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Instability of the Rayleigh problem with piecewise smooth steady states

Jiangang Qi, Bing Xie* and Shaozhu Chen

*Correspondence: xiebing@sdu.edu.cn
Department of Mathematics, Shandong University at Weihai, Weihai, 264209, P.R. China

Abstract
The present paper investigates the instability of the Rayleigh equation with piecewise smooth steady states. An existence result of a number of unstable modes for the Rayleigh problem is obtained.

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1 Introduction

Consider a plane shear flow $\mathbf{U} = \mathbf{i}U(y)$ within the channel, $x \in (-\infty, \infty)$ and $y \in [-1, 1]$, a parallel shear flow in the x-direction. The linearized vorticity equation for a two-dimensional disturbance can be written as the equation (see [1] or [2])

$$\partial_t w + U(y)\partial_x w - U''(y)\partial_x \psi = 0, \tag{1.1}$$

where $w = w(x, y, t)$ is the vorticity perturbation, $\psi = \psi(x, y, t)$ is the associated stream function, which is related by

$$w = \nabla^2 \psi = \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi$$

and the boundary condition $\psi(x, \pm 1, t) = 0$ for any $x \in (-\infty, \infty)$.

By the normal mode method, or seeking the solutions of the form

$$\psi(x, y, t) = \varphi(y)e^{i\alpha(x-ct)}$$

with α the wave number (positive real) in the x-direction and $c = c_r + \mathrm{i}\, c_i$ the complex wave speed, we obtain the Rayleigh equation

$$\big(U(y) - c\big)\big(\varphi'' - \alpha^2\varphi\big) - U''(y)\varphi = 0, \quad \varphi = \varphi(y), y \in (-1, 1), \tag{1.2}$$

with the boundary condition

$$\varphi(-1) = 0, \qquad \varphi(1) = 0. \tag{1.3}$$

So for shear flows, the instability problem is reduced to study the Rayleigh eigenvalue problems (1.2) and (1.3). This problem has a long history, going back to scientists such as Rayleigh and Kelvin in the 19th century (see [3–5]).

The flow is linearly unstable if some nontrivial solutions to (1.2) and (1.3) exist, with the imaginary part of c satisfying $c_i := \operatorname{Im} c > 0$. A classical result of Rayleigh [1] is the necessary condition for instability that the basic velocity profile should have an inflection point at some points $y = y_s$, that is, $U''(y_s) = 0$. This condition was later improved by Fjörtoft [6]. However, it is far more difficult to obtain effective sufficient conditions for instability.

In 1935, Tollmien [7] obtained an unstable solution to (1.2) by formally perturbing around a neutral mode (*i.e.*, c is a real number) for symmetric flows. In 1964, Rosenbluth and Simon [8] gave a necessary and sufficient integral condition for the monotone flows. Recently, some instability criteria for the special flows $U(y) = \cos my$ in [9] and $U(y) = \sin my$ in [10] have been obtained. These results were much improved and extended by Lin to more general odd symmetric flows in [11] and other classes of shear flows in [12].

We call a function $U(y)$ *point symmetric* with respect to $c_0 \in \mathbb{R}$ if $U(y) + U(-y) \equiv 2c_0$. For such $U(y)$ we define

$$K(y) := \frac{U''(y)}{U_0(y)}, \quad U_0(y) = U(y) - c_0. \tag{1.4}$$

In the recent paper [13], the instability results were extended to the case where K is allowed to be unbounded, but integrable on $[-1,1]$.

Proposition 1.1 (*cf.* Theorem 1.2 in [13]) *Suppose that $U(y)$ is point symmetric, $U(y) + U(-y) = 2c_0$, $U \in C^2[-1,1]$, and $K \in L^1[-1,1]$. If $-\mathrm{d}^2/\mathrm{d}y^2 + K(y)$ with the boundary condition* (1.3) *has negative eigenvalues*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N < 0,$$

then there is an unstable mode for every wave number $\alpha \in \bigcup_{k=1}^{m_0}(\alpha_{2k}, \alpha_{2k-1})$, where $m_0 = [(N+1)/2]$, the largest integer less than or equal to $(N+1)/2$, $\alpha_j = \sqrt{-\lambda_j}$ for $1 \le j \le N$, and $\alpha_{N+1} = 0$.

The present paper mainly focuses on the instability of the Rayleigh equation with piecewise smooth and point-symmetric velocity profiles $U(y)$, satisfying the requirement that $U'(y)$ exists continuously on $[-1,1]$ and $U''(y)$ exists on $[-1,1]$ except for a finite number of points. More general cases with piecewise smooth velocity profile were studied in [13].

Proposition 1.2 (*cf.* Theorem 1.4 in [13]) *Suppose that $U(y)$ is point symmetric, $U \in C[-1,1]$, and there exist $2N+1$ points $\{y_j\}_{-N}^{N}$ in $[-1,1]$ such that $0 = y_0 < y_1 < \cdots < y_N = 1$, $y_{-j} = -y_j$, and $U(y)$ is twice continuously differentiable for $y \neq y_j$, $-N \le j \le N$. If there exist constants $C_j > 0$ and $0 \le \rho_j < 1$, $0 \le j \le N$, such that*

$$\left|U_0(y)\right| \ge C_j|y - y_j|^{\rho_j} \quad \text{for } y \text{ near } y_j,$$

then there exists an $\alpha_c > 0$ such that the Rayleigh problem of (1.2) *with* (1.3) *has at least one unstable mode for every $\alpha \in (0, \alpha_c)$.*

Set

$$L_{\mathrm{loc}}\big([-1,0)\cup(0,1]\big) := \big\{f \text{ is a measurable function on } [-1,0)\cup(0,1]: \\ \text{for any compact subset } I\subset[-1,0)\cup(0,1], f\in L^1(I)\big\}.$$

Clearly, $L^1[-1,1]$ is a proper subset of $L_{\mathrm{loc}}([-1,0)\cup(0,1])$. Then we can give the following instability criterion, which is the main result of this paper.

Theorem 1.3 *Let $U\in\mathbf{C}[-1,1]$ be point symmetric with respect to $c_0\in\mathbb{R}$ and there exist $2N+1$ points $\{y_j\}_{-N}^{N}$ in $[-1,1]$ such that $0=y_0<y_1<\cdots<y_N=1$, $y_{-j}=-y_j$, and $U(y)$ is twice continuously differentiable for $y\neq y_j$, $-N\le j\le N$. Let U_0 and K be defined in* (1.4). *Suppose that $U_0'U_0^k$ is bounded on $[-1,1]$ for sufficiently large k and*

$$\begin{aligned} &\text{(i)}\quad K\in L_{\mathrm{loc}}\big([-1,0)\cup(0,1]\big),\qquad U_0(\pm1)\neq0,\\ &\text{(ii)}\quad U_0(y)\sim ay^{\rho}\quad \text{as } y\to0, 3/2>\rho>1/2, \rho\neq1, a\neq0. \end{aligned}\tag{1.5}$$

Let $H(0)$ be the differential operator in $L^2(0,1]$ associated to

$$-\varphi''+K\varphi=\lambda\varphi,\qquad \lim_{y\to0+}\big(\varphi'(y)U_0(y)-\varphi(y)U_0'(y)\big)=0,\qquad \varphi(1)=0.\tag{1.6}$$

If $H(0)$ has negative eigenvalues arranged in the order

$$\lambda_1<\lambda_2<\cdots<\lambda_N<0,$$

then there is an unstable mode of the Rayleigh problem (1.2) *and* (1.3) *for every wave number $\alpha\in\bigcup_{k=1}^{m_0}(\alpha_{2k},\alpha_{2k-1})$, where $m_0=[(N+1)/2]$, the largest integer less than or equal to $(N+1)/2$, $\alpha_j=\sqrt{-\lambda_j}$ for $1\le j\le N$, and $\alpha_{N+1}=0$.*

Note that Proposition 1.1 studies the case where $K\in L^1[-1,1]$ and in Theorem 1.3 the function K is allowed not to be in $L^1[-1,1]$ ($K\in L_{\mathrm{loc}}([-1,0)\cup(0,1])$). Besides, Proposition 1.2 only gives the existence of an unstable mode in one interval of wave numbers, while Theorem 1.3 gives the existence of many unstable modes in a number of intervals of wave number if $H(0)$ has more than one negative eigenvalue. At the end of the present paper, we give an illustrative example in which $H_1(0)\oplus H_2(0)$ has more than one negative eigenvalue.

