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Solving Robin problems in multiply connected regions via an integral equation with the generalized Neumann kernel

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Abstract

This paper presents a boundary integral equation method for finding the solution of Robin problems in bounded and unbounded multiply connected regions. The Robin problems are formulated as Riemann-Hilbert problems which lead to systems of integral equations and the related differential equations are also constructed that give rise to unique solutions, which are shown. Numerical results on several test regions are presented to illustrate that the approximate solution when using this method for the Robin problems when the boundaries are sufficiently smooth are accurate.

Keywords: Robin problem; Riemann-Hilbert problem; integral equation; generalized Neumann kernel; multiply connected region

1 Introduction

A boundary value problem is a problem that involves finding the solution of a differential equation or system of differential equation which meets certain specified requirements or boundary conditions at the end points or along a boundary, usually connected with the physical condition for certain values of the independent variable. This paper considers Laplace's equation $\Delta u = 0$ in both bounded and unbounded multiply connected regions Ω with a linear combination of Dirichlet and Neumann boundary conditions on the boundary $\Gamma = \partial\Omega$, generally known as a mixed boundary value problem and commonly called the Robin problem.

Applications of mixed boundary value problems exist in large numbers in classical mathematical physics, physical geodesy, electro-magnetics, analysis of measurement [1, 2], and specific boundary problems such as the Dirichlet problem and the Neumann problem [3]. The applications of the mixed boundary value problem in potential theory can be found in [4].

It has been shown that the problem of conformal mapping, the Dirichlet problem, the Neumann problem, and the mixed Dirichlet-Neumann problem can all be treated as Riemann Hilbert (briefly RH) problems as discussed in [5–7]. The interplay of RH problems and the boundary Fredholm integral equation with the generalized Neumann kernel has been investigated in [8] for simply connected regions with smooth and piecewise smooth boundaries and in [9] for bounded and unbounded multiply connected regions.

Earlier, the well-known integral equations for RH problem have been employed for solving the Dirichlet problem and the Neumann problem [5] and the mixed Dirichlet-Neumann problem [6]. They are uniquely solvable Fredholm integral equations of the second kind. However, the problems in [5] and [6] are not Robin problems, since the Dirichlet condition and Neumann condition are given separately.

This paper solves the Robin problem by reducing it to a RH problem and hence providing a related system of boundary integral equations. Additional conditions are given to obtain a unique solution to the Robin problem.

This paper is organized as follows: After the presentation of some notations and auxiliary material in Section 2, we present in Section 3, the reduction of the Robin problems in bounded and unbounded multiply connected regions to the RH problem. We then show how to construct the integral equations and differential equations related to the Robin problems. Conditions for obtaining unique solution for the Robin problems are also given. In Section 4, we show how to treat the integral equations and differential equations numerically. In Section 5, we give some numerical examples for several test regions. In Section 6 we give a short conclusion.

2 Notations and auxiliary material

Consider a multiply connected region Ω in the extended complex plane $\overline{\mathbb{C}}$ of the following two types [5]:

(a) A bounded region Ω of connectivity $m + 1 \geq 1$ with boundary $\Gamma = \bigcup_{j=0}^{m} \Gamma_j$ consisting of $m + 1$ smooth closed Jordan curves $\Gamma_j, j = 1, 2, \ldots, m$. The curve Γ_0 contains the other curves $\Gamma_1, \ldots, \Gamma_m$. The complement $\Omega^- := \overline{\mathbb{C}} \setminus \overline{\Omega}$ consists of m bounded simply connected components Ω_j interior to $\Gamma_j, j = 1, 2, \ldots, m$, and an unbounded simply connected component Ω_0 exterior to Γ_0 (see Figure 1). We assume that $0 \in \Omega$.

(b) An unbounded region Ω of connectivity $m \geq 1$ with boundary $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ consisting of m smooth closed Jordan curves $\Gamma_j, j = 1, 2, \ldots, m$. The complement $\Omega^- := \overline{\mathbb{C}} \setminus \overline{\Omega}$ consists of m bounded simply connected components Ω_j interior to $\Gamma_j, j = 1, 2, \ldots, m$ (see Figure 2). We assume that $0 \notin \Omega$.

The orientation of the boundary $\Gamma = \partial\Omega$ is such that Ω is always on the left of Γ. Thus, the curves $\Gamma_1, \ldots, \Gamma_m$ always have clockwise orientations. For bounded Ω, the curve Γ_0 has a counterclockwise orientation. The curves Γ_j are parameterized by 2π-periodic twice

Figure 1 A bounded multiply connected region Ω of connectivity $m + 1$.
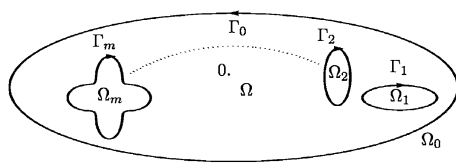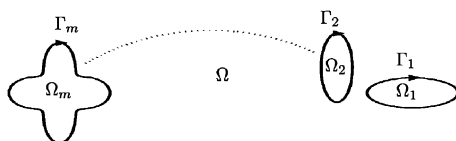

Figure 2 An unbounded multiply connected region Ω of connectivity m.

continuously differentiable complex functions $\eta_j(t)$ with non-vanishing first derivatives,

$$\dot{\eta}_j(t) = \frac{d\eta_j(t)}{dt} \neq 0, \quad t \in J_j := [0, 2\pi],$$

$j = 0$ (for bounded Ω), $1, 2, \ldots, m$. We represent the disjoint union of the intervals J_j by J as a total parameter domain. Automatically, the whole boundary Γ will be parameterized by a complex function η defined on J by

$$\eta(t) := \begin{cases} \eta_0(t), & t \in J_0 \text{ (for bounded } \Omega), \\ \eta_1(t), & t \in J_1, \\ \vdots \\ \eta_m(t), & t \in J_m. \end{cases}$$

Assume that H be the space of all real Hölder continuous functions on the boundary Γ. Since η is smooth, a function $\phi \in H$ can be interpreted via $\hat{\phi}(t) := \phi(\eta(t))$, $t \in J$, as a real Hölder continuous 2π-periodic functions $\hat{\phi}(t)$ of the parameter $t \in J$, *i.e.*,

$$\hat{\phi}(t) := \begin{cases} \hat{\phi}_0(t), & t \in J_0 \text{ (for bounded } \Omega), \\ \hat{\phi}_1(t), & t \in J_1, \\ \vdots \\ \hat{\phi}_m(t), & t \in J_m, \end{cases}$$

with real Hölder continuous 2π-periodic functions $\hat{\phi}_j$ defined on J_j.

From now on, for a complex-valued or a real-valued function $\psi \in H$ defined on the boundary Γ and for $t \in J$, we will not distinguish between $\psi(\eta(t))$ and $\psi(t)$. For $t \in J_k$, the values $\psi(t)$ will be denoted by $\psi_k(t)$.

For given functions $\alpha \in H$, $\beta \in H$, $l \in H$, a Robin problem is a boundary value problem for determining a harmonic function $u(x, y)$ harmonic in Ω and continuous on $\Omega \cup \Gamma$ and satisfying the Robin boundary condition [10]

$$\alpha(t)u\big(\eta(t)\big) + \beta(t)\frac{\partial u(\eta(t))}{\partial \mathbf{n}} = l(t), \quad \alpha(t) \neq 0, \beta(t) \neq 0, \eta(t) \in \Gamma, \tag{1}$$

where $\mathbf{n}$ is an exterior normal to Γ. In particular, if $t \in J_k$, then (1) becomes

$$\alpha_k(t)u\big(\eta_k(t)\big) + \beta_k(t)\frac{\partial u(\eta_k(t))}{\partial \mathbf{n}} = l_k(t), \quad \alpha_k(t) \neq 0, \beta_k(t) \neq 0, \eta_k(t) \in \Gamma, t \in J_k. \tag{2}$$

For unbounded Ω, the function u is also required to satisfy $u(z) \rightarrow C$ as $|z| \rightarrow \infty$ with a constant C. If $\frac{\alpha_k(t)}{\beta_k(t)} > 0$, $t \in J_k$ (briefly $\frac{\alpha(t)}{\beta(t)} > 0$), then the Robin problem is uniquely solvable (see *e.g.* [11], p.163, [12], p.103, [13], pp.314-315, and [14], p.141). For some integral equations related to the Robin problem, see [13], p.314, and [15]. Without loss of generality, we assume $C = 0$.

In this paper, we shall relate the Robin problem with the RH problem. The RH problem consists of finding a function g analytic in Ω, continuous in its closure $\overline{\Omega}$ and having boundary values

$$\text{Re}[Ag] = \gamma, \tag{3}$$

where $\gamma \in H$ and $A(t)$ is a complex continuously differentiable 2π-periodic function with $A \neq 0$ for all t such that

$$A(t) := \begin{cases} A_0(t), & t \in J_0 \text{ (for bounded } \Omega), \\ A_1(t), & t \in J_1, \\ \vdots \\ A_m(t), & t \in J_m. \end{cases}$$

For unbounded Ω, the function g is also required to satisfy $g(\infty) = 0$.

