Lee and Kim *Boundary Value Problems* (2016) 2016:95
DOI 10.1186/s13661-016-0603-x

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Multiplicity results for nonlinear Neumann boundary value problems involving p-Laplace type operators

Jongrak Lee[1] and Yun-Ho Kim[2*]

*Correspondence:
kyh1213@smu.ac.kr
[2]Department of Mathematics Education, Sangmyung University, Seoul, 110-743, Republic of Korea

Abstract

We consider the existence of at least two or three distinct weak solutions for the nonlinear elliptic equations

$$\begin{cases} -\operatorname{div}(\varphi(x,\nabla u)) + |u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ \varphi(x,\nabla u)\frac{\partial u}{\partial n} = \lambda g(x,u) & \text{on } \partial\Omega. \end{cases}$$

Here the function $\varphi(x,v)$ is of type $|v|^{p-2}v$ and the functions f, g satisfy a Carathéodory condition. To do this, we give some critical point theorems for continuously differentiable functions with the Cerami condition which are extensions of the recent results in Bonanno (Adv. Nonlinear Anal. 1:205-220, 2012) and Bonanno and Marano (Appl. Anal. 89:1-10, 2010) by applying Zhong's Ekeland variational principle.

MSC: 35B38; 35D30; 35J20; 35J60; 35J66

Keywords: p-Laplace type operator; weak solutions; Cerami condition; multiple critical points

1 Introduction

In the present paper, we are concerned with multiple solutions for the nonlinear Neumann boundary value problem associated with p-Laplacian type,

$$\begin{cases} -\operatorname{div}(\varphi(x,\nabla u)) + |u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ \varphi(x,\nabla u)\frac{\partial u}{\partial n} = \lambda g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where the function $\varphi(x,v)$ is of type $|v|^{p-2}v$ with a real constant $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u with respect to $\partial\Omega$, and the functions f, g satisfy a Carathéodory condition. Concerning elliptic equations with nonlinear boundary conditions, we refer to [3–7].

Ricceri's three-critical-points theorems which are important to obtain the existence of at least three weak solutions for nonlinear elliptic equations have been extensively studied by various researchers; see [8–11]. It is well known that Ricceri's theorems in [12, 13] gave no further information on the size and location of an interval of values λ in $\mathbb{R}$ for the existence of at least three critical points.

Based on [1, 2], Bonanno and Chinnì [14] obtained the existence of at least two or three distinct weak solutions for nonlinear elliptic equations with the variable exponents whenever the parameter λ belongs to a precise positive interval. To obtain the existence of two distinct weak solutions for this problem, they assumed that the nonlinear term f satisfies the Ambrosetti and Rabinowitz condition (the (AR) condition, for short) in [15]:

(AR) There exist positive constants M and θ such that $\theta > p$ and

$$0 < \theta F(x,t) \le f(x,t)t, \quad \text{for } x \in \Omega \text{ and } |t| \ge M,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$ and Ω is a bounded domain in $\mathbb{R}^N$.

Moreover, in order to determine the precise positive interval of the parameter for the existence of multiple solutions of the nonlinear elliptic equations, they observed an embedding constant of the variable exponent Sobolev spaces into the variable exponent Lebesgue spaces by using Talenti's inequality (see [16]).

The goal of the present paper is to establish the existence of at least two or three weak solutions for the problem (P) whenever the parameter λ belongs to a precise positive interval. To do this, we give a theorem which is an extension of the recent critical point theorem in [1] by considering Zhong's Ekeland variational principle. Roughly speaking, we give this result under the Cerami condition which is another compactness condition of the Palais-Smale type introduced by Cerami [17]. First we show the existence of at least two weak solutions for (P) without assuming that f satisfies the (AR) condition. In recent years, some authors in [18–23] have tried to drop the (AR) condition that is crucial to guarantee the boundedness of the Palais-Smale sequence of the Euler-Lagrange functional which plays a decisive role in applying the critical point theory. In this respect, we observe that the energy functional associated with (P) satisfies the Cerami condition when the nonlinear term f does not satisfies (AR) condition. This together with the best Sobolev trace constant given in [3] yields the existence of at least two weak solutions for (P). Finally, as an application of the recent three-critical-points theorem introduced by [2], we show that the problem (P) has at least three weak solutions provided that λ is suitable.

This paper is organized as follows. In Section 2, by using Zhong's Ekeland variational principle, we state some critical point theorems for continuously differentiable functions with the Cerami condition. In Section 3, we state some basic results for the integral operators corresponding to the problem (P) under certain conditions on φ, f, and g. In Sections 4 and 5, we establish the existence of at least two or three distinct weak solutions for the problem (P) using some properties in Sections 2 and 3.

2 Abstract critical point theorem

Let $(X, \|\cdot\|)$ be a real Banach space. We denote the dual space of X by X^*, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $I : X \to \mathbb{R}$ is called locally Lipschitz when, with every $u \in X$, there corresponds a neighborhood U of u and a constant $L \ge 0$ such that

$$\left|I(v) - I(w)\right| \le L\|v - w\|_X \quad \text{for all } v, w \in U.$$

If $u, v \in X$, the symbol $\langle J'(u), v\rangle$ indicates the generalized directional derivative of I at point u along direction v, namely

$$\langle J'(u), v\rangle := \limsup_{w\to u, t\to 0^+} \frac{I(w+tv) - I(w)}{t}.$$

The generalized gradient of the function I at u, denoted by $\partial I(u)$, is the set

$$\partial I(u) := \{u^* \in X^* : \langle u^*, v\rangle \le \langle J'(u), v\rangle \text{ for all } v \in X\}.$$

A function $I : X \to \mathbb{R}$ is called Gâteaux differentiable at $u \in X$ if there is $\varphi \in X^*$(denoted by $I'(u)$) such that

$$\lim_{t\to 0^+} \frac{I(u+tv) - I(u)}{t} = I'(u)(v)$$

for any $v \in X$. It is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the function $u \to I'(u)$ is a continuous map from X to its dual X^*. We recall that if I is continuously Gâteaux differentiable then it is locally Lipschitz and one has $\langle J'(u), v\rangle = I'(u)(v)$ for all $u, v \in X$.

Definition 2.1 Let X be a real Banach space and let $I : X \to \mathbb{R}$ be a Gâteaux differentiable function.

(i) I satisfies the Palais-Smale condition ((PS)-condition for short), if any sequence $\{u_n\} \subset X$ such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|_{X^*} \to 0$ as $n \to \infty$ has a convergent subsequence.

(ii) I satisfies the Cerami condition (the (C)-condition for short), if any sequence $\{u_n\} \subset X$ such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0$ as $n \to \infty$ has a convergent subsequence.

Now, let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions; put

$$I = \Phi - \Psi$$

and fix $\mu \in [-\infty, +\infty]$, we say that the function I verifies the Cerami condition cut off upper at μ (in short, the $(C)^{[\mu]}$-condition) if any sequence $\{u_n\}$ such that I satisfies the Cerami condition and $\Phi(u_n) < \mu$, for any $n \in \mathbb{N}$, has a convergent subsequence.

As a key tool, recall the following lemma, which is a generalization of Ekeland's variational principle [24] due to Zhong in [25], Theorem 2.1.

Lemma 2.2 [26] *Let $h : [0, \infty) \to [0, \infty)$ be a continuous nondecreasing function such that $\int_0^\infty 1/(1 + h(r))\,dr = +\infty$. Let $\mathcal{M}$ be a complete metric space and x_0 be a fixed point of $\mathcal{M}$. Suppose that $f : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, not identically $+\infty$, bounded from below. Then, for every $\varepsilon > 0$ and $y \in \mathcal{M}$ such that*

$$f(y) < \inf_{\mathcal{M}} f + \varepsilon,$$

and every $\lambda > 0$, *there exists some point* $z \in \mathcal{M}$ *such that*

$$f(z) \leq f(y), \quad d(z, x_0) \leq \bar{r} + r_0,$$

and

$$f(x) \geq f(z) - \frac{\varepsilon}{\lambda(1 + h(d(x_0, z)))} d(x, z) \quad \text{for all } x \in \mathcal{M},$$

where $d(x, y)$ *is the distance of two points* $x, y \in \mathcal{M}$, $r_0 = d(x_0, y)$, *and* $\bar{r}$ *is such that*

$$\int_{r_0}^{r_0+\bar{r}} \frac{1}{1 + h(r)} \, dr \geq \lambda.$$

Remark 2.3 [26] To employ Lemma 2.2, we give a specific function h. If we define the function h by $h(r) := r + d(x_0, 0)$, then it is continuous and nondecreasing. Setting $x_0 := y$ and $d(y, z) := \|y - z\|_X$. Then we see that $h(d(y, z)) = d(y, z) + d(y, 0) = \|y - z\|_X + \|y\|_X \geq \|z\|_X$ and hence

$$-\frac{\varepsilon}{\lambda(1 + h(d(y, z)))} \geq -\frac{\varepsilon}{\lambda(1 + \|z\|_X)}.$$

Since $r_0 = 0$ and

$$\lambda \leq \int_0^{\bar{r}} \frac{1}{1 + h(r)} \, dr = \int_0^{\bar{r}} \frac{1}{1 + r + \|y\|_X} \, dr = \left[\ln\left|1 + r + \|y\|_X\right|\right]_0^{\bar{r}} = \ln\left|\frac{1 + \bar{r} + \|y\|_X}{1 + \|y\|_X}\right|,$$

if we take, for each $n \in \mathbb{N}$,

$$\lambda_n := \ln\left(1 + \frac{1}{n(1 + \|y\|_X)}\right) > 0,$$

then we can get $\bar{r}_n = 1/n$. Also,

$$\frac{\varepsilon_n}{\lambda_n} \to 0 \quad \text{as } n \to \infty,$$

where $\varepsilon_n := \lambda_n^2 > 0$.

We point out the following consequence of Zhong's Ekeland variational principle in Lemma 2.2.

Lemma 2.4 *Let* X *be a real Banach space and let* $I : X \to \mathbb{R}$ *be a locally Lipschitz function bounded from below. Then, for all minimizing sequences of* I, $\{u_n\}_{n\in\mathbb{N}} \subseteq X$, *there exists a minimizing sequence of* I, $\{v_n\}_{n\in\mathbb{N}} \subseteq X$, *such that for any* $n \in \mathbb{N}$

$$I(v_n) \leq I(u_n) \quad \text{and} \quad \left\langle I'(v_n), h\right\rangle \geq \frac{-\varepsilon_n \|h\|_X}{1 + \|v_n\|_X}$$

for all $h \in X$, *and* $n \in \mathbb{N}$, *where* $\varepsilon_n \to 0^+$.

Using Lemma 2.4, we obtain the following result; see [27] for the case of (PS)-condition cut off upper at μ. The proof of this theorem proceeds in the analogous way to that of Theorem 3.1 in [27].

