K$  has an inverse mapping  $\mathcal{L}_K^{-1} : W_0^* \rightarrow W_0$ . Now we check the continuity of  $\mathcal{L}_K^{-1}$ . Assume  $g_n, g \in W_0^*$ , with  $g_n \rightarrow g$  in  $W_0^*$ . Let  $u_n = \mathcal{L}_K^{-1} g_n$  and  $u = \mathcal{L}_K^{-1} g$ , then  $\mathcal{L}_K u_n = g_n$  and  $\mathcal{L}_K u = g$ . Clearly,  $\{u_n\}$  is bounded in  $W_0$ . Thus there exist  $u_0 \in W_0$  and a subsequence of  $\{u_n\}$  still denoted by  $\{u_n\}$  such that  $u_n \rightharpoonup u_0$ . Since  $g_n \rightarrow g$ , we have

$$\lim_{n \rightarrow \infty} \langle \mathcal{L}_K u_n - \mathcal{L}_K u_0, u_n - u_0 \rangle = \lim_{n \rightarrow \infty} \langle g_n, u_n - u_0 \rangle = 0.$$

In view of  $\mathcal{L}_K$  is of type  $(S_+)$ , we get  $u_n \rightarrow u_0$  in  $W_0$ . Moreover,  $u = u_0$  a.e. in  $\Omega$ . Hence  $u_n \rightarrow u$  in  $W_0$ , so that  $\mathcal{L}_K^{-1}$  is continuous. Thus, we complete the proof.  $\square$

To prove the existence of solutions for problem (1.1), we need the following theorem.

**Theorem 3.1** (Alternative of Leray-Schauder; see [30]) *Let  $B(0, R)$  denote the closed ball in a Banach space  $X$ ,  $\{u \in X : \|u\| \leq R\}$ , and let  $I : B(0, R) \rightarrow X$  be a compact operator. Then either:*

- (i) *the equation  $\lambda Iu = u$  has a solution in  $B(0, R)$  for  $\lambda = 1$  or*
- (ii) *there exists  $u \in X$  with  $\|u\| = R$  satisfying  $\lambda Iu = u$  for some  $\lambda \in (0, 1)$ .*

*Proof of Theorem 1.1* Following the idea of [31], for simplicity of notation, we set

$$Y = L^q(\Omega), \quad Y^* = L^{q'}(\Omega), \quad \|\cdot\|_Y = \|\cdot\|_{L^q(\Omega)}.$$

By Lemma 2.1 and  $1 < q < p_s^*$ ,  $W_0$  is compactly embedded in  $Y$ . Denote by  $i$  the compact injection of  $W_0$  in  $Y$  and by  $i^* : Y^* \rightarrow W_0^*$ ,  $i^*v = v \circ i$  for all  $v \in Y^*$ , its adjoint. It follows

from assumption  $(f_1)$  that the Nemytskii operator  $N_f$  generated by  $f$ ,  $(N_f u)(x) = f(x, u(x))$ , is well defined from  $Y$  into  $Y^*$ , continuous, and bounded (see for example [32]). In order to prove that problem (1.1) has a weak solution in  $W_0$  it is sufficient to prove that the equation

$$\mathcal{L}_K u = (i^* N_f i) u \tag{3.4}$$

has a solution in  $W_0$ . Indeed, if  $u \in W_0$  satisfies (3.4), then for all  $v \in W_0$ , one has

$$\langle \mathcal{L}_K u, v \rangle = \langle (i^* N_f i) u, v \rangle = \langle N_f(iu), iv \rangle,$$

which can be rewritten

$$\iint_Q |u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y)) K(x - y) dx dy = \int_{\Omega} f(x, u) v dx,$$

this means that  $u$  is a weak solution in  $W_0$  to problem (1.1).