The RH problem can be solved using a boundary integral equation with the generalized Neumann kernel. Define the real kernels M and N as real and imaginary parts [8, 9]:

$$M(\tau, t) = \frac{1}{\pi} \operatorname{Re}\left[\frac{A(\tau)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(\tau)} \right], \quad \tau \neq t,$$

$$N(\tau, t) = \frac{1}{\pi} \operatorname{Im}\left[\frac{A(\tau)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(\tau)} \right], \quad \tau \neq t. \tag{4}$$

The kernel $N(\tau, t)$ is called the generalized Neumann kernel formed with A and η. When $A = 1$, it reduces to the classical Neumann kernel,

$$N(\tau, t) = \frac{1}{\pi} \operatorname{Im}\left[\frac{\dot{\eta}(t)}{\eta(t) - \eta(\tau)} \right], \quad \tau \neq t.$$

The generalized Neumann kernel (4) is continuous at $t = \tau$ with

$$N(t, t) = \frac{1}{\pi} \operatorname{Im}\left[\frac{1}{2}\left(\frac{\ddot{\eta}(t)}{\dot{\eta}(t)} \right) - \frac{\dot{A}(t)}{A(t)} \right].$$

The kernel $M(\tau, t)$ has the representation

$$M(\tau, t) = -\frac{1}{2\pi} \cot \frac{\tau - t}{2} + M_1(\tau, t), \quad t \neq \tau, \tag{5}$$

with the continuous kernel M_1, which takes on the diagonal the values

$$M_1(t, t) = \frac{1}{\pi} \operatorname{Re}\left[\frac{1}{2}\left(\frac{\ddot{\eta}(t)}{\dot{\eta}(t)} \right) - \frac{\dot{A}(t)}{A(t)} \right].$$

For details, see [8, 9].

Let $\mathbf{N}$ and $\mathbf{M}_1$ be the Fredholm integral operators associated with the continuous kernels N and M_1, *i.e.*,

$$(\mathbf{N}\mu)(\tau) = \int_J N(\tau, t)\mu(t)\,dt, \quad \tau \in J,$$

$$(\mathbf{M}_1\mu)(\tau) = \int_J M_1(\tau, t)\mu(t)\,dt, \quad \tau \in J.$$

Let $\mathbf{M}$ and $\mathbf{K}$ be the singular integral operators

$$(\mathbf{M}\mu)(\tau) = \int_J M(\tau, t)\mu(t)\,dt, \quad \tau \in J, \tag{6}$$

$$(\mathbf{K}\mu)(\tau) = \frac{1}{2\pi} \int_{J_k} \cot\left(\frac{\tau - t}{2}\right) \mu(t)\, dt, \quad \tau \in J_k, k = 0, 1, \ldots, m. \tag{7}$$

The integrals (6) and (7) are principal value integrals. The operator $\mathbf{K}$ is known as the conjugation operator. It is also known as the Hilbert transform [8]. It follows from equation (5) that

$$\mathbf{M} = \mathbf{M}_1 - \mathbf{K}.$$

Theorem 1 ([9, 16]) *If g is a solution of the RH problem equation* (3) *with boundary values*

$$Ag = \gamma + i\mu, \tag{8}$$

then the imaginary part μ in (8) *satisfies the integral equation*

$$\mu - \mathbf{N}\mu = -\mathbf{M}\gamma,$$

and the real part γ in (8) *satisfies the integral equation*

$$\gamma - \mathbf{N}\gamma = \mathbf{M}\mu.$$

The solvability of boundary integral equations with the generalized Neumann kernel is determined by the index (winding number in other terminology) of the function $A(t)$ [9].

Theorem 2 ([17] Cauchy integral formula) *Let f be a function that is analytic everywhere in Ω and on a simple closed contour Γ. Then the Cauchy integral formula is given by*

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\eta)}{\eta - z}\, d\eta = \begin{cases} f(z), & z \in \Omega, \\ 0, & z \notin \Omega \end{cases} \quad [\textit{Bounded } \Omega], \tag{9}$$

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\eta)}{\eta - z}\, d\eta = \begin{cases} f(z) - f(\infty), & z \in \Omega, \\ -f(\infty), & z \notin \Omega \end{cases} \quad [\textit{Unbounded } \Omega]. \tag{10}$$

3 Reduction Robin problems in bounded and unbounded multiply connected regions to Riemann-Hilbert problem

We consider the Robin problem (1) either bounded or unbounded. The unit exterior normal vector is given by $\mathbf{n}(\eta(t)) = e^{-i\frac{\pi}{2}} \mathbf{T}(\eta(t)) = -i\frac{\dot{\eta}(t)}{|\dot{\eta}(t)|} = e^{i\theta(\eta(t))} = \cos\theta(\eta(t)) + i\sin\theta(\eta(t))$.

Then

$$\frac{\partial u(\eta(t))}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n} = (u_x \vec{i} + u_y \vec{j}) \cdot \left(\cos\theta\big(\eta(t)\big)\vec{i} + i\sin\theta\big(\eta(t)\big)\vec{j}\right)$$

$$= \cos\theta\big(\eta(t)\big) \frac{\partial u(\eta(t))}{\partial x} + \sin\theta\big(\eta(t)\big) \frac{\partial u(\eta(t))}{\partial y}$$

$$= \mathrm{Re}\left[e^{i\theta(\eta(t))}\left(\frac{\partial u(\eta(t))}{\partial x} - i\frac{\partial u(\eta(t))}{\partial y}\right)\right].$$

Assume $u(z) = \mathrm{Re}[f(z)]$, where $f(z) = u + iv$, is analytic in Ω. For bounded Ω, we further assume $\mathrm{Im}[f(0)] = 0$, for unbounded Ω, we assume $f(\infty) = 0$. Then by the Cauchy-Riemann

equations we have $f'(z) = u_x - iu_y$. Thus,

$$\frac{\partial u(\eta(t))}{\partial \mathbf{n}} = \mathrm{Re}\left[\frac{-i\dot{\eta}(t)f'(\eta(t))}{|\dot{\eta}(t)|}\right].$$

Substituting this result into the Robin condition (1), we get

$$\alpha(t)\,\mathrm{Re}\big[f\big(\eta(t)\big)\big] - \beta(t)\,\mathrm{Re}\left[\frac{i\dot{\eta}(t)f'(\eta(t))}{|\dot{\eta}(t)|}\right] = l(t). \tag{11}$$

It is sometimes assumed that $\eta_j(t)$ are the arc length parameterization of the boundary Γ_j, which implies $|\eta_j(t)| = 1$. These assumptions, although convenient for theoretical work, but in numerical aspect, it introduces an additional sources of error [18]. If Ω is a unit disk (11) yields the same result as in Petrila [10]. Multiplying both sides of (11) by $|\dot{\eta}(t)|$ we get

$$\mathrm{Re}\big[\alpha(t)\big|\dot{\eta}(t)\big|f\big(\eta(t)\big) - i\beta(t)\dot{\eta}(t)f'\big(\eta(t)\big)\big] = l(t)\big|\dot{\eta}(t)\big|,$$

or, equivalently,

$$\mathrm{Re}\left[-i\beta(t)\left\{\frac{d}{dt}\big(f\big(\eta(t)\big)\big) + ie(t)f\big(\eta(t)\big)\right\}\right] = l(t)\big|\dot{\eta}(t)\big|,$$

where

$$e(t) = \frac{\alpha(t)}{\beta(t)}\big|\dot{\eta}(t)\big|, \quad t \in J.$$

Hence

$$-i\beta(t)\left[\frac{d}{dt}\big(f\big(\eta(t)\big)\big) + ie(t)f\big(\eta(t)\big)\right] = l(t)\big|\dot{\eta}(t)\big| + i\mu(t), \quad t \in J,$$

where

$$\mu(t) = \begin{cases} \hat{\mu}_0(t), & t \in J_0 \text{ (for bounded } \Omega\text{)}, \\ \hat{\mu}_1(t), & t \in J_1, \\ \vdots & \\ \hat{\mu}_m(t), & t \in J_m, \end{cases}$$

is an unknown function. By means of integrating factor, we obtain

$$-i\beta(t)A(t)\left[\frac{d}{dt}\big(f\big(\eta(t)\big)\big) + ie(t)f\big(\eta(t)\big)\right] = l(t)\big|\dot{\eta}(t)\big|e^{i\zeta(t)} + i\mu(t)e^{i\zeta(t)}, \quad t \in J, \tag{12}$$

$$A(t) = e^{i\int_0^t e(\tau)\,d\tau} = e^{i\zeta(t)},$$

$$\zeta(t) = \int_0^t e(\tau)\,d\tau.$$

Then equation (12) becomes

$$\frac{d}{dt}\big[A(t)(-i)\big(f\big(\eta(t)\big)\big)\big] = \frac{l(t)|\dot{\eta}(t)|e^{i\zeta(t)} + i\mu(t)e^{i\zeta(t)}}{\beta(t)}.$$

Letting $g = -if$, which is analytic on Ω, we obtain

$$\frac{d}{dt}\big[A(t)g\big(\eta(t)\big)\big] = \frac{l(t)|\dot{\eta}(t)|e^{i\zeta(t)} + i\mu(t)e^{i\zeta(t)}}{\beta(t)}.\tag{13}$$