Theorem 2.5 *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below. Put*

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r \in \mathbb{R}$, with $\mu > \Phi(x_0)$, such that

$$\sup_{u\in\Phi^{-1}((-\infty,\mu))} \Psi(u) \leq \mu - \Phi(x_0) + \Psi(x_0). \tag{2.1}$$

Moreover, assume that I satisfies $(C)^{[\mu]}$-condition. Then there is a $u_0 \in \Phi^{-1}((-\infty,\mu))$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}((-\infty,\mu))$ and $I'(u_0) = 0$.

Proof Put

$$\begin{aligned} &M = \mu - \Phi(x_0) + \Psi(x_0), \qquad (2.2)\\ &\Psi_M(u) = \begin{cases} \Psi(u) & \text{if } \Psi(u) < M, \\ M & \text{if } \Psi(u) \geq M, \end{cases}\\ &I_M = \Phi - \Psi_M. \end{aligned}$$

Then it is obvious that I_M is locally Lipschitz and bounded from below. Now, given a sequence $\{u_n\}_{n\in\mathbb{N}}$ in X satisfying $\lim_{n\to\infty} I_M(u_n) = \inf_X I_M$, according to Lemma 2.4 there is a sequence $\{v_n\}_{n\in\mathbb{N}}$ in X such that

$$\lim_{n\to\infty} I_M(v_n) = \inf_X I_M \quad \text{and} \quad \langle I_M'(v_n), h\rangle \geq -\frac{\varepsilon_n\|h\|_X}{1+\|v_n\|_X}$$

for all $h \in X$ and for all $n \in \mathbb{N}$, where $\varepsilon_n \to 0^+$. If $I_M(x_0) = \inf_X I_M$ then x_0 verifies the consequence. Indeed, if $u \in \Phi^{-1}((-\infty,\mu))$, the inequality (2.1) implies that $\Psi(u) \leq M$ and $I_M(u) = I(u)$ for all $u \in \Phi^{-1}((-\infty,\mu))$, and hence $I(x_0) = I_M(x_0) \leq I_M(u) = I(u)$ for all $u \in \Phi^{-1}((-\infty,\mu))$. So, we assume $\inf_X I_M < I_M(x_0)$. Therefore, there is an $N \in \mathbb{N}$ such that $I_M(v_n) < I_M(x_0)$ for all $n > N$. Now, we claim that $\Phi(v_n) < \mu$ for all $n > N$. Since $\Phi(v_n) - \Psi_M(v_n) < \Phi(x_0) - \Psi_M(x_0)$, we assert that

$$\Phi(v_n) < \Psi_M(v_n) + \Phi(x_0) - \Psi(x_0) \leq M + \Phi(x_0) - \Psi(x_0) = \mu,$$

as claimed.

Hence, one has $I_M(v_n) = I(v_n)$ and $\langle I_M'(v_n), h\rangle = I'(v_n)(h)$ for all $n > N$. Therefore it follows from Lemma 2.4 that

$$\lim_{n\to\infty} I(v_n) = \lim_{n\to\infty} I_M(v_n) = \inf_X I_M \quad \text{and} \quad I'(v_n)(h) \geq -\frac{\varepsilon_n\|h\|_X}{1+\|v_n\|_X},$$

that is, $\lim_{n\to\infty}(1 + \|v_n\|_X)\|I'(v_n)\|_{X^*} = 0$. Since I satisfies $(C)^{[\mu]}$-condition, $\{v_n\}$ admits a subsequence strongly converging to v^* in X as $n \to \infty$. So, $I(v^*) = \inf_X I_M \le I_M(u) = I(u)$ for all $u \in \Phi^{-1}((-\infty,\mu))$, that is,

$$I(v^*) \le I(u) \tag{2.3}$$

for all $u \in \Phi^{-1}((-\infty,\mu))$. Since $\Phi(v_n) < \mu$ for all $n > N$, the continuity of Φ implies that $v^* \in \Phi^{-1}((-\infty,\mu])$.

If $v^* \in \Phi^{-1}((-\infty,\mu))$, by the inequality (2.3) the conclusion holds. If $\Phi(v^*) = \mu$, first we observe that $\Psi(v^*) \le M$. In fact, taking into account that $I(v^*) = I_M(v^*)$, we have $\mu - \Psi(v^*) = \mu - \Psi_M(v^*)$ and $\Psi(v^*) = \Psi_M(v^*) \le M$. Next, we prove that $I(v^*) = I(x_0)$. Indeed, suppose that $I(v^*) < I(x_0)$; from (2.2) we have

$$I(v^*) = \mu - \Psi(v^*) \ge \mu - M = \Phi(x_0) - \Psi(x_0) = I(x_0),$$

that is, $I(v^*) \ge I(x_0)$ and this contradicts with the assumption. Hence, from the inequality (2.3) we have $I(x_0) \le I(u)$ for all $u \in \Phi^{-1}((-\infty,\mu))$ and also in this case the conclusion is achieved. □

The next result is an immediate consequence of Theorem 2.5. This is crucial to get the existence of at least two distinct weak solutions for the problem (P) in the next section.

Theorem 2.6 *Let X be a real Banach space, $\Phi,\Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $\mu > 0$ and assume that, for each*

$$\lambda \in \left(0, \frac{\mu}{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u)}\right),$$

the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (C)-condition for all $\lambda > 0$ and it is unbounded from below. Then, for each

$$\lambda \in \left(0, \frac{\mu}{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u)}\right),$$

the functional I_λ admits two distinct critical points.

Proof Fix λ as in the conclusion. One has $\frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u)}{\mu} < \frac{1}{\lambda}$, so there is an element x_0 in $\Phi^{-1}((-\infty,\mu))$ such that

$$\frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u) - \Psi(x_0)}{\mu - \Phi(x_0)} < \frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u)}{\mu} < \frac{1}{\lambda}.$$

This implies

$$\sup_{u\in\Phi^{-1}((-\infty,\mu))} \lambda\Psi(u) < \mu - \Phi(x_0) + \lambda\Psi(x_0).$$

Hence, it follows from Theorem 2.5 that I_λ admits a local minimum. Since I_λ is unbounded from below, it is not strictly global and the mountain pass theorem ensures the conclusion. □

Combining [27], Remark 7.1, with [2], Theorem 3.6, we get the following assertion.

Theorem 2.7 *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Assume that the functional I_λ satisfies (C)-condition for all $\lambda > 0$ and that there exist a positive constant μ and an element $\tilde{u} \in X$, with $\mu < \Phi(\tilde{u})$, such that

(A1) $\frac{\sup_{\Phi(u)\le\mu} \Psi(u)}{\mu} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$;

(A2) *for each $\lambda \in \Lambda_\mu := (\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{\mu}{\sup_{\Phi(u)\le\mu} \Psi(u)})$, the functional I_λ is coercive.*

Then, for each $\lambda \in \Lambda_\mu$, the functional I_λ has at least three distinct critical points in X.

Proof The proof is essentially the same as in that of [2]. □

This is an immediate result of Theorem 2.7. This plays an important role in obtaining the fact that the problem (P) admits at least three distinct weak solutions.

Corollary 2.8 *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist a positive constant μ and an element $\tilde{u} \in X$, with $\mu < \Phi(\tilde{u})$, such that

(A1) $\frac{\sup_{\Phi(u)\le\mu} \Psi(u)}{\mu} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$;

(A2) *for each $\lambda \in \Lambda_\mu$, the functional I_λ is coercive.*

Then, for each $\lambda \in \Lambda_\mu$, the functional I_λ has at least three distinct critical points in X.

Proof Since Gâteaux derivative of Φ admits a continuous inverse and I_λ is coercive, I_λ satisfies (C)-condition. Hence, applying Theorem 2.7 to the function I_λ the conclusion is obtained. □

Corollary 2.9 *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

If I_λ is bounded from below and satisfies (C)-condition for any $\lambda > 0$, and there exist $\mu > 0$ and $\tilde{u} \in X$, with $\mu < \Phi(\tilde{u})$, such that

$$\frac{\sup_{\Phi(u)\le\mu} \Psi(u)}{\mu} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

then, for each $\lambda \in \Lambda_\mu$, the functional I_λ has at least three distinct critical points in X.

Proof Since I_λ is bounded from below and satisfies (C)-condition, I_λ is coercive; see [28]. Hence, by Corollary 2.8 the conclusion is obtained. □

3 Basic concepts and preliminary results

In this section, we first collect some preliminary properties that will be used later. Throughout this paper, consider the Sobolev space $X := W^{1,p}(\Omega)$ with the usual norm

$$\|u\|_X = \left(\int_\Omega |\nabla u|^p + |u|^p \, dx\right)^{\frac{1}{p}}.$$

Lemma 3.1 [8, 29] *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Let $p \in [1, N)$ be a constant. Then there is a continuously embedding $X \hookrightarrow L^{p^*}(\Omega)$ where $p^* = \frac{Np}{N-p}$. Moreover, for every $q \in [1, p^*)$ the embedding $X \hookrightarrow L^q(\Omega)$ is compact.*

Lemma 3.2 [8] *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Let $p \in [1, N)$ be a constant. Then there is a continuous boundary trace embedding $X \hookrightarrow L^{p^\partial}(\partial\Omega)$ where $p^\partial = \frac{(N-1)p}{N-p}$. Moreover, for every $r \in [1, p^\partial)$ the trace embedding $X \hookrightarrow L^r(\partial\Omega)$ is compact.*

Definition 3.3 We say that $u \in X$ is a weak solution of the problem (P) if

$$\int_\Omega \varphi(x, \nabla u) \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} u v \, dx = \lambda \int_\Omega f(x, u) v \, dx + \lambda \int_{\partial\Omega} g(x, u) v \, dS$$

for all $v \in X$, where dS is the measure on the boundary.

We assume that $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is the continuous derivative with respect to v of the mapping $\Phi_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $\Phi_0 = \Phi_0(x, v)$, that is, $\varphi(x, v) = \frac{d}{dv}\Phi_0(x, v)$. Suppose that φ and Φ_0 satisfy the following assumptions:

(J1) The equality

$$\Phi_0(x, \mathbf{0}) = 0$$

holds for almost all $x \in \Omega$.

(J2) There is a function $a \in L^{p'}(\Omega)$ and a nonnegative constant b such that

$$\left|\varphi(x, v)\right| \le a(x) + b|v|^{p-1}$$

for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$ where $1/p + 1/p' = 1$.

(J3) $\Phi_0(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \Omega$.

(J4) The relations

$$d|v|^p \le \varphi(x,v)\cdot v \quad \text{and} \quad d|v|^p \le p\Phi_0(x,v)$$

hold for all $x \in \Omega$ and $v \in \mathbb{R}^N$, where d is a positive constant.