Since $U''(y)$ is allowed to be discontinuous at finite points in $[-1,1]$, this will result in different definitions of solutions of the Rayleigh equation and different Sturm-Liouville problems. For this reason, we will put much of attention on the properties of such solutions and corresponding eigenvalue problems. See Lemmas 2.1, 2.2, 2.4, and 2.5.

The main tools in the proof of Theorem 1.3 are the perturbation theory of operators in Hilbert spaces and the spectral theory of $\mathcal{PT}$-symmetric differential operators.

Following this section, Section 2 presents some preliminary knowledge about the properties of solutions to the Rayleigh equation with piecewise smooth velocity profiles, the spectral properties of singular Sturm-Liouville problems with one singular end point and regular Sturm-Liouville problems with $\mathcal{PT}$-symmetric potentials. The proof of the main result is given in Section 3 and the illustrative example is also given at the end of that section.

2 Preliminary knowledge

In this section, we first introduce properties of solutions of the Rayleigh equation with piecewise smooth velocity profiles and spectral properties of regular and singular Sturm-Liouville problems, especially the regular one with $\mathcal{PT}$-symmetric potentials. We note that the assumptions of all lemmas in Section 2 and Section 3 are the same as in Theorem 1.3 and so are omitted for the sake of brevity.

2.1 Solutions of the Rayleigh equations

Since $U''(y)$ may be discontinuous at some junctions inside the interval, we must redefine the solution of the governing equation (1.2) and the eigenfunction to the Rayleigh eigenvalue problem in a weaker sense than the classical one.

For fixed α^2 and $c = c_r + \mathrm{i}\,c_i$ with $c_i > 0$, consider the differential equation

$$-\varphi'' + \left(\frac{U''(y)}{U(y) - c} + \alpha^2\right)\varphi = \lambda\varphi, \quad y \in (-1,1), \tag{2.1}$$

involving the spectral parameter λ (*cf.* [14]). Set $I_j = (y_j, y_{j+1})$, $-N \le j \le N-1$. A continuous function φ is called a *solution* of (2.1) on $[-1,1]$ if φ satisfies (2.1) on each I_j and

$$\bigl(B(c)\varphi\bigr)(y) := (U - c)\varphi' - U'\varphi \tag{2.2}$$

can be extended to be absolutely continuous on $[-1,1]$ (see [13]). If, for a λ, φ is a nonzero solution of (2.1) and satisfies the boundary condition (1.3), then we call λ an *eigenvalue* and φ the corresponding *eigenfunction* to (2.1) with the boundary (1.3) and, by convention, we call $c = c_r + \mathrm{i}\,c_i$ an eigenvalue and φ the corresponding eigenfunction of the Rayleigh problem (2.1) with (1.3).

Lemmas 3.1 and 3.2 in [13] show that such a solution is also uniquely determined by an initial value condition.

Lemma 2.1 (*cf.* Lemmas 3.1 and 3.2 in [13]) *For each initial value condition*

$$\varphi(\xi) = a_1, \qquad (B\varphi)(\xi) = a_2, \quad a_1, a_2 \in \mathbb{C}, \xi \in [-1,1], \tag{2.3}$$

there exists a unique solution $\varphi(y,c,\lambda)$ *of* (2.1) *in* $\mathbf{C}[-1,1] \cup \mathbf{C}^1([-1,0)\cup(0,1])$. *Furthermore,* $\varphi(y,c,\lambda)$ *is analytic in* $c_i > 0$ *and* λ.

Note, Lemma 2.1 implies that the corresponding eigen-subspace of every eigenvalue of (2.1) and (1.3) has exactly one dimension.

Lemma 2.2 *Let* φ_1, φ_2 *be two solutions of* (2.1) *with* $\operatorname{Im} c \ne 0$ *and define the Wronskian* $W[\varphi_1,\varphi_2]$ *of* φ_1 *and* φ_2 *for* $y \ne 0$:

$$W[\varphi_1,\varphi_2](y) = \varphi_1(y)\varphi_2'(y) - \varphi_1'(y)\varphi_2(y).$$

Then there exists a constant $C \in \mathbb{C}$ *such that* $W[\varphi_1,\varphi_2](y) \equiv C$ *for* $y \ne 0$.

Proof Since φ_1, φ_2 are classical solutions of (2.1), $W[\varphi_1,\varphi_2](y) \equiv C_{\pm j}$ on $I_{\pm j}$, $0 \le j \le N$. On the other hand, $B\varphi_1$ and $B\varphi_2$ are both continuous on $[-1,1]$ and $U_0\,W[\varphi_1,\varphi_2](y) = \varphi_1 B\varphi_2 -$

$\varphi_2 B\varphi_1$ is continuous by the continuity of φ_k on $[-1,1]$, $k = 1,2$, and hence $U(y) - c \neq 0$ implies that $W[\varphi_1,\varphi_2]$ is continuous on $[-1,1]$. As a result, $C_{\pm j} \equiv C$ for $1 \le j \le N$. This proves Lemma 2.2. □

By Lemma 2.2 we can define the linearly dependent(resp. independent) solutions. Let φ_1, φ_2 be solutions of (2.1). If $W[\varphi_1,\varphi_2](y_0) = 0$ for some $y_0 \in [-1,1]$, then we say that φ_1, φ_2 are *linearly dependent*. Otherwise we say they are *linearly independent*.

Lemma 2.3 *Set*

$$F(y,t) = t^{1/\rho-1}\int_{yt^{-1/\rho}}^{t^{-1/\rho}} \frac{\mathrm{d}s}{1+s^{2\rho}}, \quad y \in (0,1].$$

If $\rho \in (0,3/2)$, *then there exists an* $F_0 \in L^2(0,1]$ *such that* $F(y,t) \le F_0(y)$ *for all* $t \in (0,1]$.