Hence, it follows from (13) that

$$A(t)g\big(\eta(t)\big) = \int_0^t \frac{l(\tau)|\dot{\eta}(\tau)|\cos\zeta(\tau)}{\beta(\tau)}\,d\tau + i\int_0^t \frac{l(\tau)|\dot{\eta}(\tau)|\sin\zeta(\tau)}{\beta(\tau)}\,d\tau$$
$$+ i\int_0^t \frac{\mu(\tau)\cos\zeta(\tau)}{\beta(\tau)}\,d\tau - \int_0^t \frac{\mu(\tau)\sin\zeta(\tau)}{\beta(\tau)}\,d\tau + c_1 + ic_2$$
$$= \gamma_1(t) + i\gamma_2(t) + i\mu_1(t) - \mu_2(t) + c_1 + ic_2$$
$$= \big(\gamma_1(t) - \mu_2(t) + c_1\big) + i\big(\gamma_2(t) + \mu_1(t) + c_2\big), \quad t \in J,\tag{14}$$

where c_1, c_2, are unknown piecewise real constants, and

$$\gamma_1(t) := \int_0^t \frac{l(\tau)|\dot{\eta}(\tau)|\cos\zeta(\tau)}{\beta(\tau)}\,d\tau, \quad t \in J,\tag{15}$$

$$\gamma_2(t) := \int_0^t \frac{l(\tau)|\dot{\eta}(\tau)|\sin\zeta(\tau)}{\beta(\tau)}\,d\tau, \quad t \in J,\tag{16}$$

are known functions, and

$$\mu_1(t) := \int_0^t \frac{\mu(\tau)\cos\zeta(\tau)}{\beta(\tau)}\,d\tau, \quad t \in J,\tag{17}$$

$$\mu_2(t) := \int_0^t \frac{\mu(\tau)\sin\zeta(\tau)}{\beta(\tau)}\,d\tau, \quad t \in J,\tag{18}$$

are unknown functions. Define

$$c_1 = \begin{cases} c_{01}, & t \in J_0 = [0,2\pi]\ (\text{for bounded }\Omega), \\ c_{11}, & t \in J_1 = [0,2\pi], \\ \vdots \\ c_{m1}, & t \in J_m = [0,2\pi], \end{cases}$$

$$c_2 = \begin{cases} c_{02}, & t \in J_0 = [0,2\pi]\ (\text{for bounded }\Omega), \\ c_{12}, & t \in J_1 = [0,2\pi], \\ \vdots \\ c_{m2}, & t \in J_m = [0,2\pi], \end{cases}$$

$$\gamma_1(t) = \begin{cases} \gamma_{01}(t), & t \in J_0 = [0,2\pi]\ (\text{for bounded }\Omega), \\ \gamma_{11}(t), & t \in J_1 = [0,2\pi], \\ \vdots \\ \gamma_{m1}(t), & t \in J_m = [0,2\pi], \end{cases}$$

$$\gamma_2(t) = \begin{cases} \gamma_{02}(t), & t \in J_0 = [0, 2\pi] \text{ (for bounded } \Omega\text{)}, \\ \gamma_{12}(t), & t \in J_1 = [0, 2\pi], \\ \vdots \\ \gamma_{m2}(t), & t \in J_m = [0, 2\pi], \end{cases}$$

$$\mu_1(t) = \begin{cases} \mu_{01}(t), & t \in J_0 = [0, 2\pi] \text{ (for bounded } \Omega\text{)}, \\ \mu_{11}(t), & t \in J_1 = [0, 2\pi], \\ \vdots \\ \mu_{m1}(t), & t \in J_m = [0, 2\pi], \end{cases}$$

$$\mu_2(t) = \begin{cases} \mu_{02}(t), & t \in J_0 = [0, 2\pi] \text{ (for bounded } \Omega\text{)}, \\ \mu_{12}(t), & t \in J_1 = [0, 2\pi], \\ \vdots \\ \mu_{m2}(t), & t \in J_m = [0, 2\pi]. \end{cases}$$

Then we can write (14) briefly as

$$A(t)g(t) = \gamma(t) + i\mu(t), \quad t \in J, \tag{19}$$

where $g(t) = g(\eta(t))$,

$$\gamma(t) = \gamma_1(t) - \mu_2(t) + c_1, \tag{20}$$

$$\mu(t) = \gamma_2(t) + \mu_1(t) + c_2. \tag{21}$$

The real part of (19) yields the RH problem. The function $A(t) = e^{i\zeta(t)}$ is in general not periodic. To apply the result of Theorem 1, $A(t)$ must be periodic. The function $A(t)$ is periodic if we assume $\zeta(2\pi) - \zeta(0) = 2\pi$. Thus, Theorem 1 implies that (19) can be reformulated as

$$(\mathbf{I} - \mathbf{N})\big(\gamma_2(t) + \mu_1(t) + c_2\big) = -\mathbf{M}\big(\gamma_1(t) - \mu_2(t) + c_1\big), \quad t \in J, \tag{22}$$

$$(\mathbf{I} - \mathbf{N})\big(\gamma_1(t) - \mu_2(t) + c_1\big) = \mathbf{M}\big(\gamma_2(t) + \mu_1(t) + c_2\big), \quad t \in J. \tag{23}$$

We next show that the above system of integral equations are linearly independent.

Theorem 3 *Let* $\frac{\alpha(t)}{\beta(t)} > 0$. *Then the following system of integral equations are linearly independent*:

$$(\mathbf{I} - \mathbf{N})x = -\mathbf{M}y, \tag{24}$$

$$(\mathbf{I} - \mathbf{N})y = \mathbf{M}x \tag{25}$$

for $x, y \in H$.

Proof We define the functions S and T as

$$S(x, y) = (\mathbf{I} - \mathbf{N})x + \mathbf{M}y,$$

$$T(x, y) = (\mathbf{I} - \mathbf{N})y - \mathbf{M}x$$

for all $x, y \in H$. Suppose that equations (24) and (25) are linearly dependent, then there exists $d \in \mathbb{R} \setminus \{0\}$ such that $S(x, y) = dT(x, y)$ for all $x, y \in H$ or

$$(\mathbf{I} - \mathbf{N})x + \mathbf{M}y = d\big((\mathbf{I} - \mathbf{N})y - \mathbf{M}x\big), \quad \forall x, y \in H.$$

It follows from this that

$$(\mathbf{I} - \mathbf{N})x = -d\mathbf{M}x, \tag{26}$$

$$\mathbf{M}y = d(\mathbf{I} - \mathbf{N})y \tag{27}$$

for all $x, y \in H$. Note that the Robin problem is uniquely solvable subject to $\frac{\alpha(t)}{\beta(t)} > 0$. Therefore the system of integral equations

$$(\mathbf{I} - \mathbf{N})\mu = -\mathbf{M}\gamma, \tag{28}$$

$$(\mathbf{I} - \mathbf{N})\gamma = \mathbf{M}\mu \tag{29}$$

has the solution (μ, γ), where $\gamma(t) = \gamma_1(t) - \mu_2(t) + c_1$ and $\mu(t) = \gamma_2(t) + \mu_1(t) + c_2$. Equations (26) and (27) are true for $x = \mu$ and $y = \gamma$, *i.e.*,

$$(\mathbf{I} - \mathbf{N})\mu = -d\mathbf{M}\mu, \tag{30}$$

$$\mathbf{M}\gamma = d(\mathbf{I} - \mathbf{N})\gamma. \tag{31}$$

Using (28) and (29), equations (30) and (31) become

$$\mathbf{M}\gamma = d\mathbf{M}\mu, \tag{32}$$

$$-(\mathbf{I} - \mathbf{N})\mu = d(\mathbf{I} - \mathbf{N})\gamma. \tag{33}$$

Multiplying both sides of (33) by $(\mathbf{I} - \mathbf{N})^{-1}$, we get

$$-\mu = d\gamma.$$

Then from (32), we have

$$\mathbf{M}\gamma = -d^2 \mathbf{M}\gamma. \tag{34}$$

Since (28) and (29) has non-trivial solutions (μ, γ) and $(\mathbf{I} - \mathbf{N})$ is invertible, $\mathbf{M}\gamma \neq 0$. Hence by (34)

$$d^2 = -1.$$

It contradicts $d \in \mathbb{R}$. The theorem is proven. $\qquad\square$

Equations (22) and (23) imply

$$(\mathbf{I} - \mathbf{N})\mu_1(t) - \mathbf{M}\mu_2(t) + \mathbf{M}c_1 + (\mathbf{I} - \mathbf{N})c_2$$

$$= -\mathbf{M}\gamma_1(t) - (\mathbf{I} - \mathbf{N})\gamma_2(t), \quad t \in J, \tag{35}$$

$$-\mathbf{M}\mu_1(t) - (\mathbf{I} - \mathbf{N})\mu_2(t) + (\mathbf{I} - \mathbf{N})c_1 - \mathbf{M}c_2$$

$$= -(\mathbf{I} - \mathbf{N})\gamma_1(t) + \mathbf{M}\gamma_2(t), \quad t \in J. \tag{36}$$

Applying the definitions of the integral operators $\mathbf{N}$ and $\mathbf{M}$, we get

$$\mu_1(\tau) - \int_J N\big(\eta(\tau), \eta(t)\big)\mu_1(t)\,dt - \int_J M\big(\eta(\tau), \eta(t)\big)\mu_2(t)\,dt$$

$$+ \int_J M\big(\eta(\tau), \eta(t)\big)c_1\,dt + \left(c_2 - \int_J N\big(\eta(\tau), \eta(t)\big)c_2\,dt\right)$$

$$= -\int_J M\big(\eta(\tau), \eta(t)\big)\gamma_1(t)\,dt - \left(\gamma_2(\tau) - \int_J N\big(\eta(\tau), \eta(t)\big)\gamma_2(t)\,dt\right), \quad \tau \in J, \tag{37}$$

$$- \int_J M\big(\eta(\tau), \eta(t)\big)\mu_1(t)\,dt - \mu_2(\tau) + \int_J N\big(\eta(\tau), \eta(t)\big)\mu_2(t)\,dt$$

$$+ \left(c_1 - \int_J N\big(\eta(\tau), \eta(t)\big)c_1\,dt\right) - \int_J M\big(\eta(\tau), \eta(t)\big)c_2\,dt$$

$$= -\left(\gamma_1(\tau) - \int_J N\big(\eta(\tau), \eta(t)\big)\gamma_1(t)\,dt\right) + \int_J M\big(\eta(\tau), \eta(t)\big)\gamma_2(t)\,dt, \quad \tau \in J. \tag{38}$$