(J5) There exists a constant $\mu_1 \ge 0$ such that

$$\mathcal{H}(x,sv) \le \mathcal{H}(x,v) + \mu_1$$

for $v \in \mathbb{R}^N$ and $s \in [0,1]$, where $\mathcal{H}(x,v) = p\Phi_0(x,v) - \varphi(x,v)\cdot v$ for almost all $x \in \Omega$.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \Phi_0(x,\nabla u)\,dx + \int_\Omega \frac{1}{p}|u|^p\,dx.$$

Under assumptions (J1)-(J2) and (J4), it follows from [30] that the functional Φ is well defined on X, $\Phi \in C^1(X,\mathbb{R})$, and its Fréchet derivative is given by

$$\langle \Phi'(u), v\rangle = \int_\Omega \varphi(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega |u|^{p-2}uv\,dx,$$

for any $\varphi \in X$.

Example 3.4 Let us consider

$$\varphi(x,v) = \left(1 + \frac{|v|^p}{\sqrt{1+|v|^{2p}}}\right)|v|^{p-2}v$$

and

$$\Phi_0(x,v) = \frac{1}{p}\left(|v|^p + \sqrt{1+|v|^{2p}} - 1\right)$$

for all $v \in \mathbb{R}^N$. Then the direct calculation shows that $\mathcal{H}(x,sv) \le \mathcal{H}(x,v) + 1$ for all $s \in [0,1]$, and so the assumption (J5) holds for $\mu_1 \ge 1$.

The following assertion can be found in [31]; see also [32].

Lemma 3.5 [31] *Assume that* (J1)-(J4) *hold. Then the functional* $\Phi : X \to \mathbb{R}$ *is convex and weakly lower semicontinuous on* X. *Moreover, the operator* Φ' *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *as* $n \to \infty$ *and* $\limsup_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \le 0$, *then* $u_n \to u$ *in* X *as* $n \to \infty$.

Corollary 3.6 *Assume that* (J1)-(J4) *hold. Then the operator* $\Phi' : X \to X^*$ *is strictly monotone, coercive and hemicontinuous on* X. *Furthermore, the operator* Φ' *is a homeomorphism onto* X^*.

Proof It is obvious that the operator Φ' is strictly monotone, coercive, and hemicontinuous on X. By the Browder-Minty theorem, the inverse operator $(\Phi')^{-1}$ exists; see Theorem 26.A in [33]. Since Φ' is a mapping of type (S_+), by Lemma 3.5, the proof of continuity of the inverse operator $(\Phi')^{-1}$ is obvious. □

Next we need the following assumptions for f and g. Denoting $F(x,t) = \int_0^t f(x,s)\,ds$ and $G(x,t) = \int_0^t g(x,s)\,ds$, then we assume that:

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\cdot,t)$ is measurable for all $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous for almost all $x \in \Omega$.

(F2) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist two constants $d_1 \geq 0$ and $d_2 > 0$ such that

$$\left|f(x,t)\right| \leq d_1 + d_2|t|^{\alpha-1},$$

for all $x \in \Omega$ and for all $t \in \mathbb{R}$, where $p < \alpha < p^*$.

(G1) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two constants $d_3 \geq 0$ and $d_4 > 0$ such that

$$\left|g(x,t)\right| \leq d_3 + d_4|t|^{\beta-1},$$

for all $x \in \partial\Omega$ and for all $t \in \mathbb{R}$, where $p < \beta < p^\partial$.

Under assumptions (F1), (F2), and (G1), we define the functionals $\Psi_1, \Psi_2 : X \to \mathbb{R}$ by

$$\Psi_1(u) = \int_\Omega F(x,u)\,dx, \qquad \Psi_2(u) = \int_{\partial\Omega} G(x,u)\,dS \quad \text{and} \quad \Psi(u) = \Psi_1(u) + \Psi_2(u).$$

Then it is easy to check that $\Psi_1, \Psi_2 \in C^1(X,\mathbb{R})$ and these Fréchet derivatives are

$$\left\langle \Psi_1'(u), v\right\rangle = \int_\Omega f(x,u)v\,dx \quad \text{and} \quad \left\langle \Psi_2'(u), v\right\rangle = \int_{\partial\Omega} g(x,u)v\,dS$$

for any $u, v \in X$. Next we define the functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Then it follows that the functional $I_\lambda \in C^1(X,\mathbb{R})$ and its Fréchet derivative is

$$\left\langle I_\lambda'(u), v\right\rangle = \int_\Omega \varphi(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega |u|^{p-2}uv\,dx - \lambda\int_\Omega f(x,u)v\,dx - \lambda\int_{\partial\Omega} g(x,u)v\,dS$$

for any $u, v \in X$.

Lemma 3.7 *Assume that* (F1)-(F2) *and* (G1) *hold. Then* Ψ *and* Ψ' *are weakly-strongly continuous on* X.

Proof Proceeding like the analogous argument in [34], it follows that functionals Ψ and Ψ' are weakly-strongly continuous on X. □

To localize precisely the intervals of λ for which the problem (P) has at least two or three distinct weak solutions, we consider the following eigenvalue problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = \lambda m(x)|u|^{q-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{E}$$

Definition 3.8 We say that $\lambda \in \mathbb{R}$ is an eigenvalue of the eigenvalue problem (E) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx + \int_\Omega |u|^{p-2}uv\,dx = \lambda \int_\Omega m(x)|u|^{q-2}uv\,dx$$

holds for any $v \in X$ and $p < q < p^*$. Then u is called an eigenfunction associated with the eigenvalue λ.

Now we obtain the positivity of the infimum of all eigenvalues for the problem (E). Although the idea of the proof is completely the same as in that of Lemma 3.1 in [29], for the sake of convenience, we give the proof of the following proposition.

Proposition 3.9 *Assume that*
(H1) $m(x) > 0$ *for all* $x \in \Omega$ *and* $m \in L^{\frac{\gamma}{\gamma-q}}(\Omega)$ *with some* γ *satisfying* $q < \gamma < p^*$.
Then the eigenvalue problem (E) *has a pair* (λ_1, u_1) *of a principal eigenvalue* λ_1 *and an eigenfunction* u_1 *with* $\lambda_1 > 0$ *and* $0 < u_1 \in X$.

Proof Set

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla v|^p + |v|^p\,dx\right\}$$

the infimum being taken over all v such that $\int_\Omega m(x)|v|^q\,dx = 1$. We shall prove that λ_1 is the least eigenvalue of (E). The expression for λ_1 presented above will be referred to as its variational characterization. Obviously $\lambda_1 \geq 0$. Let $\{v_n\}_{n=1}^\infty$ be the minimizing sequence for λ_1, *i.e.*,

$$\int_\Omega m(x)|v_n|^q\,dx = 1 \quad \text{and} \quad \int_\Omega |\nabla v_n|^p + |v_n|^p\,dx = \lambda_1 + \delta_n \tag{3.1}$$

with $\delta_n \to 0^+$ for $n \to \infty$. It follows from (3.1) that $\|v_n\|_X \leq c$ for some constant $c > 0$. The reflexivity of X yields the weak convergence $v_n \rightharpoonup u_1$ in X for some u_1 (at least for some subsequence of $\{v_n\}$). The compact embedding $X \hookrightarrow L^\gamma(\Omega)$ implies the strong convergence $v_n \to u_1$ in $L^\gamma(\Omega)$. It follows from (H1), (3.1), and the Minkowski and Hölder inequalities that

$$\begin{aligned}
1 &= \lim_{n\to\infty}\left(\int_\Omega m(x)|v_n|^q\,dx\right)^{\frac{1}{q}} \\
&\leq \lim_{n\to\infty}\left(\int_\Omega m(x)|v_n - u_1|^q\,dx\right)^{\frac{1}{q}} + \left(\int_\Omega m(x)|u_1|^q\,dx\right)^{\frac{1}{q}} \\
&\leq \lim_{n\to\infty}\left(\int_\Omega (m(x))^{\frac{\gamma}{\gamma-q}}\,dx\right)^{\frac{\gamma-q}{q\gamma}}\left(\int_\Omega |v_n - u_1|^\gamma\,dx\right)^{\frac{1}{\gamma}} + \left(\int_\Omega m(x)|u_1|^q\,dx\right)^{\frac{1}{q}} \\
&= \left(\int_\Omega m(x)|u_1|^q\,dx\right)^{\frac{1}{q}},
\end{aligned}$$

and analogously

$$\left(\int_\Omega m(x)|u_1|^q\,dx\right)^{\frac{1}{q}} \le \lim_{n\to\infty}\left(\int_\Omega \bigl(m(x)\bigr)^{\frac{\gamma}{\gamma-q}}\,dx\right)^{\frac{\gamma-q}{q\gamma}}\left(\int_\Omega |u_1-v_n|^\gamma\,dx\right)^{\frac{1}{\gamma}} + \lim_{n\to\infty}\left(\int_\Omega m(x)|v_n|^q\,dx\right)^{\frac{1}{q}} = 1.$$

Hence

$$\int_\Omega m(x)|u_1|^q\,dx = 1.$$

In particular, $u_1 \not\equiv 0$. The weak lower semicontinuity of the norm in X yields

$$\begin{aligned}\lambda_1 &\le \int_\Omega |\nabla u_1|^p + |u_1|^p\,dx = \|u_1\|_X^p \le \liminf_{n\to\infty}\|v_n\|_X^p \\ &= \liminf_{n\to\infty}\left\{\int_\Omega |\nabla v_n|^p + |v_n|^p\,dx\right\} = \liminf_{n\to\infty}(\lambda_1+\delta_n) = \lambda_1,\end{aligned}$$

i.e.,

$$\lambda_1 = \int_\Omega |\nabla u_1|^p + |u_1|^p\,dx. \tag{3.2}$$

It follows from (3.2) that $\lambda_1 > 0$ and it is easy to see that λ_1 is the least eigenvalue of (E) with the corresponding eigenfunction u_1.

Moreover, if u is an eigenfunction corresponding to λ_1 then $|u|$ is also an eigenfunction corresponding to λ_1. Hence we can suppose that $u_1 > 0$ a.e. in Ω. □

4 Existence of two weak solutions

In this section, we present the existence of at least two distinct weak solutions for the problem (P). To do this, we assume that

(F3) $\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^p} = \infty$ uniformly for almost all $x \in \Omega$.

(F4) There is a constant $\mu_2 > 0$ such that

$$\mathcal{F}(x,t) \le \mathcal{F}(x,s) + \mu_2$$

for any $x \in \Omega$, $0 < t < s$ or $s < t < 0$, where $\mathcal{F}(x,t) = tf(x,t) - pF(x,t)$.

(F5) $\limsup_{s\to 0}\frac{|f(x,s)|}{m(x)|s|^{\xi_1-1}} < \infty$ uniformly for almost all $x \in \Omega$, where $\xi_1 \in \mathbb{R}$ with $q < \xi_1 < p^*$.

(F6) $\limsup_{|s|\to\infty}\left(\operatorname{ess\,sup}_{x\in\Omega}\frac{|f(x,s)|}{|s|^{q-1}}\right) < \infty$, where $q \in \mathbb{R}$ satisfies $p < q < p^*$.

(G2) $\lim_{|t|\to\infty}\frac{G(x,t)}{|t|^p} = \infty$ uniformly for almost all $x \in \partial\Omega$.