Proof For $\rho \in (0,1/2)$, we have, for $t \in (0,1]$,

$$\begin{aligned} F(y,t) &\le t^{1/\rho-1}\int_{yt^{-1/\rho}}^{t^{-1/\rho}} \frac{\mathrm{d}s}{s^{2\rho}} = \frac{1}{1-2\rho}t^{1/\rho-1}\bigl[\bigl(t^{-1/\rho}\bigr)^{1-2\rho} - \bigl(yt^{-1/\rho}\bigr)^{1-2\rho}\bigr] \\ &\le \frac{1}{1-2\rho}t^{1/\rho-1}t^{-1/\rho+2} = \frac{t}{1-2\rho} \le \frac{1}{1-2\rho}. \end{aligned}$$

For $\rho = 1/2$, we have, for $t \in (0,1]$,

$$F(y,t) = t\ln\frac{1+t^{-2}}{1+yt^{-2}} = t\ln\frac{1+t^2}{y+t^2} \le M, \quad y \in (0,1].$$

For $1/2 < \rho \le 1$, we have, for $t \in (0,1]$,

$$F(y,t) \le t^{1/\rho-1}\int_0^\infty \frac{\mathrm{d}s}{1+s^{2\rho}} \le M.$$

For $3/2 > \rho > 1$, we have, for $t \in (0,1]$,

$$\begin{aligned} F(y,t) &= t^{1/\rho-1}\int_{yt^{-1/\rho}}^{t^{-1/\rho}} \frac{1}{\rho s^{\rho-1}}\frac{\mathrm{d}s^\rho}{1+s^{2\rho}} \\ &\le \frac{y^{1-\rho}}{\rho}t^{1/\rho-1}\bigl(t^{-1/\rho}\bigr)^{1-\rho}\int_0^\infty \frac{\mathrm{d}u}{1+u^2} \le \frac{\pi y^{1-\rho}}{2\rho}. \end{aligned}$$

Since $\rho < 3/2$, one sees that $y^{1-\rho} \in L^2(0,1]$. This completes the proof. □

2.2 Spectral properties of the corresponding differential operators

Consider the Sturm-Louville eigenvalue problem associated of (2.1) and (1.3) for $c = c_0 + \mathrm{i}\,c_i$ with $c_i > 0$ in $L^2[-1,1]$, *i.e.*,

$$-\varphi'' + \bigl[K(y,c_i) + \alpha^2\bigr]\varphi = \lambda\varphi, \qquad \varphi(\pm 1) = 0, \quad K(y,c_i) = \frac{U''(y)}{U_0(y) - \mathrm{i}\,c_i}. \tag{2.4}$$

Since $U'' \in L[-1,1]$ and $|K(y,c_i)| \le U''/|c_i|$, the problem is regular. Denote by $H(c_i)$ the associated operator. If $U(y)$ is essentially odd, it is easy to see that $\overline{K(y,c_i)} = K(-y,c_i)$. This ensures that the operator $H(c_i)$ is $\mathcal{PT}$*-symmetric* for every $c_i \in (0,\infty)$, that is, $H(c_i)$ is densely defined and

$$H(c_i)\mathcal{PT} \subset \mathcal{PT}H(c_i)$$

where $\mathcal{P}$ is the parity transformation: $\varphi(y) \to \varphi(-y)$ and $\mathcal{T}$ is the time reversal transformation: $\varphi(y) \to \overline{\varphi(y)}$. It is well known that the spectrum of a $\mathcal{PT}$-symmetric operator is symmetric with respect to the real axis. As a result we have, for $c_i > 0$,

$$\lambda \in \sigma\bigl(H(c_i)\bigr) \quad \Longrightarrow \quad \overline{\lambda} \in \sigma\bigl(H(c_i)\bigr), \quad c_i \in [0,\infty), \tag{2.5}$$

where $\sigma(H(c_i))$ is the spectrum of $H(c_i)$.

Since (2.4) is regular, the operator $H(c_i)$ has only countable discrete eigenvalues. The properties of eigenvalues were studied in [13] and it has been proved that the real parts of eigenvalues of (2.4) are bounded from below. The eigenvalues can be arranged in the dictionary order according to their real parts and imaginable parts. Then we write the countably many eigenvalues of $H(c_i)$ as $\{\lambda_n(c_i)\}_{n\geq 1}$ in such an order and conclude from Theorem 2.2 of [13] the following.

Lemma 2.4 (*cf.* Theorem 2.2 of [13]) *We have*

$$\lambda_n(c_i) = \left(\frac{n\pi}{2}\right)^2 + r(n,c_i), \quad \bigl|r(n,c_i)\bigr| \leq C(c_i)n, \tag{2.6}$$

where the positive constant $C(c_i)$ *satisfies*

$$C(c_i) \leq C_0 \int_{-1}^{1} \bigl|K(y,c_i)\bigr| \,\mathrm{d}y, \tag{2.7}$$

and the constant C_0 *is independent of* c_i. *Moreover, there exists a positive integer* $N(c_i)$ *depending only on* c_i *such that, for* $n > N(c_i)$, $\lambda_n(c_i)$ *is algebraically simple.*

Let $H(0)$ be the associated operator to

$$\begin{aligned} &E_j\colon -\varphi'' + K(y)\varphi = \lambda\varphi, \\ &\lim_{y\to 0+} \bigl(\varphi'(y)U_0(y) - \varphi(y)U_0'(y)\bigr) = 0, \qquad \varphi(1) = 0 \end{aligned} \tag{2.8}$$

in $L^2(0,1]$, where $K(y) = U''(y)/U_0(y)$. The operator $H(0)$ maybe singular with the possible singular end point $y = 0$ since U''/U_0 is allowed to be not integrable on $(0,1]$. We prepare some basic results about the spectrum σ_0 of $H(0)$.

Lemma 2.5 *The operator* $H(0)$ *is self-adjoint and* σ_0 *contains only discrete, real and algebraically simple eigenvalues.*

Proof By the condition (ii) of (1.5) there exists a $\delta > 0$ such that $U(y) \neq 0$ for $y \in (0,\delta]$. Let φ_0 be the solution of $-\varphi'' + K\varphi = 0$ satisfying

$$\varphi_0(\delta) = 0, \qquad \varphi_0'(\delta) = \frac{1}{U_0(\delta)}.$$

Since $K \in L^1_{\text{loc}}(0,1]$ in (i) of (1.5), we know that φ_0 exists on $(0,1]$ and is uniquely determined by the above initial value condition. Clearly $\varphi_0 \in L^2[\delta,1]$. Since $U_0(y) \neq 0$ on $(0,\delta]$, the explicit expression of φ_0 on $(0,\delta]$ is given by

$$\varphi_0(y) = U_0(y) \int_\delta^y \frac{1}{U_0^2}, \quad y \in (0,\delta]. \tag{2.9}$$

Applying $U_0(y) \sim ay^\rho$ as $y \to 0$ and $1/2 < \rho < 3/2$ in (ii) of (1.5) we find that

$$\left|\varphi_0(y)\right| \leq C_1 y^\rho + C_2 y^{1-\rho} \in L^2[0,\delta] \tag{2.10}$$

with some constants $C_1, C_2 > 0$. Thus $\varphi_0 \in L^2[0,1]$. Clearly, U_0 is the other linearly independent solution of $-\varphi'' + K\varphi = 0$ such that $U_0 \in L^2[0,1]$, and hence all the solutions of $-\varphi'' + K\varphi = \lambda\varphi$ belong to $L^2[0,1]$ for $\lambda \in \mathbb{C}$ by the variation of constants formula. This means that the equation in (2.8) is of limit circle type at $y = 0$ by the classification of Weyl; see [15]. Therefore, $H(0)$ with the separated and self-adjoint boundary conditions in (2.8) is self-adjoint, and hence all the conclusions of Lemma 2.2 are valid by the spectral theory of symmetric Sturm-Liouville differential operators. This completes the proof. □

Note that $U(\pm 1) \neq 0$. Denote by $\varphi_+(y)$, $\varphi_-(y)$ the solutions of $-\varphi'' + K\varphi = 0$ on $(0,1]$, $[-1,0)$, respectively, satisfying

$$\varphi_\pm(\pm 1) = 0, \qquad \varphi_\pm'(\pm 1) = 1/U_0(\pm 1). \tag{2.11}$$

Since $\varphi_+(1) = 0$ and the Wronsikian $W[U_0,\varphi_+]$ of the two linearly independent solutions, U_0 and φ_+ satisfy

$$B\varphi_+(y) = W[U_0,\varphi_+](y) = \varphi_+'(y)U_0(y) - \varphi_+(y)U_0'(y) \equiv 1 \quad \text{on } (0,1]$$

by Lemma 2.2, one sees that φ_+ is not an eigenfunction of $H(0)$. This means that $\lambda = 0$ is not an eigenvalue of $H(0)$, and hence the resolvent $(H(0) - z)^{-1}$ of $H(0)$ at $z = 0$ exists. By a computation, the integral expression of $H^{-1}(0)$ is given by

$$H^{-1}(0)f(y) = \int_0^1 G_0(y,t)f(t)\,\mathrm{d}t, \tag{2.12}$$

where $G_0(y,t)$ is the Green function associated to $H(0)$ at 0 given by

$$G_0(y,t) = -\begin{cases} U_0(t)\varphi_+(y), & 1 > y > t > 0, \\ U_0(y)\varphi_+(t), & 0 < y < t < 1. \end{cases} \tag{2.13}$$

Equations (2.12) and (2.13) will be used in the proof of Lemma 3.3.