By Lemma 3.1,  $\mathcal{L}_K$  is a homeomorphism of  $W_0$  onto  $W_0^*$ . Equation (3.4) can be equivalently rewritten

$$u = \mathcal{L}_K^{-1} (i^* N_f i) u. \tag{3.5}$$

Therefore, proving problem (1.1) has a weak solution in  $W_0$  reduces to proving that the compact operator

$$\mathcal{L} = \mathcal{L}_K^{-1} (i^* N_f i) : W_0 \rightarrow W_0^*$$

has a fixed point.

By Theorem 3.1, a sufficient condition for  $\mathcal{L}$  to have a fixed point is that there exists a constant  $R > 0$  such that

$$S = \{u \in W_0 : u = \lambda \mathcal{L} u \text{ for some } \lambda \in [0, 1]\} \subset B(0, R).$$

Since for  $\lambda = 0$  the only solution of equation  $u = \lambda \mathcal{L} u$  is  $u = 0$ , it is enough to show that there exists a constant such that any  $u \in W_0$  which satisfies

$$u = \lambda \mathcal{L}_K^{-1} [(i^* N_f i) u] \tag{3.6}$$

for some  $\lambda \in (0, 1]$  belongs to  $B(0, R)$ .

Indeed, if  $u \in W_0$  satisfies (3.6) for some  $\lambda \in (0, 1]$ , then we have

$$\langle \mathcal{L}_K u, u \rangle = \lambda \langle (i^* N_f i) u, u \rangle. \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned} \|u\|_{W_0}^p &= \lambda \langle (i^* N_f i) u, u \rangle \leq \langle (i^* N_f i) u, u \rangle \\ &\leq \|i^*\| \|N_f(iu)\|_{Y^*} \|u\|_{W_0}. \end{aligned} \tag{3.8}$$

In order to estimate  $\|N_f(iu)\|_{Y^*}$ , we deduce first from assumption  $(f_1)$  that

$$|(N_f v)(x)| = |f(x, v)| \leq c_1 |v(x)|^{q/q'} + a(x).$$

Hence,

$$\|N_f v\|_{Y^*} \leq \|c_1 |v(x)|^{q/q'} + a(x)\|_{Y^*} \leq c_1 \|v\|_Y^{q-1} + \|a(x)\|_{Y^*}. \tag{3.9}$$

By taking (3.9) for  $v = iu, u \in W_0$ , we have

$$\|N_f(iu)\|_{Y^*} \leq c_1 \|i\|^{q-1} \|u\|_{W_0}^{q-1} + \|a\|_{Y^*}. \tag{3.10}$$

In particular, if  $u \in W_0$  and satisfies (3.6) for some  $\lambda \in (0, 1]$ , we derive from (3.8) and (3.10) that

$$\|u\|_{W_0}^p \leq \|i^*\| (c_1 \|i\|^{q-1} \|u\|_{W_0}^{q-1} + \|a\|_{Y^*}) \|u\|_{W_0} = C_1 \|u\|_{W_0}^q + C_2 \|u\|_{W_0}, \tag{3.11}$$

where  $C_1 = c_1 \|i^*\| \|i\|^q, C_2 = \|i^*\| \|a\|_{Y^*}$ . Applying the Young inequality, we get

$$\|u\|_{W_0}^q C_1 \leq \frac{1}{4} \|u\|_{W_0}^p + 4^{\frac{q}{p-q}} C_1^{\frac{p}{p-q}}$$

and

$$\|u\|_{W_0} C_2 \leq \frac{1}{4} \|u\|_{W_0}^p + 4^{\frac{1}{p-1}} C_2^{\frac{p}{p-1}}.$$

Inserting these two inequalities into (3.11), one has

$$\frac{1}{2} \|u\|_{W_0}^p \leq 4^{\frac{q}{p-q}} C_1^{\frac{p}{p-q}} + 4^{\frac{1}{p-1}} C_2^{\frac{p}{p-1}}.$$

Set

$$R = \left( 2^{\frac{p+q}{p-q}} C_1^{\frac{p}{p-q}} + 2^{\frac{p+1}{p-1}} C_2^{\frac{p}{p-1}} \right)^{1/p}.$$

Then  $\|u\|_{W_0} \leq R$ , this implies the set  $S$  is bounded. □

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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