(G3) There is a constant $\mu_3 > 0$ such that

$$\mathcal{G}(x,t) \le \mathcal{G}(x,s) + \mu_3,$$

for any $x \in \partial\Omega$, $0 < t < s$ or $s < t < 0$, where $\mathcal{G}(x,t) = tg(x,t) - pG(x,t)$.

(G4) $\limsup_{s\to 0} \frac{|g(x,s)|}{|s|^{\xi_2-1}} < \infty$ uniformly for almost all $x \in \partial\Omega$, where $\xi_2 \in \mathbb{R}$ with $q < \xi_2 < p^\partial$.

(G5) $\limsup_{|s|\to\infty} \left(\operatorname{ess\,sup}_{x\in\partial\Omega} \frac{|g(x,s)|}{|s|^{r-1}} \right) < \infty$, where $r \in \mathbb{R}$ satisfies $p < r < p^\partial$.

With the help of Lemmas 3.5 and 3.7, we prove that the energy functional I_λ satisfies the (C)-condition for any $\lambda > 0$. This plays an important role in obtaining our first main result.

Lemma 4.1 *Assume that* (J1)-(J5), (F1)-(F4), *and* (G1)-(G3) *hold. Then the functional* I_λ *satisfies the* (C)*-condition for any* $\lambda > 0$.

Proof Note that Ψ_1' and Ψ_2' are the mapping of type (S_+) by Lemma 3.7. Let $\{u_n\}$ be a (C)-sequence in X, *i.e.*, $\{I_\lambda(u_n)\}$ is bounded and $\|I_\lambda'(u_n)\|_{X^*}(1 + \|u\|_X) \to 0$ as $n \to \infty$, so that $\sup|I_\lambda(u_n)| \le M$ for some $M > 0$ and $\langle I_\lambda'(u_n), u_n\rangle = o(1)$. Since X is reflexive and I_λ' is the mapping of type (S_+), it suffices to verify that $\{u_n\}$ is bounded in X. Indeed, if $\{u_n\}$ is unbounded in X, we may assume that $\|u_n\|_X > 1$ and $\|u_n\|_X \to \infty$ as $n \to \infty$. We define a sequence $\{w_n\}$ by $w_n = u_n/\|u_n\|_X$, $n = 1, 2, \ldots$. It is clear that $\{w_n\} \subset X$ and $\|w_n\|_X = 1$ for any n. Therefore, up to a subsequence, still denoted by $\{w_n\}$, we see that $\{w_n\}$ converges weakly to $w \in X$ and, by Lemmas 3.1 and 3.2, we have

$$\begin{aligned} &w_n(x) \to w(x) \quad \text{a.e. in } \Omega \text{ and } \partial\Omega, \\ &w_n \to w \quad \text{in } L^\alpha(\Omega) \text{ and } L^\beta(\partial\Omega) \quad \text{as } n \to \infty. \end{aligned} \tag{4.1}$$

Let $\Omega_0 = \{x \in \overline{\Omega} : w(x) \neq 0\}$. If $x \in \Omega_0 \cap \Omega$, then it follows from (4.1) that $|u_n(x)| = |w_n(x)|\|u_n\|_X \to \infty$ as $n \to \infty$. Similarly we know by (4.1) that $|u_n(x)| \to \infty$ as $n \to \infty$ for $x \in \Omega_0 \cap \partial\Omega$. According to (F3) and (G2), we have

$$\lim_{n\to\infty} \frac{F(x,u_n(x))}{\|u_n\|_X^p} = \lim_{n\to\infty} \frac{F(x,u_n(x))}{|u_n(x)|^p} \left|w_n(x)\right|^p = \infty, \quad x \in \Omega \cap \Omega_0, \tag{4.2}$$

and

$$\lim_{n\to\infty} \frac{G(x,u_n(x))}{\|u_n\|_X^p} = \lim_{n\to\infty} \frac{G(x,u_n(x))}{|u_n(x)|^p} \left|w_n(x)\right|^p = \infty, \quad x \in \partial\Omega \cap \Omega_0. \tag{4.3}$$

In addition, the condition (F3) implies that there exists $t_0 > 1$ such that $F(x,t) > |t|^p$ for all $x \in \Omega$ and $|t| > t_0$. Since $F(x,t)$ is continuous on $\overline{\Omega} \times [-t_0, t_0]$ by (F2), there exists a positive constant C_1 such that $|F(x,t)| \le C_1$ for all $(x,t) \in \overline{\Omega} \times [-t_0, t_0]$. Therefore we can choose a real number C_2 such that $F(x,t) \ge C_2$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, and thus

$$\frac{F(x,u_n(x)) - C_2}{\|u_n\|_X^p} \ge 0,$$

for all $x \in \Omega$ and for all $n \in \mathbb{N}$. Similarly, using the assumption (G2), we see that there exists a constant $C_3 \in \mathbb{R}$ such that

$$\frac{G(x,u_n(x)) - C_3}{\|u_n\|_X^p} \ge 0,$$

for all $x \in \partial\Omega$ and for all $n \in \mathbb{N}$. Also, using the assumption (J2), we get

$$\begin{aligned}
I_\lambda(u_n) &= \int_\Omega \Phi_0(x,\nabla u_n)\,dx + \int_\Omega \frac{1}{p}|u_n|^p\,dx - \lambda\int_\Omega F(x,u_n)\,dx - \lambda\int_{\partial\Omega} G(x,u_n)\,dS \\
&\le \int_\Omega a(x)|\nabla u_n|\,dx + \frac{b}{p}\int_\Omega |\nabla u_n|^p\,dx + \frac{1}{p}\int_\Omega |u_n|^p\,dx - \lambda\int_\Omega F(x,u_n)\,dx \\
&\quad - \lambda\int_{\partial\Omega} G(x,u_n)\,dS \\
&\le \bigl(2\|a\|_{L^{p'}(\Omega)} + b + 1\bigr)\|u_n\|_X^p - \lambda\int_\Omega F(x,u_n)\,dx - \lambda\int_{\partial\Omega} G(x,u_n)\,dS.
\end{aligned}$$

It follows that

$$2\|a\|_{L^{p'}(\Omega)} + b + 1 \ge \frac{1}{\|u_n\|_X^p}\biggl(I_\lambda(u_n) + \lambda\int_\Omega F(x,u_n)\,dx + \lambda\int_{\partial\Omega} G(x,u_n)\,dS\biggr) \tag{4.4}$$

for n large enough. We claim that $|\Omega_0| = 0$. In fact, if $|\Omega_0| \neq 0$, then by (4.2), (4.3), (4.4), and the Fatou lemma, we have

$$\begin{aligned}
&2\|a\|_{L^{p'}(\Omega)} + b + 1 \\
&\quad\ge \liminf_{n\to\infty} \frac{1}{\|u_n\|_X^p}\biggl(I_\lambda(u_n) + \lambda\int_\Omega F(x,u_n)\,dx + \lambda\int_{\partial\Omega} G(x,u_n)\,dS\biggr) \\
&\quad= \liminf_{n\to\infty}\int_\Omega \frac{F(x,u_n(x)) - C_2}{\|u_n\|_X^p}\,dx + \liminf_{n\to\infty}\int_{\partial\Omega} \frac{G(x,u_n(x)) - C_3}{\|u_n\|_X^p}\,dS \\
&\quad\ge \int_{\Omega\cap\Omega_0} \liminf_{n\to\infty} \frac{F(x,u_n(x)) - C_2}{\|u_n\|_X^p}\,dx + \int_{\partial\Omega\cap\Omega_0} \liminf_{n\to\infty} \frac{G(x,u_n(x)) - C_3}{\|u_n\|_X^p}\,dS \\
&\quad= \int_{\Omega\cap\Omega_0} \liminf_{n\to\infty} \frac{F(x,u_n(x))}{|u_n(x)|^p}\bigl|w_n(x)\bigr|^p\,dx - \int_{\Omega\cap\Omega_0} \limsup_{n\to\infty} \frac{C_2}{\|u_n\|_X^p}\,dx \\
&\quad\quad + \int_{\partial\Omega\cap\Omega_0} \liminf_{n\to\infty} \frac{G(x,u_n(x))}{|u_n(x)|^p}\bigl|w_n(x)\bigr|^p\,dS - \int_{\partial\Omega\cap\Omega_0} \limsup_{n\to\infty} \frac{C_3}{\|u_n\|_X^p}\,dS \\
&\quad= \infty,
\end{aligned}$$

which is a contradiction. This shows that $|\Omega_0| = 0$ and thus $w(x) = 0$ almost everywhere in $\overline{\Omega}$.

Since $I_\lambda(tu_n)$ is continuous in $t \in [0,1]$, for each $n \in \mathbb{N}$ there exists t_n in [0,1] such that

$$I_\lambda(t_n u_n) := \max_{t\in[0,1]} I_\lambda(tu_n).$$

Let $\{R_k\}$ be a positive sequence of real numbers such that $\lim_{k\to\infty} R_k = \infty$ and $R_k > 1$ for any k. Then $\|R_k w_n\|_X = R_k > 1$ for any k and n. For fixed k, we derive from the continuity of the Nemytskii operator that $F(x,R_k w_n) \to 0$ in $L^1(\Omega)$ and $G(x,R_k w_n) \to 0$ in $L^1(\partial\Omega)$ as $n \to \infty$, respectively. Hence we assert that

$$\lim_{n\to\infty}\int_\Omega F(x,R_k w_n)\,dx = 0 \quad\text{and}\quad \lim_{n\to\infty}\int_{\partial\Omega} G(x,R_k w_n)\,dS = 0. \tag{4.5}$$