2.3 Convergence properties of solutions of (2.4) as $c_i \to 0+$

Set

$$\varphi_\pm(y,c_i) = \left(U_0(y) - \mathrm{i}\,c_i\right) \int_{\pm 1}^y \frac{1}{(U_0 - \mathrm{i}\,c_i)^2}, \tag{2.14}$$

$I(\delta) = [\delta, 1]$ and $I(-\delta) = [-1, -\delta]$ for $\delta \in (0,1)$. Note that $\varphi_\pm(y; c_i)$ belong to both $L^2[-1,1]$ and $\mathbf{C}[-1,1]$ for $c_i > 0$. The following result gives the convergence properties for $\varphi_\pm(y; c_i)$ in $\mathbf{C}$-norm and L^2-norm.

Lemma 2.6 *Let $\varphi_\pm(y; c_i)$ be defined in* (2.14) *and $\varphi_\pm(y)$ in* (2.11), *then as $c_i \to 0+$,*

$$\varphi_+(\cdot; c_i) \to \varphi_+ \quad \text{in } C\bigl(I(\delta)\bigr), \qquad \varphi_-(\cdot; c_i) \to \varphi_- \quad \text{in } C\bigl(I(-\delta)\bigr) \tag{2.15}$$

for every $\delta \in (0,1)$ and

$$\varphi_+(\cdot; c_i) \to \varphi_+ \quad \text{in } L^2(0,1], \qquad \varphi_-(\cdot; c_i) \to \varphi_- \quad \text{in } L^2[-1,0). \tag{2.16}$$

Proof Since $K \in L(I(\pm\delta))$ by the assumptions in Theorem 1.3, one can verify that $K(\cdot, c_i) \to K(\cdot)$ in $L(I(\pm\delta))$ as $c_i \to 0+$ for $\delta \in (0,1)$ by Lebesgue dominated convergence theorem. Note that

$$\varphi_\pm(\pm 1, c_i) = 0, \qquad \varphi'_\pm(\pm 1, c_i) = \frac{1}{U_0(\pm 1) - \mathrm{i}\, c_i} \to \frac{1}{U_0(\pm 1)} = \varphi'_\pm(\pm 1),$$

as $c_i \to 0$. Then (2.15) is true by the theory of ordinary differential equations.

Let $\delta > 0$ be sufficiently small such that $|U_0(y)| \geq |a| y^\rho / 2$ for $0 < y \leq \delta$ by (ii) of (1.5). For $y \in [\delta, 1]$, we see from (2.15), for $0 < c_i \leq 1$, that there exists $M(\delta) > 0$, independent of c_i, such that

$$\bigl|\varphi_+(y; c_i)\bigr| \leq M(\delta), \quad y \in [\delta, 1]. \tag{2.17}$$

For $y \in (0, \delta)$, we write $\varphi_+(y, c_i)$ into

$$\begin{aligned} \varphi_+(y; c_i) &= \bigl(U_0(y) - \mathrm{i}\, c_i\bigr) \biggl(\int_1^\delta + \int_\delta^y \biggr) \frac{1}{(U_0 - \mathrm{i}\, c_i)^2} \\ &= \frac{U_0(y) - \mathrm{i}\, c_i}{U_0(\delta) - \mathrm{i}\, c_i} \varphi_+(\delta; c_i) + \bigl(U_0(y) - \mathrm{i}\, c_i\bigr) \int_\delta^y \frac{1}{(U_0 - \mathrm{i}\, c_i)^2}. \end{aligned} \tag{2.18}$$

The second term of the last expression in (2.18) is dominated by

$$\left| U_0(y) \int_\delta^y \frac{1}{(U_0 - \mathrm{i}\, c_i)^2} \right| + c_i \left| \int_\delta^y \frac{1}{(U_0 - \mathrm{i}\, c_i)^2} \right| \leq \varphi_0(y) + c_i \int_y^\delta \frac{1}{U_0^2 + c_i^2},$$

where $\varphi_0(y)$ is defined in (2.9). Since $|U_0(y)| \geq |a| y^\rho / 2$ for $0 < y \leq \delta$,

$$\begin{aligned} c_i \int_y^\delta \frac{1}{U_0^2 + c_i^2} &= \frac{2\hat{c}}{|a|} \int_y^\delta \frac{\mathrm{d}t}{t^{2\rho} + \hat{c}^2} \leq \frac{2c_i}{|a|} \int_y^{|a|\delta/2} \frac{\mathrm{d}t}{t^2 + c_i^2} \\ &= \frac{2\hat{c}^{1/\rho - 1}}{|a|} \int_{y\hat{c}^{-1/\rho}}^{\delta c^{-1/\rho}} \frac{\mathrm{d}s}{s^{2\rho} + 1}, \end{aligned}$$

where $\hat{c} = 2c_i / |a|$. Then by Lemma 2.3 there exists an $F_0 \in L^2(0,1]$ such that

$$c_i \int_y^\delta \frac{1}{U_0^2 + c_i^2} \leq F_0(y), \quad y \in (0,1]$$

for all $c_i \in [-|a|/2, |a|/2]$. Therefore,

$$\left|\bigl(U_0(y) - \mathrm{i}\,c_i\bigr)\int_\delta^y \frac{1}{(U_0 - \mathrm{i}\,c_i)^2}\right| \le \bigl|\varphi_0(y)\bigr| + F_0(y) \tag{2.19}$$

for $y \in (0, \delta]$ and all sufficiently small $c_i > 0$.

Since $U_0(\delta) \neq 0$, the first term of the last expression in (2.18) is clearly dominated by $C_1|U_0(y)| + C_2$ for $c_i \le 1$ by (2.17). Therefore we conclude from (2.17), (2.18), and (2.19) that, for $y \in (0, 1]$,

$$\bigl|\varphi_+(y; c_i)\bigr| \le C_1\bigl|U_0(y)\bigr| + \bigl|\varphi_0(y)\bigr| + F_0(y) + C_2 \in L^2(0, 1] \tag{2.20}$$

for all sufficiently small $c_i > 0$ by (2.10), where the constants C_1 and C_2 are independent of c_i. This together with (2.15) yields that $\varphi_+(\cdot; c_i) \to \varphi_+(\cdot)$ in $L^2(0, 1]$ as $c_i \to 0+$ by Lebesgue dominated convergence theorem. Similarly, one can prove that $\varphi_-(\cdot; c_i) \to \varphi_-(\cdot)$ in $L^2[-1, 0)$ as $c_i \to 0+$. The proof of Lemma 2.6 is finished. □

Lemma 2.7 *Let $\varphi(y; \lambda, c_i)$ be the solution of* (2.1) *such that $\varphi(1; \lambda, c_i) = 0$ and $\varphi'(1; \lambda, c_i) = 1/U_0(1)$ for $c_i \ge 0$. Then $\varphi(\cdot; \lambda, c_i) \to \varphi(\cdot; \lambda, 0)$ as $c_i \to 0+$ in $L^2[0, 1]$.*