The system of integral equations (37) and (38) is in two unknown real functions $\mu_1(t)$, $\mu_2(t)$ and two unknown real constants c_1, c_2. Furthermore, by the definitions of the functions $\mu_1(t)$, $\mu_2(t)$ given in (17), (18), we have a condition in the form of a differential equation,

$$\sin\big(\zeta(t)\big)\mu_1'(t) - \cos\big(\zeta(t)\big)\mu_2'(t) = 0, \quad t \in J, \tag{39}$$

and the conditions

$$\mu_1(0) = 0, \quad \text{and} \quad \mu_2(0) = 0. \tag{40}$$

For bounded region Ω, since $\operatorname{Im} f(0) = 0$, we have $\operatorname{Re}[g(0)] = 0$. Applying the Cauchy integral formula for the bounded region Ω in Theorem 2 together with (19), (20), and (21), yields

$$g(0) = \frac{1}{2\pi i}\int_\Gamma \frac{g(\eta)}{\eta}\,d\eta = \frac{1}{2\pi i}\int_\Gamma \frac{g(\eta(t))}{\eta(t)}\dot\eta(t)\,dt$$

$$= \frac{1}{2\pi i}\int_J \frac{[\gamma_1(t) - \mu_2(t) + c_1 + i(\gamma_2(t) + \mu_1(t) + c_2)]\dot\eta(t)}{A(t)\eta(t)}\,dt, \quad t \in J.$$

Thus $\operatorname{Re}[g(0)] = 0$ implies that

$$\int_J \left[\big(\gamma_1(t) - \mu_2(t) + c_1\big)\operatorname{Im}\frac{\dot\eta(t)}{A(t)\eta(t)} + \big(\gamma_2(t) + \mu_1(t) + c_2\big)\operatorname{Re}\frac{\dot\eta(t)}{A(t)\eta(t)}\right]dt = 0, \quad t \in J,$$

which gives rise to the following condition for $\mu_1(t)$, $\mu_2(t)$, c_1, and c_2:

$$\int_J \operatorname{Re}\frac{\dot\eta(t)}{A(t)\eta(t)}\mu_1(t)\,dt - \int_J \operatorname{Im}\frac{\dot\eta(t)}{A(t)\eta(t)}\mu_2(t)\,dt + \int_J \operatorname{Im}\frac{\dot\eta(t)}{A(t)\eta_0(t)}c_1\,dt$$

$$+ \int_J \operatorname{Re}\frac{\dot\eta(t)}{A(t)\eta(t)}c_2\,dt = -\int_J \operatorname{Im}\frac{\dot\eta(t)}{A(t)\eta(t)}\gamma_1(t)\,dt - \int_J \operatorname{Re}\frac{\dot\eta(t)}{A(t)\eta_0(t)}\gamma_2(t)\,dt. \tag{41}$$

In summary to solve the Robin problem (1) on a bounded multiply connected regions, we solve for $\mu_1(t)$, $\mu_2(t)$, c_1, and c_2 from (37), (38), (40), and (41). Then we compute $g(\eta(t))$ from (14). Using the relation $g = -if$ we get $f(\eta(t)) = ig(\eta(t))$ and hence $u(\eta(t)) = \text{Re}[f(\eta(t))]$. The interior values of f can be computed via the Cauchy integral formula (9), which yields $u(z) = \text{Re}[f(z)]$ for $z \in \Omega$.

For an unbounded region Ω, since $f(\infty) = 0$, we have $g(\infty) = 0$. By using the Cauchy integral formula for the unbounded region Ω in Theorem 2, we have

$$\frac{1}{2\pi i} \int_\Gamma \frac{g(\eta)}{\eta - z} \, d\eta = \begin{cases} g(z) - g(\infty), & z \in \Omega, \\ -g(\infty), & z \notin \Omega. \end{cases}$$

Since $z = 0 \notin \Omega$,

$$\frac{1}{2\pi i} \int_\Gamma \frac{g(\eta)}{\eta} \, d\eta = -g(\infty) = 0.$$

Applying (19), (20), and (21), we get

$$\frac{1}{2\pi i} \int_J \frac{\gamma_1(t) - \mu_2(t) + c_1 + i(\gamma_2(t) + \mu_1(t) + c_2)}{A(t)} \frac{\dot\eta(t) \, dt}{\eta(t)} = 0, \quad t \in J,$$

or, equivalently,

$$\int_J \left[\gamma_1(t) - \mu_2(t) + c_1 + i(\gamma_2(t) + \mu_1(t) + c_2)\right]\left[\text{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)} + i\,\text{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\right] dt = 0,$$

which implies

$$\int_J \left[(\gamma_1(t) - \mu_2(t) + c_1)\,\text{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)} \right.$$
$$+ (\gamma_2(t) + \mu_1(t) + c_2)\,\text{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\Big] dt$$
$$- i\int_J \left[(\gamma_1(t) - \mu_2(t) + c_1)\,\text{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\right.$$
$$- (\gamma_2(t) + \mu_1(t) + c_2)\,\text{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\Big] dt = 0, \quad t \in J. \tag{42}$$

The real part and imaginary part of equation (42), respectively, yield

$$\int_J \left[(\gamma_1(t) - \mu_2(t) + c_1)\,\text{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)} + (\gamma_2(t) + \mu_1(t) + c_2)\,\text{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\right] dt = 0, \quad t \in J,$$

and

$$\int_J \left[(\gamma_1(t) - \mu_2(t) + c_1)\,\text{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)} - (\gamma_2(t) + \mu_1(t) + c_2)\,\text{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\right] dt = 0, \quad t \in J,$$

which gives rise to the conditions

$$
\int_J \mathrm{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\mu_1(t)\,dt - \int_J \mathrm{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\mu_2(t)\,dt
$$

$$
+ \int_J \mathrm{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}c_1\,dt + \int_J \mathrm{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}c_2\,dt
$$

$$
= -\int_J \mathrm{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\gamma_1(t)\,dt - \int_J \mathrm{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\gamma_2(t)\,dt, \quad t \in J, \tag{43}
$$

and

$$
-\int_J \mathrm{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\mu_1(t)\,dt - \int_J \mathrm{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\mu_2(t)\,dt
$$

$$
+ \int_J \mathrm{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}c_1\,dt - \int_J \mathrm{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}c_2\,dt
$$

$$
= -\int_J \mathrm{Re}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\gamma_1(t)\,dt - \int_J \mathrm{Im}\,\frac{\dot\eta(t)}{A(t)\eta(t)}\gamma_2(t)\,dt, \quad t \in J. \tag{44}
$$

In summary to solve the Robin problem (1) on an unbounded multiply connected regions, we solve for $\mu_1(t)$, $\mu_2(t)$, c_1, and c_2 from (37), (38) (40), (43), and (44). Then we compute $g(\eta(t))$ from (14). Using the relation $g = -if$ we get $f(\eta(t)) = ig(\eta(t))$ and hence $f(\eta(t)) = u(\eta(t)) + iv(\eta(t))$. The exterior values of f can be computed via the Cauchy integral formula (10), which yields $u(z) = \mathrm{Re}[f(z)]$ for $z \in \Omega$.

4 Numerical implementation

Since the functions $A(t)$ and $\eta(t)$ are 2π-periodic, the integral equations in (35) and (36) can be best discretized on an equidistant grid by the Nyström method with trapezoidal rule using n equidistant nodes [19]. Since $\mathbf{M} = \mathbf{M}_1 - \mathbf{K}$, the integrals involving the singular kernel $K(\tau, t)$ are discretized using the Wittich method [20]. Define the n equidistant collocation points t_i by

$$
t_i = \frac{2\pi(i-1)}{n}, \quad i = 1, 2, 3, \dots, n.
$$

Discretizing the integral equations (37) and (38) on J, we obtain, respectively, the linear systems

$$
\mu_1(t_i) - \frac{2\pi}{n}\sum_{j=1}^{n} N\big(\eta(t_i),\eta(t_j)\big)\mu_1(t_j) + \sum_{j=1}^{n} K(t_i,t_j)\mu_2(t_j) - \frac{2\pi}{n}\sum_{j=1}^{n} M_1\big(\eta(t_i),\eta(t_j)\big)\mu_2(t_j)
$$

$$
- \sum_{j=1}^{n} K(t_i,t_j)c_1 + \frac{2\pi}{n}\sum_{j=1}^{n} M_1\big(\eta(t_i),\eta(t_j)\big)c_1 + c_2 - \frac{2\pi}{n}\sum_{j=1}^{n} N\big(\eta(t_i),\eta(t_j)\big)c_2
$$

$$
= -\left(\gamma_2(t_i) - \frac{2\pi}{n}\sum_{j=1}^{n} N\big(\eta(t_i),\eta(t_j)\big)\gamma_2(t_j)\right)
$$

$$
+ \sum_{j=1}^{n} K(t_i,t_j)\gamma_1(t_j) - \frac{2\pi}{n}\sum_{j=1}^{n} M_1\big(\eta(t_i),\eta(t_j)\big)\gamma_1(t_j)
$$

and

$$\sum_{j=1}^{n} K(t_i, t_j)\mu_1(t_j) - \frac{2\pi}{n} \sum_{j=1}^{n} M_1\big(\eta(t_i), \eta(t_j)\big)\mu_1(t_j) - \mu_2(t_i)$$

$$+ \frac{2\pi}{n} \sum_{j=1}^{n} N\big(\eta(t_i), \eta(t_j)\big)\mu_2(t_j) + c_1 - \frac{2\pi}{n} \sum_{j=1}^{n} N\big(\eta(t_i), \eta(t_j)\big)c_1$$

$$+ \sum_{j=1}^{n} K(t_i, t_j)c_2 - \frac{2\pi}{n} \sum_{j=1}^{n} M_1\big(\eta(t_i), \eta(t_j)\big)c_2$$