Since $\|u_n\|_X \to \infty$ as $n \to \infty$, we also have $\|u_n\|_X > R_k$ or $0 < \frac{R_k}{\|u_n\|_X} < 1$ for n large enough. Hence, using the assumption (J4) and (4.5) it follows that

$$\begin{aligned}
I_\lambda(t_n u_n) &\geq I_\lambda\left(\frac{R_k}{\|u_n\|_X} u_n\right) = I_\lambda(R_k w_n) \\
&= \int_\Omega \Phi_0\big(x, |\nabla R_k w_n|\big)\,dx + \int_\Omega \frac{1}{p}|R_k w_n|^p\,dx - \lambda \int_\Omega F(x, R_k w_n)\,dx \\
&\quad - \lambda \int_{\partial\Omega} G(x, R_k w_n)\,dS \\
&\quad \geq \frac{d}{p}\int_\Omega |\nabla R_k w_n|^p\,dx + \frac{1}{p}\int_\Omega |R_k w_n|^p\,dx - \lambda \int_\Omega F(x, R_k w_n)\,dx \\
&\quad - \lambda \int_{\partial\Omega} G(x, R_k w_n)\,dS \\
&\geq C_4 \|R_k w_n\|_X^p - \lambda \int_\Omega F(x, R_k w_n)\,dx - \lambda \int_{\partial\Omega} G(x, R_k w_n)\,dS \\
&\geq \frac{C_4}{2} R_k^p
\end{aligned}$$

for some positive constant C_4 and for any n large enough. Then letting n and k tend to infinity, it follows that

$$\lim_{n\to\infty} I_\lambda(t_n u_n) = \infty. \tag{4.6}$$

Since $I_\lambda(0) = 0$ and $|I_\lambda(u_n)| \leq M$, it is obvious that $t_n \in (0,1)$ and also $\langle I'_\lambda(t_n u_n), t_n u_n\rangle = o(1)$. Therefore, due to the assumptions (J5), (F4) and (G3), for n large enough, we deduce that

$$\begin{aligned}
I_\lambda(t_n u_n) &= I_\lambda(t_n u_n) - \frac{1}{p}\big\langle I'_\lambda(t_n u_n), t_n u_n\big\rangle + o(1) \\
&= \int_\Omega \Phi_0(x, t_n \nabla u_n)\,dx + \int_\Omega \frac{1}{p}|t_n u_n|^p\,dx \\
&\quad - \lambda \int_\Omega F(x, t_n u_n)\,dx - \lambda \int_{\partial\Omega} G(x, t_n u_n)\,dS - \frac{1}{p}\int_\Omega \varphi(x, t_n \nabla u_n)\cdot(t_n \nabla u_n)\,dx \\
&\quad - \frac{1}{p}\int_\Omega |t_n u_n|^p\,dx + \frac{\lambda}{p}\int_\Omega f(x, t_n u_n) t_n u_n\,dx + \frac{\lambda}{p}\int_{\partial\Omega} g(x, t_n u_n) t_n u_n\,dS + o(1) \\
&\leq \frac{1}{p}\int_\Omega \mathcal{H}(x, t_n \nabla u_n)\,dx + \int_\Omega \frac{1}{p}|t_n u_n|^p\,dx \\
&\quad - \frac{1}{p}\int_\Omega |t_n u_n|^p\,dx + \frac{\lambda}{p}\int_\Omega \mathcal{F}(x, t_n u_n)\,dx + \frac{\lambda}{p}\int_{\partial\Omega} \mathcal{G}(x, t_n u_n)\,dS + o(1) \\
&\leq \frac{1}{p}\int_\Omega \big(\mathcal{H}(x, \nabla u_n) + \mu_1\big)\,dx + \int_\Omega \frac{1}{p}|u_n|^p\,dx \\
&\quad - \frac{1}{p}\int_\Omega |u_n|^p\,dx + \frac{\lambda}{p}\int_\Omega \big(\mathcal{F}(x, u_n) + \mu_2\big)\,dx + \frac{\lambda}{p}\int_{\partial\Omega} \big(\mathcal{G}(x, u_n) + \mu_3\big)\,dS + o(1) \\
&\leq \int_\Omega \Phi_0(x, \nabla u_n)\,dx + \int_\Omega \frac{1}{p}|u_n|^p\,dx - \lambda \int_\Omega F(x, u_n)\,dx - \lambda \int_{\partial\Omega} G(x, u_n)\,dS \\
&\quad - \frac{1}{p}\bigg(\int_\Omega \varphi(x, \nabla u_n)\cdot\nabla u_n\,dx - \int_\Omega |u_n|^p\,dx - \lambda \int_\Omega f(x, u_n) u_n\,dx
\end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_{\partial\Omega} g(x,u_n)u_n \, dS\Big) \\
&+ \frac{\lambda}{p}(\mu_1+\mu_2)|\Omega| + \frac{\lambda}{p}\mu_3|\partial\Omega| + o(1) \\
&= I_\lambda(u_n) - \frac{1}{p}\langle I'_\lambda(u_n), u_n\rangle + \frac{\lambda}{p}(\mu_1+\mu_2)|\Omega| + \frac{\lambda}{p}\mu_3|\partial\Omega| + o(1) \\
&\le M + \frac{\lambda}{p}(\mu_1+\mu_2)|\Omega| + \frac{\lambda}{p}\mu_3|\partial\Omega| \quad \text{as } n \to \infty,
\end{aligned}
$$

which contradicts with (4.6). This completes the proof. □

Remark 4.2 It is easily to confirm that the conditions (F1) and (F5) imply that $f(x,0) = 0$ for almost all $x \in \Omega$. Otherwise there exists $A \subset \Omega$, $|A| > 0$ such that $|f(x,0)| > 0$ for all $x \in A$. Hence $\lim_{s\to 0} \frac{|f(x,s)|}{m(x)|s|^{\xi_1-1}} = \infty$ for all $x \in A$, contradicting (F5). In addition, we get $\limsup_{s\to 0} \frac{|F(x,s)|}{m(x)|s|^{\xi_1}} < \infty$ uniformly almost everywhere in Ω, by the L'Hôpital rule. Define the crucial value

$$
\mathcal{C}_f = \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|f(x,s)|}{m(x)|s|^{q-1}}. \tag{4.7}
$$

Then it follows from the analogous arguments in [10] that $\mathcal{C}_f$ is a positive constant. However, since the conditions of f are slightly different from those of [10], we discuss this fact. Indeed, $\mathcal{C}_f > 0$ having $f \not\equiv 0$ and $\mathcal{C}_f < \infty$, since first, by (F7)

$$
\lim_{s\to 0} \frac{|f(x,s)|}{m(x)|s|^{q-1}} = \lim_{s\to 0} \left(\frac{|f(x,s)|}{m(x)|s|^{\xi_1-1}} \right) |s|^{\xi_1-q} = 0
$$

uniformly almost everywhere in Ω, having $q < \xi_1$. This, together with the assumption (F6) yields $\mathcal{C}_f < \infty$. Similarly, we assert that $g(x,0) = 0$ for almost all $x \in \Omega$ and the crucial value

$$
\mathcal{C}_g = \operatorname*{ess\,sup}_{s\neq 0, x\in\partial\Omega} \frac{|g(x,s)|}{|s|^{p-1}} \tag{4.8}
$$

is a positive constant. Furthermore, the following relations hold:

$$
\operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|F(x,s)|}{m(x)|s|^{q}} = \frac{\mathcal{C}_f}{q}
$$

and

$$
\operatorname*{ess\,sup}_{s\neq 0, x\in\partial\Omega} \frac{|G(x,s)|}{|s|^{p}} = \frac{\mathcal{C}_g}{p}.
$$

Remark 4.3 [3, 8, 16] From the embeddings in Lemmas 3.1 and 3.2, for any $u \in X$, the following inequalities hold:

$$
s\|u\|_{L^q(\Omega)} \le \|u\|_X \quad \text{and} \quad \tilde{s}\|u\|_{L^r(\partial\Omega)} \le \|u\|_X
$$

for every $q \in [1, p^*)$ and $r \in [1, p^\partial)$. The best constants for these embeddings are the largest constants s and $\tilde{s}$ such that the above inequalities hold, that is,

$$s_q = \inf_{u \in X \setminus \{0\}} \frac{\|u\|_X}{\|u\|_{L^q(\Omega)}}$$

and

$$\tilde{s}_r = \inf_{u \in X \setminus \{0\}} \frac{\|u\|_X}{\|u\|_{L^r(\partial\Omega)}}.$$

Moreover, since these embeddings are compact by Lemmas 3.1 and 3.2, we have the existence of extremals, namely, functions where the infimum is attained.

Theorem 4.4 *Assume* (H1), (J1)-(J5), (F1)-(F6) *and* (G1)-(G5) *hold. Then*

(i) *there exists a positive constant* $\ell_* = \min\{d, 1\}\lambda_1/(\mathcal{C}_f + \lambda_1 \mathcal{C}_g \tilde{s}_p^{-1})$ *such that the problem* (P) *has only the trivial solution for all* $\lambda \in [0, \ell_*)$;

(ii) *there exists a positive constant* $\tilde{\lambda}$ *such that the problem* (P) *admits at least two distinct weak solutions in* X *for each* $\lambda \in (\ell_*, \tilde{\lambda})$.

Proof We prove the assertion (i). Let $u \in X$ be a nontrivial weak solution of the problem (P). Then it is clear that

$$\int_\Omega \varphi(x, \nabla u) \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} u v \, dx = \lambda \int_\Omega f(x, u) v \, dx + \lambda \int_{\partial\Omega} g(x, u) v \, dS$$

for any $v \in X$. If we put $v = u$, then it follows from (J4) and the definitions of $\mathcal{C}_f$ and $\mathcal{C}_g$ that

$$\begin{aligned}
&\min\{d, 1\}\lambda_1 \left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx \right) \\
&\quad \le \lambda_1 \left(\int_\Omega \varphi(x, \nabla u) \cdot \nabla u \, dx + \int_\Omega |u|^p \, dx \right) \\
&\quad = \lambda_1 \lambda \left(\int_\Omega f(x, u) u \, dx + \int_{\partial\Omega} g(x, u) u \, dS \right) \\
&\quad \le \lambda_1 \lambda \left(\int_\Omega \frac{f(x, u)}{m(x)|u|^{q-1}} m(x)|u|^q \, dx + \int_{\partial\Omega} \frac{g(x, u)}{|u|^{p-1}} |u|^p \, dS \right) \\
&\quad \le \lambda_1 \lambda \left(\mathcal{C}_f \int_\Omega m(x)|u|^q \, dx + \mathcal{C}_g \int_{\partial\Omega} |u|^p \, dS \right) \\
&\quad \le \lambda \mathcal{C}_f \left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx \right) + \lambda_1 \lambda \mathcal{C}_g \int_{\partial\Omega} |u|^p \, dS \\
&\quad \le \lambda \left(\mathcal{C}_f + \lambda_1 \mathcal{C}_g \tilde{s}_p^{-1} \right) \left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx \right).
\end{aligned}$$

Thus if u is a nontrivial weak solution of the problem (P), then necessarily $\lambda \ge \ell_* = \min\{d, 1\}\lambda_1/(\mathcal{C}_f + \lambda_1 \mathcal{C}_g \tilde{s}_p^{-1})$, as claimed.