Proof Let $\varphi_+(y; c_i)$ be defined as in (2.14). Set

$$R(y, t, c_i) = \bigl(U(y) + \mathrm{i}\,c_i\bigr)\varphi_+(t; c_i) - \bigl(U(t) + \mathrm{i}\,c_i\bigr)\varphi_+(y; c_i)$$

and $R(y, t) = R(y, t, 0)$. By the variation of constants formula, the solution $\varphi(y; \lambda, c_i)$ can be expressed as

$$\varphi(y, \lambda, c_i) = \varphi_+(y; c_i) + \lambda \int_1^y R(y, t, c_i)\varphi(t, \lambda, c_i)\,\mathrm{d}t, \quad y \in [0, 1]. \tag{2.21}$$

Since $\varphi_+(\cdot; c_i) \to \varphi_+$ in $L^2[0, 1]$ as $c_i \to 0+$ by Lemma 2.6, we see that $R(\cdot; c_i) \to R$ in $L^2([0, 1] \times [0, 1])$ as $c_i \to 0$. Therefore, by a standard method we can prove the validity of the conclusion of Lemma 2.7. □

In the sequel we will use the quantity $W(c_i)$ defined by

$$W(c_i) = \int_0^1 \frac{\mathrm{d}y}{(U_0(y) - ic_i)^2}. \tag{2.22}$$

Lemma 2.8 *Under the assumptions in Theorem* 1.3, $|W(c_i)| \to \infty$ *as* $c_i \to 0+$.

Proof Let $\varphi_\pm(y; c_i)$ be defined in (2.14) and $\varphi_\pm(y)$ be defined in (2.11). Take $\delta > 0$ sufficiently small such that $U_0(\pm\delta) \neq 0$. Since $\varphi_\pm(y; c_i) \to \varphi_\pm(y)$ as $c_i \to 0+$ for all $y > 0$ by (2.15), we get

$$\int_1^{\pm\delta} \frac{1}{(U_0 - ic_i)^2} = \frac{\varphi_\pm(\pm\delta; c_i)}{(U_0(\pm\delta) - ic_i)} \to \frac{\varphi_\pm(\pm\delta)}{U_0(\pm\delta)} \neq \infty \tag{2.23}$$

as $c_i \to 0+$. It remains to prove that as $c_i \to 0+$,

$$W(c_i,\delta) := \int_{|y|\le\delta} \frac{1}{(U_0 - ic_i)^2} \to \infty. \tag{2.24}$$

To this end we set

$$\begin{aligned} F(c_i) : &= \int_0^\delta \frac{U_0^2 - c_i^2}{(U_0^2 + c_i)^2} = \int_0^\delta \frac{1}{U_0^2 + c_i^2} - 2c_i^2 \int_0^\delta \frac{1}{(U_0^2 + c_i)^2} \\ &=: F_1(c_i) - c_i F_2(c_i). \end{aligned} \tag{2.25}$$

Clearly $W(c_i,\delta) = 2F(c_i)$ by the odd symmetry of U_0. Since $U_0(y) \sim ay^\rho$ as $y \to 0$, we can choose $\delta > 0$ sufficiently small such that $|Ay|^\rho \le |U_0(y)| \le |By|^\rho$ for $|y| \le \delta$ with $0 < A < B$.

If $1/2 < \rho < 1$, then choose $B - A$ sufficiently small such that

$$1/B - (2 - 1/\rho)/A > 0. \tag{2.26}$$

Then we have

$$F_1(c_i) \ge \int_0^\delta \frac{\mathrm{d}y}{(By)^{2\rho} + c_i^2}, \qquad c_i F_2(c_i) \le 2c_i^2 \int_0^\delta \frac{\mathrm{d}y}{((Ay)^{2\rho} + c_i^2)^2}. \tag{2.27}$$

For the sake of simplicity, we take $\delta = 1$ and $t = c_i^{1/\rho}$ in the following estimation:

$$\begin{aligned} F_1(c_i) &= \frac{t^{1-2\rho}}{B} \int_0^{B/t} \frac{\mathrm{d}s}{s^{2\rho} + 1} := \frac{t^{1-2\rho}}{B} I(B,t) := G(B,t), \\ c_i F_2(c_i) &= -c_i \frac{\partial}{\partial c_i} \left(\int_0^1 \frac{\mathrm{d}y}{(Ay)^{2\rho} + c_i^2} \right) = -c_i \frac{\partial}{\partial c_i} G(A,c_i) \\ &= -t^\rho \frac{\partial}{\partial t} G(A,t) \frac{\partial t}{\partial c_i} = -\frac{t}{\rho} \frac{\partial}{\partial t} G(A,t) \\ &= \frac{t^{1-2\rho}}{A} (2 - 1/\rho) I(A,t) + \frac{1}{\rho} \left(A^{2\rho} + t^{2\rho} \right)^{-1}. \end{aligned}$$

Therefore, $I(B,t) > I(A,t)$, $\rho > 1/2$, and (2.26) lead to

$$W(c_i,\delta) \ge t^{1-2\rho} \left(\frac{1}{B} - \frac{(2 - 1/\rho)}{A} \right) I(A,t) - \frac{1}{\rho} \left(A^{2\rho} + t^{2\rho} \right)^{-1} \to +\infty$$

as $c_i \to 0+$. For the case $1 < \rho < 3/2$, we choose A, B satisfying

$$1/A - (2 - 1/\rho)/B < 0 \tag{2.28}$$

and use similar inequalities in (2.27),

$$F_1(c_i) \le \int_0^\delta \frac{\mathrm{d}y}{(Ay)^{2\rho} + c_i^2}, \qquad c_i F_2(c_i) \ge 2c_i^2 \int_0^\delta \frac{\mathrm{d}y}{((By)^{2\rho} + c_i^2)^2}. \tag{2.29}$$

Then a similar argument to the above proves that $W(c_i) \to -\infty$ as $c_i \to 0+$. This lemma is completed. □

Now we can outline the proof of Theorem 1.3 given in the next section. Recall the definition of $\lambda_k(c_i)$, the eigenvalue of (2.4). Since there exists an unstable mode for a wave number α if and only if $\lambda_k(c_i) = 0$ for $k \geq 1$, we only need to prove that $H(c_i)$ has at least one zero eigenvalue for some $c_i > 0$ under the assumptions in Theorem 1.3.

Consider the eigenvalues $\lambda(c_i)$ of $H(c_i)$ as functions of the variable $c_i \in [0,\infty)$. Applying the Kato-Rellich theorem (see *e.g.* [3], Theorem XII.8, p.15, IV, or [16], pp.437-439) as well as the resolvent convergence of $H(c_i)$ in Lemma 3.1, we prove that $\lambda(c_i)$ is a continuous function in the sense of (3.1). The main step is to prove the continuity of $\lambda(c_i)$ at $c_i = 0$. If this is done and $\lambda_k(c_i) \neq 0$ for all $k \geq 1$ and $c_i > 0$, then we can introduce an auxiliary function $D(c_i)$ defined in (3.21) and prove that $D(c_i)$ is continuous on $(0,\infty]$, $D(c_i) > 0$ for sufficiently large $c_i > 0$, and $D(c_i) < 0$ for all sufficiently small $c_i > 0$ by using several technical lemmas in Section 3. Therefore, the desired contradiction would appear.