$$= -\gamma_1(t_i) + \frac{2\pi}{n} \sum_{j=1}^{n} N\big(\eta(t_i), \eta(t_j)\big)\gamma_1(t_j) - \sum_{j=1}^{n} K(t_i, t_j)\gamma_2(t_j)$$

$$+ \frac{2\pi}{n} \sum_{j=1}^{n} M_1\big(\eta(t_i), \eta(t_j)\big)\gamma_2(t_j),$$

where

$$K(t_i, t_j) = \begin{cases} 0, & \text{if } j - i \text{ is even}, \\ \frac{2}{n} \cot \frac{(i-j)\pi}{n}, & \text{if } j - i \text{ is odd}, \end{cases}$$

$$N\big(\eta(t_i), \eta(t_j)\big) = \begin{cases} \frac{1}{\pi} \operatorname{Im}[\frac{A(t_i)}{A(t_j)} \frac{\dot{\eta}(t_j)}{\eta(t_j) - \eta_1(t_i)}], & \text{if } t_i \neq t_j, \\ \frac{1}{\pi} \operatorname{Im}[\frac{1}{2} \frac{\ddot{\eta}(t_i)}{\dot{\eta}(t_i)} - \frac{\dot{A}(t_i)}{A(t_i)}], & \text{if } t_i = t_j, \end{cases}$$

and

$$M_1\big(\eta(t_i), \eta(t_j)\big) = \begin{cases} \frac{1}{\pi} \operatorname{Re}[\frac{A(t_i)}{A(t_j)} \frac{\dot{\eta}(t_j)}{\eta(t_j) - \eta(t_i)}] + \frac{1}{2\pi} \cot \frac{t_i - t_j}{2}, & \text{if } t_i \neq t_j, \\ \frac{1}{\pi} \operatorname{Re}[\frac{1}{2} \frac{\ddot{\eta}(t_i)}{\dot{\eta}(t_i)} - \frac{\dot{A}(t_i)}{A(t_i)}], & \text{if } t_i = t_j. \end{cases}$$

Hence, we obtain $2mn$ equations in $2mn + 2m$ variables ($m = 1, 2, \ldots$):

$$\mu_{01}(t_1), \mu_{01}(t_2), \ldots, \mu_{01}(t_n), \ldots, \mu_{0m}(t_1), \mu_{0m}(t_2) \ldots, \mu_{0m}(t_n),$$

$$\mu_{11}(t_1), \mu_{11}(t_2), \ldots, \mu_{11}(t_n), \ldots, \mu_{1m}(t_1), \mu_{1m}(t_2) \ldots, \mu_{1m}(t_n), \tag{45}$$

$$c_{01}, \ldots c_{0m}, c_{11}, \ldots, c_{1m}.$$

The condition (39) is discretized using a five-point central difference method [21] to obtain mn equations. For $t_i \in J$, we obtain for $i = 1$

$$\sin\big(\zeta(t_1)\big)\big[-25\mu_1(t_1) + 48\mu_1(t_2) - 36\mu_1(t_3) + 16\mu_1(t_4) - 3\mu_1(t_5)\big]$$

$$- \cos\big(\zeta(t_1)\big)\big[-25\mu_2(t_1) + 48\mu_2(t_2) - 36\mu_2(t_3) + 16\mu_2(t_4) - 3\mu_2(t_5)\big] = 0.$$

For $i = 2$, we have

$$\sin\big(\zeta(t_2)\big)\big[-3\mu_1(t_1) - 10\mu_1(t_2) + 18\mu_1(t_3) - 6\mu_1(t_4) + \mu_1(t_5)\big]$$

$$- \cos\big(\zeta(t_2)\big)\big[-3\mu_2(t_1) - 10\mu_2(t_2) + 18\mu_2(t_3) - 6\mu_2(t_4) + \mu_2(t_5)\big] = 0.$$

For $i = 3, \ldots, n - 2$, we have

$$\sin\big(\zeta(t_i)\big)\big[\mu_1(t_{i-2}) - 8\mu_1(t_{i-1}) + 8\mu_1(t_{i+1}) - \mu_1(t_{i+2})\big]$$
$$- \cos\big(\zeta(t_i)\big)\big[\mu_2(t_{i-2}) - 8\mu_2(t_{i-1}) + 8\mu_2(t_{i+1}) - \mu_2(t_{i+2})\big] = 0.$$

For $i = n - 1$, we have

$$\sin\big(\zeta(t_{n-1})\big)\big[-\mu_1(t_{n-4}) + 6\mu_1(t_{n-3}) - 18\mu_1(t_{n-2}) + 10\mu_1(t_{n-1}) + 3\mu_1(t_n)\big]$$
$$- \cos\big(\zeta(t_{n-1})\big)\big[-\mu_2(t_{n-4}) + 6\mu_2(t_{n-3}) - 18\mu_2(t_{n-2}) + 10\mu_2(t_{n-1}) + 3\mu_2(t_n)\big] = 0.$$

For $i = n$, we have

$$\sin\big(\zeta(t_n)\big)\big[3\mu_1(t_{n-4}) - 16\mu_1(t_{n-3}) + 36\mu_1(t_{n-2}) - 48\mu_1(t_{n-1}) + 25\mu_1(t_n)\big]$$
$$- \cos\big(\zeta(t_n)\big)\big[3\mu_2(t_{n-4}) - 16\mu_2(t_{n-3}) + 36\mu_2(t_{n-2}) - 48\mu_2(t_{n-1}) + 25\mu_2(t_n)\big] = 0.$$

We now have $3mn$ equations in $2mn + 2m$ in unknowns (45). For the Robin problem on a bounded multiply connected region, discretizing condition (41) gives

$$\sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\mu_1(t_j) - \sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\mu_2(t_j)$$

$$+ \sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]c_1 + \sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]c_2$$

$$= -\sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\gamma_1(t_j) - \sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\gamma_2(t_j).$$

For the Robin problem on an unbounded multiply connected region, discretizing conditions (43) and (44), we obtain

$$\sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\mu_1(t_j) - \sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\mu_2(t_j)$$

$$+ \sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]c_1 + \sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]c_2$$

$$= -\sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\gamma_1(t_j) - \sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\gamma_2(t_j),$$

$$-\sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\mu_1(t_j) - \sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\mu_2(t_j)$$

$$+ \sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]c_1 - \sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]c_2$$

$$= -\sum_{j=1}^{n} \mathrm{Re}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\gamma_1(t_j) - \sum_{j=1}^{n} \mathrm{Im}\left[\frac{\dot{\eta}(t_j)}{A(t_j)\eta(t_j)}\right]\gamma_2(t_j).$$

Hence, for the Robin problem in a bounded multiply connected region we obtain $(3mn)$ by a $(2mn + 2m)$ linear system, including the $2m$ condition in (40) giving a $(3mn + 2m)$ by $(2mn + 2m)$ linear system. Finally, adding the equation obtained from (41) makes a $(3mn + 2m + 1)$ by $(2mn + 2m)$ over-determined linear system which is full ranked.

For the Robin problem in an unbounded multiply connected region we obtain a $(3mn)$ by $(2mn + 2m)$ linear system, including the $2m$ condition in (40) giving a $(3mn + 2m)$ by $(2mn + 2m)$ linear system. Finally, adding the two equations obtained from (43) and (44) makes a $(3mn + 2m + 2)$ by $(2mn + 2m)$ over-determined linear system which is full ranked.

In both cases, the obtained over-determined systems are solved using the MATLAB's \operator that makes use of the QR factorization with a column pivoting method.

From the computed solutions $\mu_1(t)$, $\mu_2(t)$, $\gamma_1(t)$, $\gamma_2(t)$, c_1, and c_2, the approximate boundary values of the analytic function $f_n(\eta(t))$ are calculated using the formula

$$f_n\big(\eta(t)\big) = \frac{-(\gamma_2(t) + \mu_1(t) + c_2) + \mathrm{i}(\gamma_1(t) - \mu_2(t) + c_1)}{e^{\mathrm{i}\zeta(t)}}.$$

The approximate interior values of the function $f(z)$ are calculated via the Cauchy integral formula in the form of

$$f(z) = \begin{cases} \dfrac{\frac{1}{2\pi\mathrm{i}} \int_J \frac{f(\eta)}{\eta - z}\, d\eta}{\frac{1}{2\pi\mathrm{i}} \int_J \frac{d\eta}{\eta - z}} & [\text{Bounded region } \Omega], \\[2ex] \dfrac{f(\infty) + \frac{1}{2\pi\mathrm{i}} \int_J \frac{f(\eta)}{\eta - z}\, d\eta}{1 + \frac{1}{2\pi\mathrm{i}} \int_J \frac{d\eta}{\eta - z}} & [\text{Unbounded region } \Omega]. \end{cases} \tag{46}$$

Here $f(\infty) = 0$ for unbounded region Ω. The formula in (46) has the advantage that the denominators in this formula compensate for the error in the numerators [22, 23]. The integrals in (46) are approximated by the trapezoidal rule. The respective discretization formula for bounded and unbounded regions are

$$f_n(z) = \begin{cases} \dfrac{\sum_{i=1}^n \frac{f_n(\eta(t_i))\dot{\eta}(t_i)}{\eta(t_i) - z}}{\sum_{i=1}^n \frac{\dot{\eta}(t_i)}{\eta(t_i) - z}} & [\text{Bounded region } \Omega], \\[2ex] \dfrac{\sum_{i=1}^n \frac{f_n(\eta(t_i))\dot{\eta}(t_i)}{\eta(t_i) - z}}{\mathrm{i}n + \sum_{i=1}^n \frac{\dot{\eta}(t_i)}{\eta(t_i) - z}} & [\text{Unbounded region } \Omega]. \end{cases}$$

5 Numerical examples

We consider some examples of solving Robin problem with Robin boundary condition (1) in bounded and unbounded multiply connected regions.