Next we prove the assertion (ii). It is obvious that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. By the conditions (F3) and (G2), for any $C(M) > 0$, there exists a constant $\delta > 0$

such that

$$F(x,\eta) \geq C(M)|\eta|^p \quad \text{and} \quad G(x,\eta) \geq C(M)|\eta|^p \tag{4.9}$$

for $|\eta| > \delta$ and for almost all $x \in \Omega$. Take $v \in X \setminus \{0\}$. Then the relation (4.9) implies that

$$\begin{aligned} I_\lambda(tv) &= \int_\Omega \Phi_0(x, t\nabla v)\,dx + \int_\Omega \frac{1}{p}|tv|^p\,dx - \lambda\left(\int_\Omega F(x,tv)\,dx + \int_{\partial\Omega} G(x,tv)\,dS\right) \\ &\leq \int_\Omega \frac{d}{p}|t\nabla v|^p\,dx + \int_\Omega \frac{1}{p}|tv|^p\,dx - \lambda\left(\int_\Omega F(x,tv)\,dx + \int_{\partial\Omega} G(x,tv)\,dS\right) \\ &\leq t^p\left(\int_\Omega \frac{d}{p}|\nabla v|^p\,dx + \int_\Omega \frac{1}{p}|v|^p\,dx - \lambda C(M)\left(\int_\Omega |v|^p\,dx + \int_{\partial\Omega} |v|^p\,dS\right)\right) \end{aligned}$$

for sufficiently large $t > 1$. If $C(M)$ is large enough, then we assert that $I_\lambda(tv) \to -\infty$ as $t \to \infty$. Hence the functional I_λ is unbounded from below. Proposition 3.9 and relations (4.7) and (4.8) imply that

$$\begin{aligned} \Psi(u) &= \int_\Omega F(x,u)\,dx + \int_{\partial\Omega} G(x,u)\,dS \\ &\leq \int_\Omega \frac{|F(x,u)|}{m(x)|u|^q} m(x)|u|^q\,dx + \int_{\partial\Omega} \frac{|G(x,u)|}{|u|^p}|u|^p\,dS \\ &\leq \frac{\mathcal{C}_f}{q}\int_\Omega m(x)|u|^q\,dx + \frac{\mathcal{C}_g}{p}\int_{\partial\Omega} |u|^p\,dS \\ &\leq \left(\frac{\mathcal{C}_f}{\lambda_1 q} + \frac{\mathcal{C}_g}{p\tilde{s}_p}\right)\left(\int_\Omega |\nabla u|^p\,dx + \int_\Omega |u|^p\,dx\right) \\ &\leq \frac{1}{\min\{d,1\}}\left(\frac{\mathcal{C}_f p}{\lambda_1 q} + \frac{\mathcal{C}_g}{\tilde{s}_p}\right)\left(\int_\Omega \Phi_0(x,\nabla u)\,dx + \int_\Omega \frac{1}{p}|u|^p\,dx\right). \end{aligned}$$

For each $u \in \Phi^{-1}((-\infty,\mu))$, it follows that

$$\Phi(u) = \int_\Omega \Phi_0(x,\nabla u)\,dx + \int_\Omega \frac{1}{p}|u|^p\,dx < \mu.$$

Denote

$$\tilde{\lambda} = \min\{d,1\}\left(\frac{\mathcal{C}_f p}{\lambda_1 q} + \frac{\mathcal{C}_g}{\tilde{s}_p}\right)^{-1}.$$

Hence we assert that

$$\frac{1}{\mu}\sup_{u\in\Phi^{-1}((-\infty,\mu))} \Psi(u) < \frac{1}{\min\{d,1\}}\left(\frac{\mathcal{C}_f p}{\lambda_1 q} + \frac{\mathcal{C}_g}{\tilde{s}_p}\right) = \frac{1}{\tilde{\lambda}} < \frac{1}{\lambda}.$$

According to Lemma 4.1, we see that the functional I_λ satisfies (C)-condition for each $\lambda \in (\ell_*, \tilde{\lambda})$. Therefore, all assumptions of Theorem 2.6 are satisfied, so that, for each $\lambda \in (\ell_*, \tilde{\lambda})$, the problem (P) admits at least two distinct weak solutions in X. This completes the proof. □

5 Existence of three weak solutions

Now, we deal with the existence of at least three weak solutions for the problem (P). We start from the following conditions:

(F7) There exist a real number s_0 and a positive constant r_0 so small that

$$\int_{B_N(x_0,r_0)} F\big(x,|s_0|\big)\,dx > 0,$$

and $F(x,t) \geq 0$ for almost all $x \in B_N(x_0,r_0) \setminus B_N(x_0,\sigma r_0)$ with $\sigma \in (0,1)$ and for all $0 \leq t \leq |s_0|$, where $B_N(x_0,r_0) = \{x \in \Omega : |x - x_0| \leq r_0\}$.

(F8) f satisfies the following growth condition: for all $(x,s) \in \Omega \times \mathbb{R}$,

$$\big|f(x,s)\big| \leq a_1(x) + b_1(x)|s|^{\gamma_1 - 1},$$

where $a_1 \in L^{q'}(\Omega)$ with $p < q < p^*$, $\gamma_1 < p$, $b_1 \in L^{\nu_1}(\Omega)$ with $\nu_1 > 1$ and there exists $t_1 \in \mathbb{R}$ such that

$$p \leq t_1 \leq p^* \quad \text{and} \quad \frac{1}{\nu_1} + \frac{\gamma_1}{t_1} = 1.$$

(G6) There exists a positive constant d such that

$$G(x,s) \geq 0$$

for almost all $x \in \partial\Omega$ and for all $s \in [0,d]$.

(G7) g satisfies the Carathéodory condition and the following growth condition holds: for all $(x,s) \in \partial\Omega \times \mathbb{R}$,

$$\big|g(x,s)\big| \leq a_2(x) + b_2(x)|s|^{\gamma_2 - 1},$$

where $a_2 \in L^{q'}(\partial\Omega)$ with $p < q < p^{\partial}$, $\gamma_2 < p$, $b_2 \in L^{\nu_2}(\partial\Omega)$ with $\nu_2 > 1$, and there exists $t_2 \in \mathbb{R}$ such that

$$p \leq t_2 \leq p^{\partial} \quad \text{and} \quad \frac{1}{\nu_2} + \frac{\gamma_2}{t_2} = 1.$$

Lemma 5.1 *Assume that* (J1)-(J4), (F1), (F5), (F8), (G4), *and* (G7) *hold. Then*

$$\lim_{\mu \to 0^+} \frac{\sup_{\Phi(u) \leq \mu} \Psi(u)}{\mu} = 0.$$

Proof By the conditions (F5), (F8), (G4), and (G7), there exists a positive constant $\eta \in (0,1]$ such that

$$F(x,s) < M_1|s|^{\xi_1} \quad \text{and} \quad G(x,s) < N_1|s|^{\xi_2} \tag{5.1}$$

for positive constants M_1, N_1, for almost all $x \in \Omega$ and for all $s \in [-\eta,\eta]$. Let us consider some positive constants M_2, M_3, N_2, and N_3 given by

$$M_2 = \sup_{|s|>1} \frac{C_7(|s| + |s|^{\gamma_1})}{|s|^{\xi_1}} \quad \text{and} \quad M_3 = \sup_{\eta<|s|<1} \frac{C_7(|s| + |s|^{\gamma_1})}{|s|^{\xi_1}},$$

$$N_2 = \sup_{|s|>1} \frac{C_8(|s| + |s|^{\gamma_2})}{|s|^{\xi_2}} \quad \text{and} \quad N_3 = \sup_{\eta<|s|<1} \frac{C_8(|s| + |s|^{\gamma_2})}{|s|^{\xi_2}},$$

for some positive constants C_7 and C_8. Then it follows from (5.1), (F8) and (G7) that

$$F(x,s) < M|s|^{\xi_1} \quad \text{and} \quad G(x,s) < N|s|^{\xi_2}$$

for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$, where $M = \max\{M_1, M_2, M_3\}$ and $N = \max\{N_1, N_2, N_3\}$.

If μ satisfies $(\min\{d,1\}/p)\|u\|_X^p \le \mu < 1$, where d is the positive constant from (J4), then by Lemmas 3.1 and 3.2, for some positive constants C_9 and C_{10}, we have

$$\begin{aligned}\Psi(u) &= \int_\Omega F(x,u)\,dx + \int_{\partial\Omega} G(x,u)\,dS \\ &< M\int_\Omega |u|^{\xi_1}\,dx + N\int_{\partial\Omega} |u|^{\xi_2}\,dS \\ &\le C_9\|u\|_X^{\xi_1} + C_{10}\|u\|_X^{\xi_2} \le C_9\mu^{\frac{\xi_1}{p}} + C_{10}\mu^{\frac{\xi_2}{p}},\end{aligned}$$

where $\xi_1 > p$ and $\xi_2 > p$. It follows that

$$\lim_{\mu\to 0^+} \frac{\sup_{\frac{\min\{d,1\}}{p}\|u\|_X^p \le \mu} \Psi(u)}{\mu} = 0.$$

For any $u \in \Phi^{-1}((-\infty,\mu])$ with $\mu < \min\{d,1\}/p$, we know that $\Phi(u) \le \mu$ and so using the assumption (J4), we get

$$\frac{\min\{d,1\}}{p}\left(\int_\Omega |\nabla u|^p\,dx + \int_\Omega |u|^p\,dx\right) \le \int_\Omega \Phi_0(x,\nabla u)\,dx + \int_\Omega \frac{1}{p}|u|^p\,dx \le \mu.$$

Hence we deduce that

$$\int_\Omega |\nabla u|^p\,dx + \int_\Omega |u|^p\,dx \le \frac{p}{\min\{d,1\}}\mu < 1.$$

This inequality implies that $\|u\|_X < 1$. It follows that

$$\frac{\min\{d,1\}}{p}\|u\|_X^p \le \int_\Omega \Phi_0(x,\nabla u)\,dx + \int_\Omega \frac{1}{p}|u|^p\,dx \le \mu.$$

So we can get

$$\Phi^{-1}\big((-\infty,\mu]\big) \subset \left\{u \in X : \frac{\min\{d,1\}}{p}\|u\|_X^p \le \mu\right\}.$$

Then it follows that

$$0 \le \lim_{\mu\to 0^+} \frac{\sup_{\Phi(u)\le\mu} \Psi(u)}{\mu} \le \lim_{\mu\to 0^+} \frac{\sup_{\frac{\min\{d,1\}}{p}\|u\|_X^p \le \mu} \Psi(u)}{\mu} = 0$$

and therefore the conclusion holds. □

Theorem 5.2 *Assume that* (J1)-(J4), (F1), (F7), (F8), *and* (G6)-(G7) *hold. Then for each* $\lambda \in \Lambda_\mu$, *the problem* (P) *has at least three distinct weak solutions in* X *for each* $\lambda \in \Lambda_\mu = \left(\frac{\Phi(u_\varrho)}{\Psi(u_\varrho)}, \frac{\mu}{\sup_{\Phi(u)\in\mu}\Psi(u)}\right)$.