3 The proof of Theorem 1.3

This section gives the proof of Theorem 1.3 through proving several lemmas. Recall that $H(c_i)$ is the associated operator to (2.4) for $c_i \in (0,\infty)$ and the spectrum of $H(c_i)$ is discrete. Denote by $\lambda(c_i)$ the eigenvalue of $H(c_i)$. The main step in this section is proving the continuity of $\lambda(c_i)$ in the sense of that, if $\lambda(c_i^0)$ is an eigenvalue of $H(c_i^0)$ with algebraic multiplicity m and

$$O\bigl(\lambda\bigl(c_i^0\bigr),r\bigr) = \bigl\{\lambda : \bigl|\lambda - \lambda\bigl(c_i^0\bigr)\bigr| \leq r\bigr\} \cap \sigma\bigl(H\bigl(c_i^0\bigr)\bigr) = \bigl\{\lambda\bigl(c_i^0\bigr)\bigr\}, \tag{3.1}$$

then there exists $\delta > 0$ such that, for $|c_i - c_i^0| < \delta$, $O(\lambda(c_i),r)$ contains eigenvalues of $H(c_i)$ with total algebraic multiplicity exactly m. We remark that the above property is called 'stableness of eigenvalue' $\lambda(c_i^0)$ for the case $m = 1$ in [3], p.29.

The first lemma gives the continuity of $\lambda(c_i)$ on $(0,\infty)$.

Lemma 3.1 $\lambda(c_i)$ *is continuous on* $(0,\infty)$.

Proof Let $\varphi(y,c_i,\lambda)$ be the solution of

$$-\varphi'' + \bigl[K(y,c_i) + \alpha^2\bigr]\varphi = \lambda\varphi, \quad K(y,c_i) = \frac{U''(y)}{U_0(y) - \mathrm{i}\,c_i},$$

such that $\varphi(-1,c_i,\lambda) = 0$, $\varphi'(-1,c_i,\lambda) = 1$. Since $\varphi(1,c_i,\lambda)$ is analytic in (c_i,λ) for $c_i > 0$ by Lemma 2.1, and λ is an eigenvalue of (2.4) for a fixed $c_i > 0$ if and only if λ is a zero of $\varphi(1,c_i,\lambda)$, it follows from the continuity of the zeros of an analytic function ([17], p.248) that $\lambda_n(c_i)$ is continuous in the above sense. This lemma is completed. □

With a similar argument to the above one can prove the following.

Lemma 3.2 $\lambda(c_i)$ *is continuous at* $c_i = \infty$, *where* $H(\infty)$ *is the differential operator associated to*

$$-\varphi'' + \alpha^2\varphi = \lambda\varphi \quad \text{in } L^2(-1,1), \qquad \varphi(-1) = \varphi(1) = 0.$$

Proof Set $\eta = 1/c_i$ for $c_i > 0$. Clearly, the eigenvalue problem (2.4) is equivalent to the eigenvalue problem

$$-\varphi'' + \left(\frac{\eta U''(y)}{\eta U_0(y) - \mathrm{i}} + \alpha^2\right)\varphi = \lambda\varphi, \qquad \varphi(\pm 1) = 0, \tag{3.2}$$

in $L^2[-1,1]$. Let $\varphi(y,\eta,\lambda)$ be the solution of (3.2) which satisfies $\varphi(-1,\eta,\lambda) = 0$, $\varphi'(-1,\eta,\lambda) = 1$. Note that

$$\left|\frac{\eta U''(y)}{\eta U_0(y) - \mathrm{i}}\right| \leq \eta\left|U''\right| \in L^1[-1,1].$$

Then $\varphi(1,\eta,\lambda)$ is analytic in (η,λ) for $\eta \geq 0$, and hence a similar argument to the above proves the continuity of eigenvalues at $\eta = 0$, *i.e.*, the continuity of $\lambda(c_i)$ at $c_i = \infty$. □

The method in the proof of Lemmas 3.1 and 3.2 cannot be applied straightforwardly to the case $c_i^0 = 0$ since $K(y,0) = K(y)$ is not integrable in $L^1[-1,1]$. Then we first present necessary notations before proving the continuity of $\lambda(c_i)$ at $c_i = 0$. Let $H(0)$ be defined as in (2.8) with spectrum σ_0. Denote by $\lambda(0)$ the eigenvalue of $H(0)$. Then we can prove the following.

Lemma 3.3 *$\lambda(c_i)$ is continuous at $c_i = 0$.*

Proof For $c_i > 0$, it is easy to see from (2.14) that 0 is an eigenvalue of $H(c_i)$ if and only if

$$W(c_i) := \int_{-1}^{1} \frac{1}{(U_0 - \mathrm{i}\,c_i)^2} = 0.$$

Since $|W(c_i)| \to \infty$ as $c_i \to 0+$ by Lemma 2.8, one sees that 0 is not an eigenvalue of $H(c_i)$ for all sufficiently small $c_i > 0$, or the resolvent of $H(c_i)$ at 0, say $G(c_i)$, exists. Let $G(y,t;c_i)$ be the associated Green function. A calculation shows that

$$G(y,t;c_i) = -W^{-1}(c_i)\begin{cases} \varphi_-(y;c_i)\varphi_+(t;c_i), & -1 \leq y \leq t \leq 1, \\ \varphi_-(t;c_i)\varphi_+(y;c_i), & -1 \leq t \leq y \leq 1. \end{cases} \tag{3.3}$$

Then $H(c_i)\varphi = \lambda\varphi$ is equivalent to

$$\varphi = \lambda G(c_i)\varphi =: \lambda \int_{-1}^{1} G(y,t;c_i)\varphi(t)\,\mathrm{d}t. \tag{3.4}$$

Now we consider the limit of $G(y,t,c_i)$ in $L^2([-1,1] \times [-1,1])$ as $c_i \to 0+$.

For $y > 0$, we have from (2.15) that

$$\begin{aligned} W^{-1}(c_i)\varphi_-(y;c_i,0) &= W^{-1}(c_i)\bigl(U_0(y) - ic_i\bigr)\left(W(c_i) + \int_1^y \frac{1}{(U_0 - ic_i)^2}\right) \\ &= \bigl(U_0(y) - ic_i\bigr) + W^{-1}(c_i)\varphi_+(y,c_i,0) \to U_0(y), \end{aligned}$$

since $W(c_i) \to \infty$ as $c_i \to 0+$. Furthermore,

$$\left|W^{-1}(c_i)\varphi_-(y;c_i,0)\right| \leq 2\bigl(\left|U_0(y)\right| + 1\bigr) \in L^2[-1,1]$$

for all sufficient small c_i, we know from the Lebesgue dominated convergence theorem that $W^{-1}(c_i)\varphi_-(y;c_i,0) \to U_0(y)$ in $L^2[0,1]$ as $c_i \to 0+$. Similarly, one can prove that $W^{-1}(c_i)\varphi_+(y;c_i,0) \to U_0(y)$ in $L^2[-1,0)$ as $c_i \to 0+$.

Therefore, we conclude from (3.3) and (2.16) that

$$G(y,t;c_i,0) \to G_+(y,t) = -\begin{cases} U_0(t)\varphi_+(y;0,0), & 1 > y > t > 0, \\ U_0(y)\varphi_+(t;0,0), & 0 < y < t < 1, \end{cases} \tag{3.5}$$

in $L^2([0,1] \times [0,1])$ and

$$G(y,t;c_i,0) \to G_-(y,t) = -\begin{cases} U_0(y)\varphi_-(t;0,0), & 0 > y > t > -1, \\ U_0(t)\varphi_-(y;0,0), & -1 < y < t < 0, \end{cases} \tag{3.6}$$

in $L^2([-1,0] \times [-1,0])$.