Example 1 Consider a bounded triply connected region Ω bounded by

$$\Gamma_0 : \quad \eta_0(t) = 1.5\cos t + 0.6\mathrm{i}\sin t,$$

$$\Gamma_1 : \quad \eta_1(t) = 0.7 + 0.3e^{-\mathrm{i}t},$$

$$\Gamma_2 : \quad \eta_2(t) = -0.7 + 0.3e^{-\mathrm{i}t}, \quad 0 \le t \le 2\pi.$$

For $\alpha(t)$, $\beta(t)$, and $A(t)$ in (1), we choose

$$\alpha(t) = \begin{cases} 10 - 10\sin t, & t \in J_0, \\ 2 - 2\sin t, & t \in J_1, \\ 2\cos t - \sin 2t, & t \in J_2, \end{cases}$$

Table 1 The errors $\|u(\eta(t)) - u_n(\eta(t))\|_\infty$ on boundary Γ for Example 1

n	$\|u(\eta(t)) - u_n(\eta(t))\|_\infty$
32	1.23 (−2)
64	2.88 (−4)
128	2.28 (−5)
256	1.61 (−6)
512	1.07 (−7)
1,024	7.11 (−9)

Table 2 Absolute errors $|f(z) - f_n(z)|$ at some selected points on Ω for Example 1

n	$-1.25 + 0.2i$	$-0.5 + 0.4i$	$0.5 + 0.4i$	$1.3 - 0.1i$
32	2.10 (−3)	8.0 (−4)	8.0 (−4)	1.30 (−3)
64	2.98 (−5)	6.11 (−5)	6.04 (−5)	3.31 (−5)
128	1.80 (−6)	7.68 (−6)	6.80 (−6)	1.61 (−6)
256	1.99 (−7)	5.79 (−7)	5.10 (−7)	8.08 (−8)
512	1.57 (−8)	5.13 (−8)	3.53 (−8)	4.48 (−9)
1,024	1.15 (−9)	3.32 (−9)	2.39 (−9)	3.08 (−10)

and

$$\beta(t) = \begin{cases} (3\sqrt{21}\sin^2 t + 4), & t \in J_0, \\ 0.6, & t \in J_1, \\ 0.6\cos t, & t \in J_2. \end{cases}$$

The function $l(t)$ in (1) is obtained by choosing an exact solution $u(z) = \mathrm{Re}[f(z)]$, where $f(z) = \cos(z) - 2$. This yields the exact values $c_{01} = 0$, $c_{02} = 1.9293$, $c_{11} = 0$, $c_{12} = 1.4597$, $c_{21} = 0$, and $c_{22} = 1.0789$. For this example, $A(t) = e^{i(t+\cos t)}$, $t \in J$. The integrals in (15) and (16) are calculated by the Gauss-Legendre rule with 256 nodes.

Table 1 lists the maximum error norms $\|u(\eta(t)) - u_n(\eta(t))\|_\infty$, where n is the number of nodes and $u_n(\eta(t))$ is the numerical approximation of $u(\eta(t))$ based on our method. The errors $\|f(z) - f_n(z)\|$ at some selected points are listed in Table 2. The absolute errors $|u(z) - u_n(z)|$ for selected points in the entire domain are plotted in Figure 3. Figure 4 shows the surface plot of $u_n(z)$ with $n = 1,024$.

Example 2 Consider an unbounded triply connected region Ω with boundaries

$$\Gamma_1: \quad \eta_1(t) = 0.5\cos t - 0.3i\sin t,$$

$$\Gamma_2: \quad \eta_2(t) = -1 + (0.2 + 0.04\cos 4t)e^{-it},$$

$$\Gamma_3: \quad \eta_3(t) = 1 + (0.2 + 0.04\cos 4t)e^{-it}, \quad 0 \le t \le 2\pi.$$

For $\alpha(t)$ and $\beta(t)$ in (1), we choose

$$\alpha(t) = \begin{cases} 10 + 20\cos t, & t \in J_1, \\ 1 + 2\cos t, & t \in J_2, \\ 1 + 2\cos t, & t \in J_3, \end{cases}$$
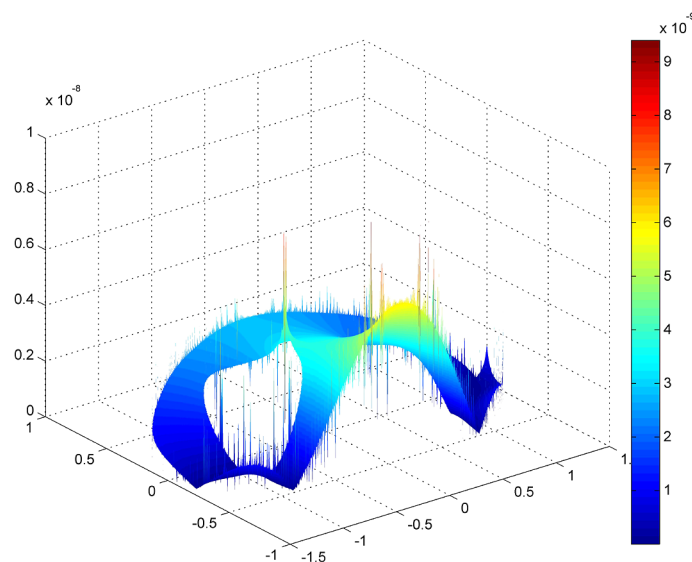
Figure 3 The absolute errors $|u(z) - u_n(z)|$ for the entire domain with $n = 1{,}024$, for Example 1.
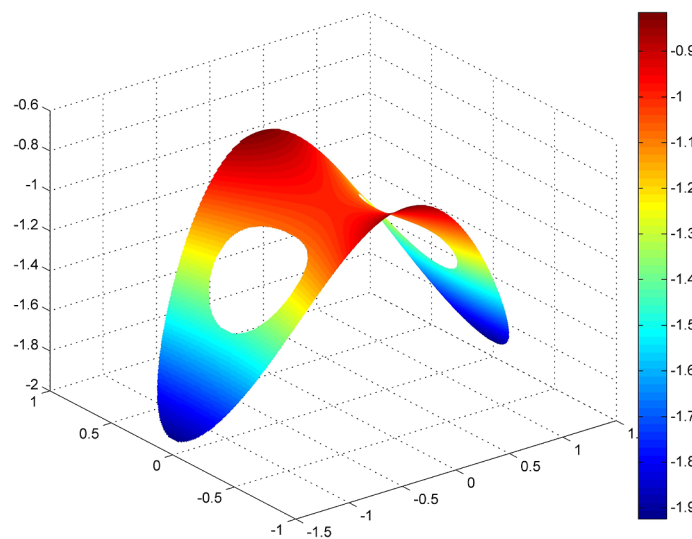

Figure 4 The surface plot of $u_n(z)$ for Example 1 with $n = 1{,}024$.

and

$$
\beta(t) = \begin{cases}
(\sqrt{25 - 16\cos^2 t}, & t \in J_1, \\
|(\frac{e^{-it}\cos 4t}{25} + \frac{1}{5})i + \frac{4\sin 4t e^{-it}}{25}|, & t \in J_2, \\
((625((\frac{\cos 5t}{10} - \frac{3\cos 3t}{50} + \frac{\cos t}{5})^2 \\
\quad + (\frac{3\sin 3t}{50} + \frac{\sin 5t}{10} + \frac{\sin t}{5})^2))^{1/2})/25, & t \in J_3.
\end{cases}
$$

The function $l(t)$ in (1) is obtained by choosing an exact solution $u(z) = \mathrm{Re}[f(z)]$, where $f(z) = \frac{1}{z}$. This yields the exact values, $c_{11} = 0$, $c_{12} = -2$, $c_{21} = 0$, $c_{22} = 1.3158$, $c_{31} = 0$, and $c_{32} = -0.8065$. For this example, $A(t) = e^{i(t+2\sin t)}$, $t \in J$. The integrals in (15) and (16) are calculated by the Gauss-Legendre rule with 256 nodes.

Table 3 The errors $\|u(\eta(t)) - u_n(\eta(t))\|_\infty$ on boundary Γ for Example 2

n	$\|u(\eta(t)) - u_n(\eta(t))\|_\infty$
32	1.38 (−2)
64	9.62 (−4)
128	6.47 (−5)
256	3.87 (−6)
512	4.73 (−7)
1,024	6.57 (−8)