Proof All assumptions in Corollary 2.8 except the conditions (A1) and (A2) hold by Corollary 3.6 and a similar argument to Lemma 3.7. Note that $s_0 \neq 0$ be from (F7). For $\varrho \in (0,1)$, define

$$u_\varrho(x) = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B_N(x_0, r_0), \\ |s_0| & \text{if } x \in B_N(x_0, \varrho r_0), \\ \frac{|s_0|}{r_0(1-\varrho)}(r_0 - |x - x_0|) & \text{if } x \in B_N(x_0, r_0) \setminus B_N(x_0, \varrho r_0). \end{cases} \tag{5.2}$$

It is clear that $0 \leq u_\varrho(x) \leq |s_0|$ for all $x \in \overline{\Omega}$, and so $u_\varrho \in X$. Moreover, we have

$$\|u_\varrho\|_X^p = \frac{|s_0|^p(1-\varrho^N)}{(1-\varrho)^p} r_0^{N-p}\omega_N > 0,$$

where ω_N is the volume of $B_N(0,1)$. Also, by using the assumption (F7), we get

$$\begin{aligned} \Psi(u_\varrho) &= \int_{B_N(x_0,\varrho r_0)} F\bigl(x, |s_0|\bigr)\,dx + \int_{B_N(x_0,r_0)\setminus B_N(x_0,\varrho r_0)} F\left(x, \frac{|s_0|}{r_0(1-\varrho)}\bigl(r_0 - |x - x_0|\bigr)\right)dx \\ &> 0. \end{aligned}$$

Let us check the assumption (A1) in Corollary 2.8. Fix a real number μ_0 such that

$$0 < \mu < \mu_0 < \frac{1}{p}\min\{d,1\}\min\bigl\{\|u_\varrho\|_X^p, 1\bigr\} \leq \frac{1}{p}\min\{d,1\},$$

where u_ϱ was defined in (5.2). By Lemma 3.2 and (J4), we have

$$\begin{aligned} \Phi(u_\varrho) &= \int_\Omega \Phi_0(x, \nabla u_\varrho)\,dx + \int_\Omega \frac{1}{p}|u_\varrho|^p\,dx \\ &\geq \int_\Omega \frac{d}{p}|\nabla u_\varrho|^p\,dx + \int_\Omega \frac{1}{p}|u_\varrho|^p\,dx \\ &\geq \frac{\min\{d,1\}}{p}\|u_\varrho\|_X^p \geq \mu_0 > \mu. \end{aligned}$$

From Lemma 5.1, we obtain

$$\sup_{u\in\Phi^{-1}((-\infty,\mu])} \Psi(u) \leq \frac{\mu}{2}\frac{\Psi(u_\varrho)}{\Phi(u_\varrho)} < \mu\frac{\Psi(u_\varrho)}{\Phi(u_\varrho)},$$

that is,

$$\sup_{u\in\Phi^{-1}((-\infty,\mu])} \Psi(u) < \mu\frac{\Psi(u_\varrho)}{\Phi(u_\varrho)}$$

and hence the condition (A1) of Corollary 2.8 is fulfilled. For $\|u\|_X$ large enough and for all $\lambda \in \mathbb{R}$, it follows from the conditions (J4), (F8), and (G7) that

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega \Phi_0(x,\nabla u)\,dx + \int_\Omega \frac{1}{p}|u|^p\,dx - \lambda\int_\Omega F(x,u)\,dx - \lambda\int_{\partial\Omega} G(x,u)\,dS \\
&\geq \frac{d}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{p}\int_\Omega |u|^p\,dx - |\lambda|\int_\Omega \big|a_1(x)\big||u|\,dx - |\lambda|\int_\Omega \frac{1}{\gamma_1}\big|b_1(x)\big||u|^{\gamma_1}\,dx \\
&\quad - |\lambda|\int_{\partial\Omega} \big|a_2(x)\big||u|\,dS - |\lambda|\int_{\partial\Omega} \frac{1}{\gamma_2}\big|b_2(x)\big||u|^{\gamma_2}\,dS \\
&\geq \frac{d}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{p}\int_\Omega |u|^p\,dx - 2|\lambda|\|a_1\|_{L^{q'}(\Omega)}\|u\|_{L^q(\Omega)} \\
&\quad - \frac{2|\lambda|}{\gamma_1}\|b_1\|_{L^{\nu_1}(\Omega)}\big\||u|^{\gamma_1}\big\|_{L^{\frac{t_1}{\gamma_1}}(\Omega)} \\
&\quad - 2|\lambda|\|a_2\|_{L^{q'}(\partial\Omega)}\|u\|_{L^q(\partial\Omega)} - \frac{2|\lambda|}{\gamma_2}\|b_2\|_{L^{\nu_2}(\partial\Omega)}\big\||u|^{\gamma_2}\big\|_{L^{\frac{t_2}{\gamma_2}}(\partial\Omega)} \\
&\geq \frac{\min\{d,1\}}{p}\left(\int_\Omega |\nabla u|^p\,dx + \int_\Omega |u|^p\,dx\right) - 2|\lambda|C_{11}\|u\|_X - \frac{2|\lambda|C_{12}}{\gamma_1}\|u\|_{L^{t_1}(\Omega)}^{\gamma_1} \\
&\quad - 2|\lambda|C_{13}\|u\|_X - \frac{2|\lambda|C_{14}}{\gamma_2}\|u\|_{L^{t_2}(\partial\Omega)}^{\gamma_2} \\
&= \frac{\min\{d,1\}}{p}\|u\|_X^p - 2|\lambda|C_{11}\|u\|_X - \frac{2|\lambda|C_{12}}{\gamma_1}\|u\|_{L^{t_1}(\Omega)}^{\gamma_1} \\
&\quad - 2|\lambda|C_{13}\|u\|_X - \frac{2|\lambda|C_{14}}{\gamma_2}\|u\|_{L^{t_2}(\partial\Omega)}^{\gamma_2} \\
&\geq \frac{\min\{d,1\}}{p}\|u\|_X^p - 2|\lambda|C_{11}\|u\|_X - \frac{2|\lambda|C_{15}}{\gamma_1}\|u\|_X^{\gamma_1} \\
&\quad - 2|\lambda|C_{13}\|u\|_X - \frac{2|\lambda|C_{16}}{\gamma_2}\|u\|_X^{\gamma_2}
\end{aligned}$$

for some constants C_{11}, C_{12}, C_{13}, C_{14}, C_{15}, and C_{16}. Since $p > \gamma_1$ and $p > \gamma_2$, we deduce that

$$\lim_{\|u\|_X\to\infty} I_\lambda(u) = \infty$$

for all $\lambda \in \mathbb{R}$. Hence the functional I_λ is coercive for all $\lambda \in \Lambda_\mu$. Consequently, Corollary 2.8 implies that the problem (P) has at least three distinct weak solutions in X for each $\lambda \in \Lambda_\mu$. □

Theorem 5.3 *Assume that* (J1)-(J4), (F1), (F5), (F8), (G4), *and* (G7) *hold. If furthermore f satisfies the following assumption*:

(F9) *There exist $\mu > 0$ and $s_0 \in \mathbb{R}$ with $\mu < (\min\{d,1\}/p)|2s_0|^p(1-2^{-N})r_0^{N-p}\omega_N$ such that* (F7) *holds and*

$$2\left(\frac{p}{\min\{d,1\}}\right)^{\frac{1}{p}}\mathcal{M}\big(2\mu^{\frac{1}{p}-1} + \mu^{\frac{\gamma_1}{p}-1} + \mu^{\frac{\gamma_2}{p}-1}\big) < \frac{pr_0^{N-\frac{N-p}{p}}\operatorname{ess\,inf}_{B_N(x_0,\frac{r_0}{2})} F(x,|s_0|)}{2^N|s_0|(2p\|a\|_{L^{p'}(\Omega)} + b|s_0|^{p-1}+1)},$$

where $\mathcal{M} = \max\{s_q^{-1}\|a_1\|_{L^{q'}(\Omega)}, s_{t_1}^{-1}\|b_1\|_{L^{\nu_1}(\Omega)}, \tilde{s}_q^{-1}\|a_2\|_{L^{q'}(\partial\Omega)}, \tilde{s}_{t_2}^{-1}\|b_2\|_{L^{\nu_2}(\partial\Omega)}\}$, *then the problem* (P) *has at least distinct three solutions for every*

$$\begin{aligned}\lambda \in \tilde{\Lambda} &:= \Bigg(\frac{2^N|s_0|(2p\|a\|_{L^{p'}(\Omega)} + b|s_0|^{p-1} + 1)}{pr_0^{N-\frac{N-p}{p}} \operatorname{ess\,inf}_{B_N(x_0,\frac{r_0}{2})} F(x,|s_0|)}, \\ &\quad \frac{1}{2}\left(\frac{\min\{d,1\}}{p}\right)^{\frac{1}{p}} \mathcal{M}^{-1}\big(2\mu^{\frac{1}{p}-1} + \mu^{\frac{\gamma_1}{p}-1} + \mu^{\frac{\gamma_2}{p}-1}\big)^{-1}\Bigg) \\ &\subset \left(\frac{\Phi(\tilde{u}_1)}{\Psi(\tilde{u}_1)}, \frac{\mu}{\sup_{\Phi(u)\le\mu}\Psi(u)}\right).\end{aligned}$$

Proof By Corollary 3.6 and a similar argument to Lemma 3.7, all assumptions in Corollary 2.8 except the conditions (A1) and (A2) are fulfilled.

Define

$$\tilde{u}_1(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B_N(x_0, r_0), \\ |s_0| & \text{if } x \in B_N(x_0, \frac{r_0}{2}), \\ \frac{2|s_0|}{r_0}(r_0 - |x - x_0|) & \text{if } x \in B_N(x_0, r_0) \setminus B_N(x_0, \frac{r_0}{2}). \end{cases}$$

Then it is clear that $0 \le \tilde{u}_1(x) \le |s_0|$ for all $x \in \overline{\Omega}$, and so $\tilde{u}_1 \in X$. Moreover, it follows from (J4) that

$$\Phi(\tilde{u}_1) \ge \frac{\min\{d,1\}}{p}|2s_0|^p\left(1 - \frac{1}{2^N}\right)r_0^{N-p}\omega_N > 0$$

and

$$\begin{aligned}\Phi(\tilde{u}_1) &= \int_\Omega \Phi_0(x, \nabla\tilde{u}_1)\,dx + \int_\Omega \frac{1}{p}|\tilde{u}_1|^p\,dx \\ &\le \int_\Omega a(x)|\nabla\tilde{u}_1| + \frac{b}{p}|\nabla\tilde{u}_1|^p\,dx + \int_\Omega \frac{1}{p}|\tilde{u}_1|^p\,dx \\ &\le 2\|a\|_{L^{p'}(\Omega)}|s_0|\omega_N r_0^{\frac{N}{p}-1} + \frac{b|s_0|^p}{p}\omega_N\left(1 - \frac{1}{2^N}\right)r_0^{N-p} + \frac{|s_0|}{2^N p}\omega_N r_0^N \\ &\le \frac{1}{p}|s_0|\omega_N r_0^{\frac{N-p}{p}}\big(2p\|a\|_{L^{p'}(\Omega)} + b|s_0|^{p-1} + 1\big).\end{aligned}$$

Owing to the assumption (F7), we deduce that

$$\begin{aligned}\Psi(\tilde{u}_1) &\ge \int_{B_N(x_0,\frac{r_0}{2})} F(x, \tilde{u}_1)\,dx \\ &\ge \operatorname*{ess\,inf}_{B_N(x_0,\frac{r_0}{2})} F\big(x, |s_0|\big)\left(\frac{\omega_N r_0^N}{2^N}\right),\end{aligned}$$

and thus

$$\frac{\Psi(\tilde{u}_1)}{\Phi(\tilde{u}_1)} \ge \frac{pr_0^{N-\frac{N-p}{p}} \operatorname{ess\,inf}_{B_N(x_0,\frac{r_0}{2})} F(x,|s_0|)}{2^N|s_0|(2p\|a\|_{L^{p'}(\Omega)} + b|s_0|^{p-1} + 1)}. \tag{5.3}$$