For the case $yt < 0$, we have

$$G(y,t;c_i) = -W^{-1}(c_i)\begin{cases} \varphi_+(y;c_i)\varphi_-(t;c_i), & y > 0, t < 0, \\ \varphi_+(t;c_i)\varphi_-(y;c_i), & y < 0, t > 0. \end{cases} \tag{3.7}$$

Since $\varphi_+(y;c_i)$ and $\varphi_-(y;c_i)$ converge to $\varphi_+(y)$ in $L^2[0,1]$ and $\varphi_-(y)$ in $L^2[-1,0]$ by (2.16), respectively, as $c_i \to 0+$, we find that $G(y,t;c_i,0) \to 0$ in L^2-norm as $c_i \to 0+$ by Lemma 2.8.

Summing up the above discussion, we get, as $c_i \to 0+$,

$$G(y,t;c_i) \to G(y,t) := \begin{cases} 0, & yt < 0, \\ G_+(y,t), & y > 0, t > 0, \\ G_-(y,t), & y < 0, t < 0, \end{cases} \tag{3.8}$$

in $L^2([-1,1] \times [-1,1])$. Applying the odd symmetry of $U_0(y)$ in $[-1,1]$ and the definitions of $\varphi_\pm(y)$, one can verify that $\varphi_+(y) = \varphi_-(-y)$ for $y \neq 0$. Therefore,

$$G(-y,-t) = G(y,t), \quad yt > 0, y,t \in [-1,1], \tag{3.9}$$

by the definitions of G_+, G_- in (3.5), (3.6), respectively. As a result, the limitation eigenvalue problem of the operators $G(c_i)$ as $c_i \to 0+$ can be expressed as

$$\varphi = \lambda G\varphi \quad \text{in } L^2[-1,1], \qquad (G\varphi)(y) := \int_{-1}^{1} G(y,t)\varphi(t)\,\mathrm{d}t, \quad y \in [-1,1]. \tag{3.10}$$

Since $G(\cdot,\cdot) \in L^2([-1,] \times [-1,1])$, this means that G is a Hilbert-Smith operator, and hence every nonzero spectral point is an isolated eigenvalue.

Now consider the eigenvalue problem $H(0)\varphi = \lambda\varphi$, or

$$\varphi(y) = \lambda \int_0^1 G_+(y,t)\varphi(t)\,\mathrm{d}t = \lambda \int_0^1 G(y,t,0)\varphi(t)\,\mathrm{d}t \tag{3.11}$$

for $y > 0$ by (2.12) and (3.8). By (3.9), one can verify that $\varphi(y)$ solves (3.11) if and only if $\varphi(-y)$ solves

$$\varphi(-y) = \lambda \int_{-1}^{0} G_-(y,t)\varphi(-t)\,\mathrm{d}t = \lambda \int_{-1}^{0} G(y,t,0)\varphi(-t)\,\mathrm{d}t \tag{3.12}$$

for $y < 0$. This clearly shows that $\varphi(y)$ solves (3.11) if and only if $\Phi(y)$ satisfies

$$\Phi(y) = \lambda \int_{-1}^{1} G(y,t,0)\Phi(t)\,\mathrm{d}t, \qquad \Phi(y) = \begin{cases} \varphi(y), & y > 0, \\ \varphi(-y), & y < 0, \end{cases} \tag{3.13}$$

which means that λ is an eigenvalue of (3.13) if and only if λ is an eigenvalue of (3.11). Since every eigenvalue of $H(0)$ is isolated and algebraically simple, it follows from [3], Lemma 1, p.29, that $\lambda(c_i)$ is continuous at $c_i = 0$. The proof is finished. □

For the convenience of applications of Lemmas 3.1-3.3 in the proof of Theorem 1.3, we give a detailed description of the continuity of $\lambda(c_i)$ by summing up the conclusions in Lemmas 3.1-3.3.

Lemma 3.4 *The eigenvalue $\lambda(c_i)$ of $H(c_i)$ is continuous on $[0,\infty]$ in the sense that, for a fixed eigenvalue $\lambda(c_i^0)$ of $H(c_i^0)$ with the algebraically multiplicity m and a neighborhood $O(\lambda(c_i^0))$ of $\lambda(c_i^0)$ such that $O(\lambda(c_i^0)) \cap \sigma(H(c_i^0)) = \{\lambda(c_i^0)\}$, there exists a neighborhood $O(c_i^0)$ of c_i^0 such that, for $c_i \in O(c_i^0)$, $O(\lambda(c_i^0))$ contains m eigenvalues $\lambda(c_i)$ of $H(c_i)$.*

In fact, Lemma 3.3 gives the resolvent convergence of $H(c_i)$ as $c_i \to 0+$. Using this fact we can prove the following.

Lemma 3.5 *If $c_i^n \to 0$ and the limit $\hat{\lambda}$ of $\lambda(c_i^n)$ is finite as $n \to \infty$, then $\hat{\lambda}$ is an eigenvalue of $H(0)$.*

Proof Choose $z_0 \notin \sigma(H(0))$. By Lemma 3.4 $z_0 \notin \sigma(H(c_i^n))$ for n sufficiently large. Set

$$G_n = \left(H\left(c_i^n\right) - z_0\right)^{-1}, \qquad G_0 = \left(H(0) - z_0\right)^{-1}, \qquad \lambda_n = \lambda\left(c_i^n\right) - z_0, \qquad \lambda_0 = \hat{\lambda} - z_0.$$

Let $\varphi_n \in L^2(-1,1)$ be the corresponding normalized eigenvector corresponding to the eigenvalue λ_n of $H(c_i^n) - z_0$, or $\varphi_n = \lambda_n G_n \varphi_n$.

Since G_0 is compact and $\|\varphi_n\| = 1$, there exists a convergent subsequence of $\{G_0\varphi_n\}$, say $\{G_0\varphi_n\}$. Then we have

$$\begin{aligned} \|\varphi_n - \varphi_m\| &= \|\lambda_n G_n \varphi_n - \lambda_m G_m \varphi_m\| \\ &\le |\lambda_n| \left\|(G_n - G_m)\varphi_n\right\| + |\lambda_n - \lambda_m| \|G_m\varphi_n\| + |\lambda_m| \left\|G_m(\varphi_n - \varphi_m)\right\|. \end{aligned} \tag{3.14}$$

However,

$$\left\|G_m(\varphi_n - \varphi_m)\right\| \le \left\|(G_m - G_0)\varphi_n\right\| + \left\|(G_0 - G_m)\varphi_m\right\| + \left\|G_0(\varphi_n - \varphi_m)\right\|. \tag{3.15}$$

Since $\|G_n - G_0\| \to 0$, $\lambda_n - \lambda_m \to 0$, $G_0(\varphi_n - \varphi_m) \to 0$ as $n \to \infty$, we conclude from (3.14) and (3.15) that $\{\varphi_n\}$ is a Cauchy sequence. Set $\varphi_n \to \varphi_0$ as $n \to \infty$. Then $\varphi_0 \neq 0$ since $\|\varphi_n\| = 1$. Clearly

$$\begin{aligned} \|\varphi_0 - \lambda_0 G_0 \varphi_0\| &\le \|\varphi_0 - \varphi_n\| + |\lambda_n| \left\|(G_n - G_0)\varphi_n\right\| \\ &\quad + |\lambda_n - \lambda_0| \|G_0\varphi_n\| + |\lambda_0| \left\|G_0(\varphi_n - \varphi_0)\right\| \to 0 \end{aligned}$$

as $n \to \infty$ gives $\varphi_0 = \lambda_0 G_0 \varphi_0$. This together with $\varphi_0 \neq 0$ implies that $\hat{\lambda}$ is an eigenvalue of $H(c_i^0)$. The proof is completed. □

The following result gives a lower bound of the real eigenvalues of $H(c_i)$.