Table 4 Absolute errors $|f(z) - f_n(z)|$ at some selected points on Ω for Example 2

n	−0.6 − 0.2i	0.75	0.8 + 0.4i	0.7 + 0.2i
32	7.0 (−4)	4.0 (−4)	1.0 (−3)	1.2 (−3)
64	1.47 (−5)	2.03 (−5)	8.64 (−5)	1.08 (−4)
128	6.51 (−7)	1.87 (−6)	6.51 (−6)	7.95 (−6)
256	2.48 (−7)	6.02 (−7)	4.56 (−7)	5.87 (−7)
512	7.30 (−8)	1.22 (−7)	4.07 (−8)	7.27 (−8)
1,024	1.29 (−8)	1.96 (−8)	5.40 (−9)	1.11 (−8)
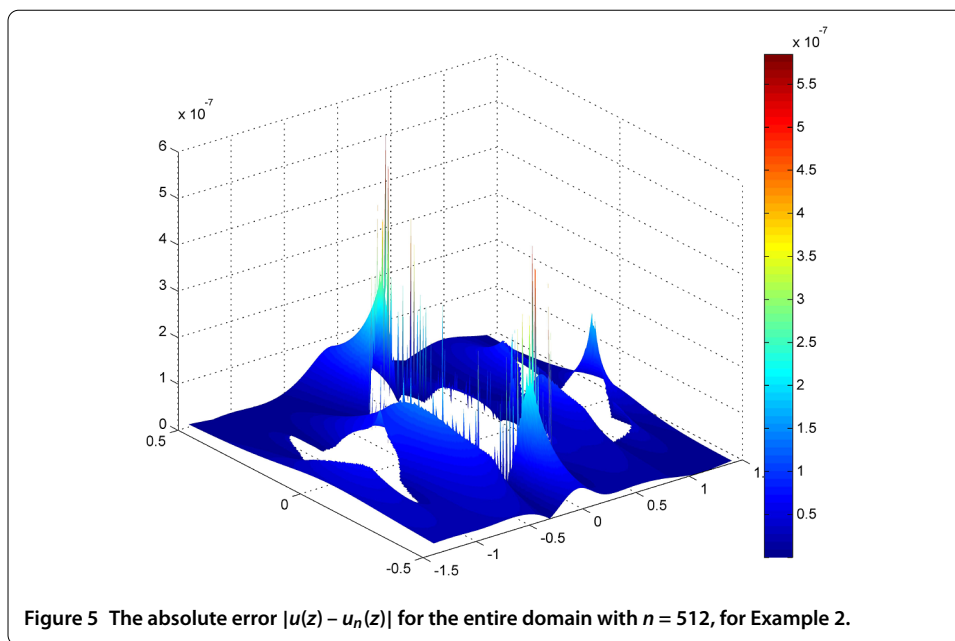
Figure 5 The absolute error $|u(z) - u_n(z)|$ for the entire domain with $n = 512$, for Example 2.

Table 3 lists the maximum error norms $\|u(\eta(t)) - u_n(\eta(t))\|_\infty$, where n is the number of nodes and $u_n(\eta(t))$ is the numerical approximation of $u(\eta(t))$ based on our method. The errors $\|f(z) - f_n(z)\|$ at some selected points are listed in Table 4. The absolute errors $|u(z) - u_n(z)|$ for selected points in the entire domain are plotted in Figure 5. Figure 6 shows the surface plot of $u_n(z)$ with $n = 512$.

Example 3 Consider a bounded 4-multiply connected region Ω with boundaries

$$\Gamma_0: \quad \eta_0(t) = 1.5\cos t + i\sin t,$$

$$\Gamma_1: \quad \eta_1(t) = -0.5 + 0.3i + 0.3e^{-it},$$

$$\Gamma_2: \quad \eta_2(t) = 0.5 + 0.3i + 0.3e^{-it},$$

$$\Gamma_3: \quad \eta_3(t) = -0.35i + 0.3e^{-it}, \quad 0 \le t \le 2\pi.$$
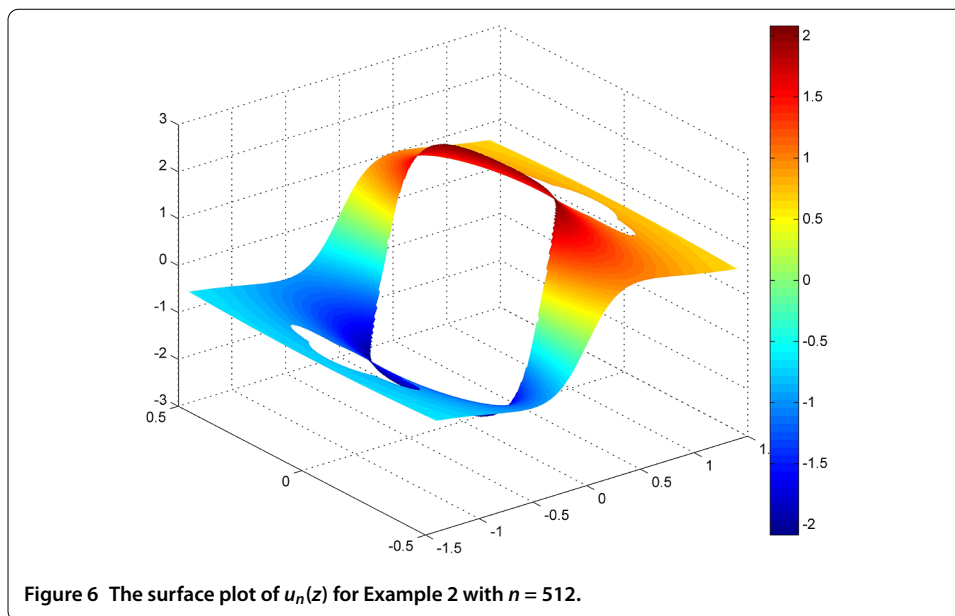
Figure 6 The surface plot of $u_n(z)$ for Example 2 with $n = 512$.

For $\alpha(t)$ and $\beta(t)$ in (1), we choose

$$\alpha(t) = \begin{cases} 2 - 2\cos t, & t \in J_0, \\ 1 - \cos t + 2\sin t - \sin 2t, & t \in J_1, \\ 1 + \cos t - 2\cos^2 t, & t \in J_2, \\ 1 - \cos t - 2\sin t + \sin 2t, & t \in J_3, \end{cases}$$

and

$$\beta(t) = \begin{cases} \sqrt{5\sin^2 + 4}, & t \in J_0, \\ 0.3 + 0.6\sin t, & t \in J_1, \\ 0.3 + 0.6\cos t, & t \in J_2, \\ 0.3 - 0.6\sin t, & t \in J_3. \end{cases}$$

The function $l(t)$ in (1) is obtained by choosing an exact solution $u(z) = \mathrm{Re}[f(z)]$, where $f(z) = z^2 - 2$. This yields the exact values, $c_{01} = 0$, $c_{02} = -0.25$, $c_{11} = -0.12$, $c_{12} = 2.05$, $c_{21} = 0.48$, $c_{22} = 1.45$, $c_{31} = -0.21$, and $c_{32} = 2.0325$. For this example, $A(t) = e^{i(t-\sin t)}$, $t \in J$. The integrals in (15) and (16) are calculated by the Gauss-Legendre rule with 256 nodes.

Table 5 lists the maximum error norms $\|u(\eta(t)) - u_n(\eta(t))\|_\infty$, where n is the number of nodes and $u_n(\eta(t))$ is the numerical approximation of $u(\eta(t))$ based on our method. The errors $\|f(z) - f_n(z)\|$ at some selected points are listed in Table 6. The absolute errors $|u(z) - u_n(z)|$ for selected points in the entire domain are plotted in Figure 7. Figure 8 shows the surface plot of $u_n(z)$ with $n = 512$.

Example 4 Consider an unbounded 4-multiply connected region Ω with boundaries

$$\Gamma_1 : \quad \eta_1(t) = 0.95 + \big(0.2 + 0.04\cos(4t)\big)e^{-it},$$
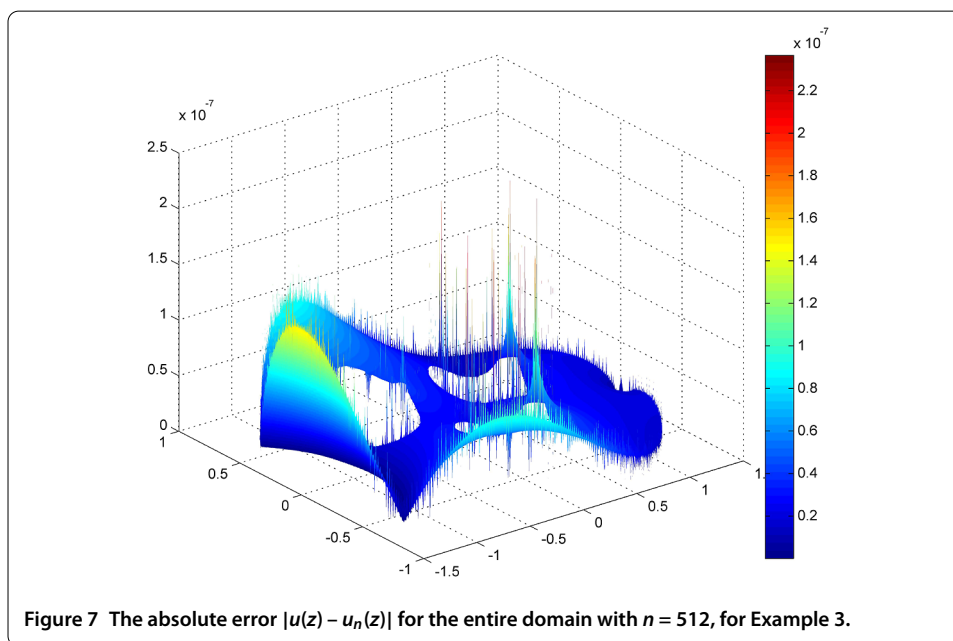
$$\Gamma_2 : \quad \eta_2(t) = 0.2e^{-it},$$

Table 5 The errors $\|u(\eta(t)) - u_n(\eta(t))\|_\infty$ on boundary Γ for Example 3

n	$\|u(\eta(t)) - u_n(\eta(t))\|_\infty$
32	1.69 (−2)
64	8.61 (−4)
128	4.98 (−5)
256	3.11 (−6)
512	1.99 (−7)
1,024	1.32 (−8)