From $\mu < \frac{\min\{d,1\}}{p}|2s_0|^p(1-\frac{1}{2^N})r_0^{N-p}\omega_N$, one has $\mu < \Phi(\tilde{u}_1)$. For each $u \in \Phi^{-1}((-\infty,\mu])$, it follows from (F8) and (G7) that

$$\begin{aligned}
\Psi(u) &= \int_\Omega F(x,u)\,dx + \lambda\int_{\partial\Omega} G(x,u)\,dS \\
&\le \int_\Omega \left|a_1(x)\right||u|\,dx + \int_\Omega \frac{1}{\gamma_1}\left|b_1(x)\right||u|^{\gamma_1}\,dx + \int_{\partial\Omega}\left|a_2(x)\right||u|\,dS \\
&\quad + \int_{\partial\Omega}\frac{1}{\gamma_2}\left|b_2(x)\right||u|^{\gamma_2}\,dS \\
&\le 2\|a_1\|_{L^{q'}(\Omega)}\|u\|_{L^q(\Omega)} + \frac{2}{\gamma_1}\|b_1\|_{L^{\nu_1}(\Omega)}\left\||u|^{\gamma_1}\right\|_{L^{\frac{t_1}{\gamma_1}}(\Omega)} \\
&\quad + 2\|a_2\|_{L^{q'}(\partial\Omega)}\|u\|_{L^q(\partial\Omega)} + \frac{2}{\gamma_2}\|b_2\|_{L^{\nu_2}(\partial\Omega)}\left\||u|^{\gamma_2}\right\|_{L^{\frac{t_2}{\gamma_2}}(\partial\Omega)} \\
&\le 2s_q^{-1}\|a_1\|_{L^{q'}(\Omega)}\|u\|_X + \frac{2\|b_1\|_{L^{\nu_1}(\Omega)}}{\gamma_1}\|u\|_{L^{t_1}(\Omega)}^{\gamma_1} \\
&\quad + 2\tilde{s}_q^{-1}\|a_2\|_{L^{q'}(\partial\Omega)}\|u\|_X + \frac{2\|b_2\|_{L^{\nu_2}(\partial\Omega)}}{\gamma_2}\|u\|_{L^{t_2}(\partial\Omega)}^{\gamma_2} \\
&\le 2s_q^{-1}\|a_1\|_{L^{q'}(\Omega)}\|u\|_X + \frac{2\|b_1\|_{L^{\nu_1}(\Omega)}}{\gamma_1 s_{t_1}}\|u\|_X^{\gamma_1} \\
&\quad + 2\tilde{s}_q^{-1}\|a_2\|_{L^{q'}(\partial\Omega)}\|u\|_X + \frac{2\|b_2\|_{L^{\nu_2}(\partial\Omega)}}{\gamma_2 \tilde{s}_{t_2}}\|u\|_X^{\gamma_2} \\
&\le 2\left(\frac{p}{\min\{d,1\}}\right)^{\frac{1}{p}}\left(s_q^{-1}\|a_1\|_{L^{q'}(\Omega)}\mu^{\frac{1}{p}} + \frac{\|b_1\|_{L^{\nu_1}(\Omega)}}{\gamma_1 s_{t_1}}\mu^{\frac{\gamma_1}{p}}\right. \\
&\quad \left. + \tilde{s}_q^{-1}\|a_2\|_{L^{q'}(\partial\Omega)}\mu^{\frac{1}{p}} + \frac{\|b_2\|_{L^{\nu_2}(\partial\Omega)}}{\gamma_2 \tilde{s}_{t_2}}\mu^{\frac{\gamma_2}{p}}\right) \\
&\le 2\left(\frac{p}{\min\{d,1\}}\right)^{\frac{1}{p}}\mathcal{M}\left(2\mu^{\frac{1}{p}} + \mu^{\frac{\gamma_1}{p}} + \mu^{\frac{\gamma_2}{p}}\right),
\end{aligned}$$

and so

$$\sup_{u\in\Phi^{-1}((-\infty,\mu])}\Psi(u) \le 2\left(\frac{p}{\min\{d,1\}}\right)^{\frac{1}{p}}\mathcal{M}\left(2\mu^{\frac{1}{p}} + \mu^{\frac{\gamma_1}{p}} + \mu^{\frac{\gamma_2}{p}}\right).$$

From (5.3) and the assumption (F5), we infer $\frac{1}{\mu}\sup\Psi(u) < \frac{\Psi(\tilde{u}_1)}{\Phi(\tilde{u}_1)}$. As seen before, the functional I_λ is coercive for each $\lambda > 0$. Taking into account that $\tilde{\Lambda} \subset (\frac{\Phi(\tilde{u}_1)}{\Psi(\tilde{u}_1)}, \frac{\mu}{\sup_{\Phi(u)\le\mu}\Psi(u)})$, Corollary 2.8 ensures that the problem (P) has at least distinct three solutions for each $\lambda \in \tilde{\Lambda}$. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

[1]Institute of Mathematical Sciences, Ewha Womans University, Seoul, 120-750, Republic of Korea. [2]Department of Mathematics Education, Sangmyung University, Seoul, 110-743, Republic of Korea.

Acknowledgements
The authors would like to thank the referees for careful reading of the manuscript. The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2009-0093827). The second author was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2014R1A1A2059536).

Received: 3 December 2015 Accepted: 3 May 2016 Published online: 09 May 2016

References

1. Bonanno, G: Relations between the mountain pass theorem and local minima. Adv. Nonlinear Anal. **1**, 205-220 (2012)
2. Bonanno, G, Marano, SA: On the structure of the critical set of non-differentiable functions with a weak compactness condition. Appl. Anal. **89**, 1-10 (2010)
3. Bonder, JF, Martínez, S, Rossi, JD: The behavior of the best Sobolev trace constant and extremals in thin domains. J. Differ. Equ. **198**, 129-148 (2004)
4. Chung, NT: Multiple solutions for quasilinear elliptic problems with nonlinear boundary conditions. Electron. J. Differ. Equ. **2008**, 165 (2008)
5. Lu, F-Y, Deng, G-Q: Infinitely many weak solutions of the p-Laplacian equation with nonlinear boundary conditions. Sci. World J. **2014**, 194310 (2014)
6. Winkert, P: Multiplicity results for a class of elliptic problems with nonlinear boundary condition. Commun. Pure Appl. Anal. **12**, 785-802 (2013)
7. Zhao, J-H, Zhao, P-H: Existence of infinitely many weak solutions for the p-Laplacian with nonlinear boundary conditions. Nonlinear Anal. **69**, 1343-1355 (2008)
8. Adams, RA, Fournier, JJF: Sobolev Spaces. Pure Appl. Math., vol. 140. Academic Press, New York (2003)
9. Arcoya, D, Carmona, J: A nondifferentiable extension of a theorem of Pucci and Serrin and applications. J. Differ. Equ. **235**, 683-700 (2007)
10. Colasuonno, F, Pucci, P, Varga, C: Multiple solutions for an eigenvalue problem involving p-Laplacian type operators. Nonlinear Anal. **75**, 4496-4512 (2012)
11. Liu, J, Shi, X: Existence of three solutions for a class of quasilinear elliptic systems involving the $(p(x),q(x))$-Laplacian. Nonlinear Anal. **71**, 550-557 (2009)
12. Ricceri, B: On three critical points theorem. Arch. Math. (Basel) **75**, 220-226 (2000)
13. Ricceri, B: Three critical points theorem revisited. Nonlinear Anal. **70**, 3084-3089 (2009)
14. Bonanno, G, Chinnì, A: Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent. J. Math. Anal. Appl. **418**, 812-827 (2014)
15. Ambrosetti, A, Rabinowitz, P: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349-381 (1973)
16. Talenti, G: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. **110**, 353-372 (1976)
17. Cerami, G: An existence criterion for the critical points on unbounded manifolds. Istit. Lombardo Accad. Sci. Lett. Rend. A **112**, 332-336 (1978) (1979) (in Italian)
18. Alves, CO, Liu, S: On superlinear $p(x)$-Laplacian equations in $\mathbb{R}^N$. Nonlinear Anal. **73**, 2566-2579 (2010)
19. Jeanjean, L: On the existence of bounded Palais-Smale sequences and application to a Landsman-Lazer type problem set on $\mathbb{R}^N$. Proc. R. Soc. Edinb. **129**, 787-809 (1999)
20. Liu, S: On superlinear problems without Ambrosetti and Rabinowitz condition. Nonlinear Anal. **73**, 788-795 (2010)
21. Li, G, Yang, C: The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition. Nonlinear Anal. **72**, 4602-4613 (2010)
22. Miyagaki, OH, Souto, MAS: Superlinear problems without Ambrosetti and Rabinowitz growth condition. J. Differ. Equ. **245**, 3628-3638 (2008)
23. Tan, Z, Fang, F: On superlinear $p(x)$-Laplacian problems without Ambrosetti and Rabinowitz condition. Nonlinear Anal. **75**, 3902-3915 (2012)
24. Ekeland, I: On the variational principle. J. Math. Anal. Appl. **47**, 324-353 (1974)
25. Zhong, C-K: A generalization of Ekeland's variational principle and application to the study of relation between the weak P.S. condition and coercivity. Nonlinear Anal. **29**, 1421-1431 (1997)
26. Bae, J-H, Kim, Y-H: A critical point theorem for nondifferentiable functionals with the Cerami condition via the generalized Ekeland variational principle. Submitted
27. Bonanno, G: A critical point theorem via the Ekeland variational principle. Nonlinear Anal. **75**, 2992-3007 (2012)
28. Motreanu, D, Motereanu, VV, Paşca, D: A version of Zhong's coercivity result for a general class of nonsmooth functionals. Abstr. Appl. Anal. **7**, 601-612 (2002)
29. Dràbek, P, Kufner, A, Nicolosi, F: Quasilinear Elliptic Equations with Degenerations and Singularities. de Gruyter, Berlin (1997)
30. Kim, IH, Kim, Y-H: Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents. Manuscr. Math. **147**, 169-191 (2015)
31. Choi, EB, Kim, Y-H: Three solutions for equations involving nonhomogeneous operators of p-Laplace type in $\mathbb{R}^N$. J. Inequal. Appl. **2014**, 427 (2014)
32. Lee, SD, Park, K, Kim, Y-H: Existence and multipliciy of solutions for equations involving nonhomogeneous operators of $p(x)$-Laplace type in $\mathbb{R}^N$. Bound. Value Probl. **2014**, 261 (2014)
33. Zeidler, E: Nonlinear Functional Analysis and Its Applications II/B. Springer, New York (1990)
34. Martínez, S, Rossi, JD: Weak solutions for the p-Laplacian with a nonlinear boundary condition at resonance. Electron. J. Differ. Equ. **2003**, 27 (2003)