Table 6 The absolute errors $|f(z) - f_n(z)|$ at some selected points on Ω for Example 3

n	−1 − 0.2i	−0.5 − 0.6i	0.2 + 0.6i	1.2 − 0.3i
32	8.0 (−4)	4.5 (−3)	3.4 (−3)	2.4 (−3)
64	9.10 (−5)	2.24 (−4)	1.48 (−4)	1.10 (−4)
128	6.68 (−6)	1.30 (−5)	8.13 (−6)	6.35 (−6)
256	4.05 (−7)	8.47 (−7)	5.16 (−7)	4.07 (−7)
512	2.33 (−8)	5.63 (−8)	3.46 (−8)	2.69 (−8)
1,024	1.34 (−9)	3.74 (−9)	2.34 (−9)	1.78 (−9)

Figure 7 The absolute error $|u(z) - u_n(z)|$ for the entire domain with $n = 512$, for Example 3.

$$\Gamma_3: \quad \eta_3(t) = (0.5 + 0.45i) + \left(0.2e^{-it}\right),$$

$$\Gamma_4: \quad \eta_4(t) = (0.5 - 0.45i) + (0.2\cos(t) - 0.1i\sin(t), \quad 0 \le t \le 2\pi.$$

For $\alpha(t)$ and $\beta(t)$ in (1), we choose

$$\alpha(t) = \begin{cases} 25 - 25\sin t, & t \in J_1, \\ 1 - \sin t + 2\cos t - \sin 2t, & t \in J_2, \\ 3(1 - \sin t - 2\cos t + \sin 2t), & t \in J_3, \\ 10 - 10\sin t, & t \in J_4, \end{cases}$$
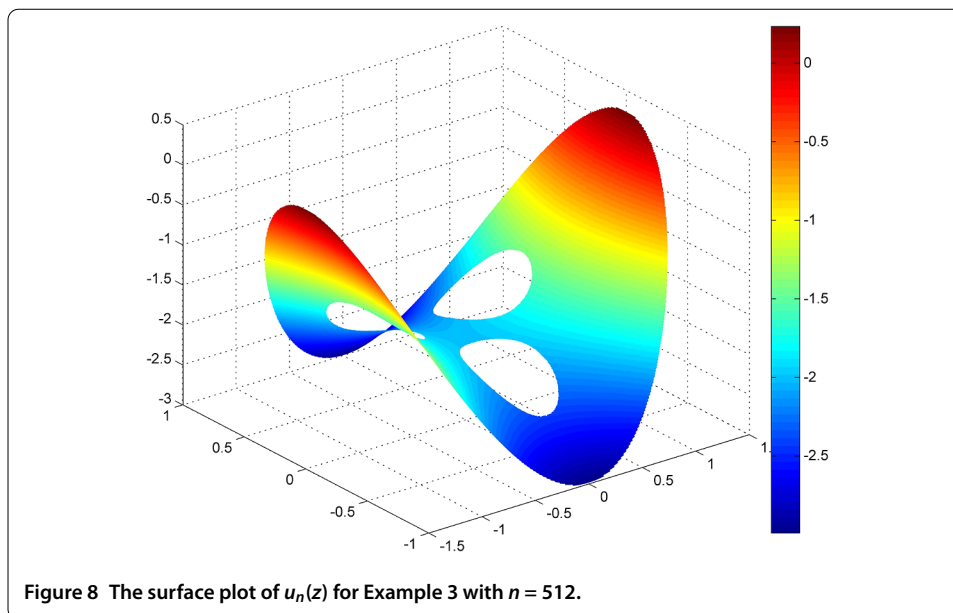
Figure 8 The surface plot of $u_n(z)$ for Example 3 with $n = 512$.

Table 7 The errors $\|u(\eta(t)) - u_n(\eta(t))\|_\infty$ on boundary Γ for Example 4

n	$\|u(\eta(t)) - u_n(\eta(t))\|_\infty$
32	1.56 (−2)
64	8.57 (−4)
128	5.16 (−5)
256	3.21 (−6)
512	1.99 (−7)
1,024	1.24 (−8)

Table 8 The absolute errors $|f(z) - f_n(z)|$ at some selected points on Ω for Example 4

n	0.2 − 0.4i	0.3 − 0.2i	0.6 + 0.2i	1 + 0.5i
32	7.0 (−4)	1.6 (−3)	1.3 (−3)	1.1 (−3)
64	3.87 (−5)	1.02 (−4)	8.92 (−5)	37.07 (−5)
128	2.04 (−6)	6.23 (−6)	5.86 (−6)	4.58 (−6)
256	9.83 (−8)	3.66 (−7)	3.76 (−7)	2.97 (−7)
512	4.34 (−9)	2.12 (−8)	2.39 (−8)	1.92 (−8)
1,024	1.87 (−10)	1.22 (−9)	1.51 (−9)	1.24 (−9)

and

$$\beta(t) = \begin{cases} ((625((\frac{\cos 5t}{10} - \frac{3\cos 3t}{50} + \frac{\cos t}{5})^2 \\ \quad + (\frac{3\sin 3t}{50} + \frac{\sin 5t}{10} + \frac{\sin t}{5})^2))^{1/2}), & t \in J_1, \\ 0.2 + 0.4\cos t, & t \in J_2, \\ 0.6 - 1.2\cos t, & t \in J_3, \\ \sqrt{3\sin^2 t + 1}, & t \in J_4. \end{cases}$$

The function $l(t)$ in (1) is obtained by choosing an exact solution $u(z) = \mathrm{Re}[f(z)]$, where $f(z) = \frac{1}{z}$. This yields the exact values, $c_{11} = 0$, $c_{12} = -0.8403$, $c_{21} = 0$, $c_{22} = -5$, $c_{31} = -0.6498$, $c_{32} = -1.0108$, $c_{41} = 0.6498$, and $c_{42} = -1.0108$. For this example, $A(t) = e^{i(t+\cos t)}$, $t \in J$. The integrals in (15) and (16) are calculated by the Gauss-Legendre rule with 256 nodes.
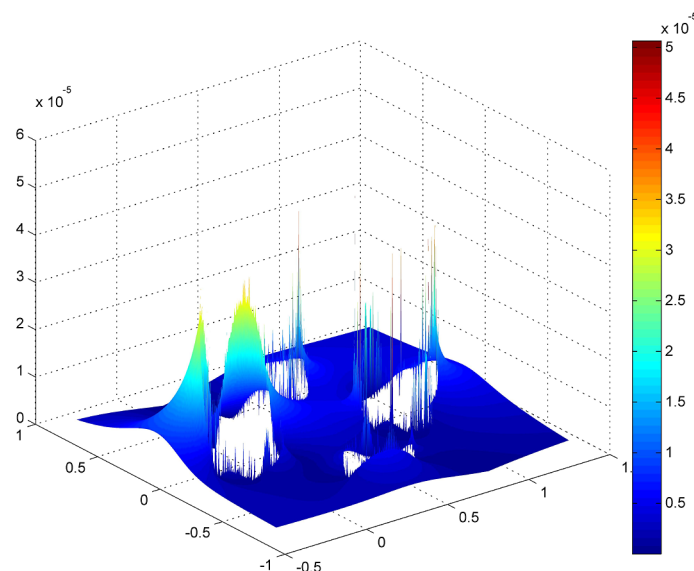
Figure 9 The absolute error $|u(z) - u_n(z)|$ for the entire domain with $n = 128$, for Example 4.
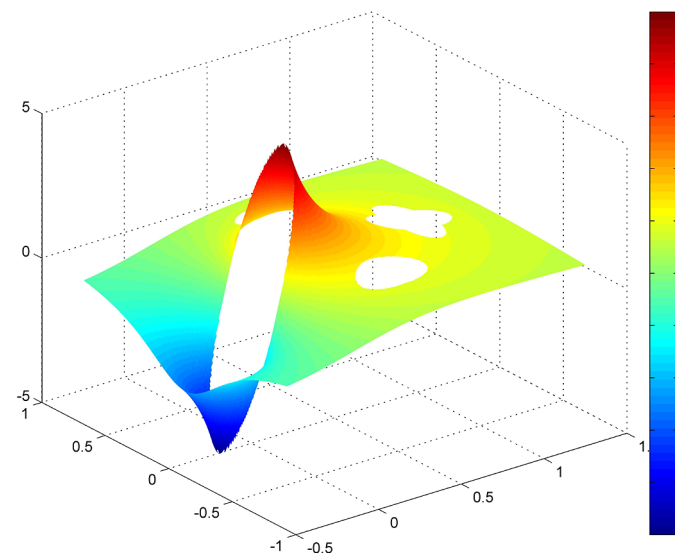

Figure 10 The surface plot of $u_n(z)$ for Example 4 with $n = 128$.

Table 7 lists the maximum error norms $\|u(\eta(t)) - u_n(\eta(t))\|_\infty$, where n is the number of nodes and $u_n(\eta(t))$ is the numerical approximation of $u(\eta(t))$ based on our method. The errors $\|f(z) - f_n(z)\|$ at some selected points are listed in Table 8. The absolute errors $|u(z) - u_n(z)|$ for selected points in the entire domain are plotted in Figure 9. Figure 10 shows the surface plot of $u_n(z)$ with $n = 128$.

6 Conclusion

A boundary integral equation for solving both bounded and unbounded Robin problem in multiply connected regions is presented. Unlike previous work on the mixed boundary value problem where the Dirichlet and Neumann conditions are given separately, see [5]

and [6], the Robin problem has a single boundary condition that is a linear combination of the Dirichlet and Neumann. Here, the method reduces the Robin problem to a RH problem, which leads to a system of integral equations. The proof that these integral equations are linearly independent was shown here. Differential equations were also constructed to provide additional conditions to make the Robin problem uniquely solvable. The integral equations were discretized by the Nyström method with the trapezoidal rule and Wittich's method, while the differential equations were discretized by the five-point central difference method. The presented numerical results illustrate that the proposed method can be used to produce approximations of high accuracy.

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