Lemma 3.6 *There exists $M > 0$ such that for any real eigenvalue $\lambda(c_i)$ of $H(c_i)$ for $c_i > 0$ we have $\lambda(c_i) \geq -M$.*

Proof Let $\varphi(y, c_i)$ be the eigenfunction corresponding to $\lambda(c_i)$ such that $\varphi(1, c_i) = 0$, $\varphi'(1, c_i) = 1$. Choose m sufficiently large such that $U_0' U_0^{m-1}$ is bounded in $[0,1]$, *i.e.*, there exists $M > 0$ such that

$$\left|U_0' U_0^{m-1}(y)\right| \leq M.$$

Since $\lambda(c_i)$ is real, multiplying both sides of (2.1) by $U_0^{2m}\overline{\varphi}$ and integrating by parts on $[0,1]$ we find that the real part is given by

$$\begin{aligned} &2m \int_0^1 U_0^{2m-1} U_0' \operatorname{Re}(\varphi' \varphi) + \int_0^1 \left(U_0^{2m} |\varphi'|^2 + \frac{U'' U_0^{2m+1}}{U_0^2 + c_i^2} |\varphi|^2 \right) \\ &\quad = \lambda(c_i) \int_0^1 U_0^{2m} |\varphi|^2. \end{aligned} \tag{3.16}$$

Here $U_0(0) = 0$ is used. By Lemma 2.7 $\varphi(y, c_i) \to g(y)$ in $L^2(0,1]$ as $c_i \to 0$, where $g(y)$ is the solution of

$$-\phi'' + \left(\frac{U''}{U_0} + \alpha^2 \right) \phi = \lambda \phi$$

such that $g(1) = 0$, $g'(1) = 1$.

Set

$$C_1 = \int_0^1 \left|g(y)\right|^2 \, dy, \qquad C_2 = \int_0^1 U_0^{2m}(y) \left|g(y)\right|^2 \, dy. \tag{3.17}$$

Then there exists $\delta > 0$ such that, for $c_i < \delta$,

$$\int_0^1 \left|\varphi(\cdot, c_i)\right|^2 \leq 2C_1, \qquad \int_0^1 U_0^{2m} \left|\varphi(\cdot, c_i)\right|^2 > C_2/2. \tag{3.18}$$

Since $U_0(0) = 0$ and $\varphi(1, c_i) = 0$, integration by parts shows that

$$\begin{aligned} \int_0^1 \frac{U'' U_0^{2m+1}}{U_0^2 + c_i^2} |\varphi|^2 &= \frac{U' U_0^{2m+1}}{U_0^2 + c_i^2} |\varphi|^2 \Big|_0^1 - \int_0^1 U' \left(\frac{U_0^{2m+1}}{U_0^2 + c_i^2} |\varphi|^2 \right)' \\ &= -2 \int_0^1 \frac{U' U_0^{2m+1}}{U_0^2 + c_i^2} \operatorname{Re}(\varphi' \varphi) - \int_0^1 U' \left(\frac{U_0^{2m+1}}{U_0^2 + c_i^2} \right)' |\varphi|^2. \end{aligned}$$

A calculation shows that (by the continuity of U_0)

$$\left| U' \left(\frac{U_0^{2m+1}}{U_0^2 + c_i^2} \right)' \right| \leq C_m$$

and hence

$$\int_0^1 \frac{U'' U_0^{2m+1}}{U_0^2 + c_i^2} |\varphi|^2 \geq -2 \int_0^1 \left| U_0^{2m-1} U_0' \varphi' \varphi \right| - C_m \int_0^1 |\varphi|^2. \tag{3.19}$$

For the first term in the above inequality we have

$$\begin{aligned} \int_0^1 \left| U_0^{2m-1} U_0' \varphi' \varphi \right| &\leq \left(\int_0^1 \left(U' U_0^{m-1} \right)^2 |\varphi|^2 \right)^{1/2} \left(\int_0^1 U_0^{2m} \left| \varphi' \right|^2 \right)^{1/2} \\ &\leq \frac{1}{2m+2} \int_0^1 U_0^{2m} \left| \varphi' \right|^2 + (m+1) \int_0^1 \left(U' U_0^{m-1} \right)^2 |\varphi|^2. \end{aligned} \tag{3.20}$$

Inserting the inequalities (3.18)-(3.20) into (3.16) we have $\lambda(c_i) \geq -2(m+1)^2 C_1 C_m / C_2$. The proof of Lemma 3.6 is finished. □

Proof of Theorem 1.3 Let $\lambda_n(c_i)$ be defined as above with the multiplicity $n_k(c_i)$. Recall that $H(0)$ has N negative eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_N < 0$, and $m_0 = [(N+1)/2]$, $\alpha_j = \sqrt{-\lambda_j}$, $1 \leq j \leq N$, $\alpha_{N+1} = 0$. We proceed to show that, for every $\alpha \in \bigcup_{k=1}^{m_0} (\alpha_{2k}, \alpha_{2k-1})$, there exist $c_i > 0$ and $k \geq 1$ such that $\lambda_k(c_i) = 0$. Assume to the contrary that, for an $\alpha \in (\alpha_{2j}, \alpha_{2j-1})$ with $1 \leq j \leq m_0$, $\lambda_k(c_i) \neq 0$ for all $c_i \in (0, \infty)$, and $k \geq 1$. Then, by Lemma 2.4, we can define

$$D(c_i) = \prod_{k=1}^{\infty} \left(1 - e^{-\lambda_k(c_i)}\right)^{n_k(c_i)}, \quad c_i \in (0, \infty), \tag{3.21}$$

to be a function because the infinite product converges to a finite nonzero number for every $c_i > 0$ and $c_i = \infty$. Fixing c^0 with $c_i^0 > 0$, for all $c_i \in (c^0, \infty)$, since the remainder in (2.6) satisfies $|r(n, c_i)| \leq C(c_i)n$ and $C(c_i)$ is dominant by $C_0 \int_{-1}^1 |K(y, c_i)|\,\mathrm{d}y$ and $|K(y, c_i)| \leq |K(y, c_i^0)|$, the infinite product in (3.21) converges uniformly on (c_i^0, ∞). Moreover, since $\lambda_k(c_i)$ appears in complex conjugate pairs, $D(c_i)$ is real-valued.

Note that $1 - e^{-\lambda_k(\infty)} > 0$ for $k \geq 1$ since $\lambda_k(\infty) = (\frac{k\pi}{2})^2 + \alpha^2 > 0$ by the definition of $H(\infty)$ in Lemma 3.2. Then $D(\infty) > 0$. If $D(c_i) < 0$ for some $c_i > 0$ and $D(c_i)$ is continuous on $(0, \infty)$, then we will have the desired contradiction.

First of all we prove that $D(c_i) < 0$ for some $c_i > 0$. Precisely, we will prove that there exists $\delta > 0$ such that

$$D(c_i) < 0, \quad c_i \in (0, \delta).$$

Since for $c_i = 0$, $\lambda_k(0) = \lambda_k + \alpha^2$ and $n_k(c_i) = 1$ for all k. Then for $\alpha \in (\alpha_{2j}, \alpha_{2j-1})$, it is easy to see that the first $2j-1$ eigenvalues $\lambda_1(0), \ldots, \lambda_{2j-1}(0)$ are negative and the rest are all positive. Choose $\varepsilon > 0$ small enough such that

$$O_k = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \lambda_k(0) \right| < \varepsilon \right\}, \quad 1 \leq k \leq 2j-1, O_k \cap O_m = \emptyset, k \neq m.$$

Then by Lemma 3.4, there exists $\delta > 0$ such that O_k contains exactly one simple eigenvalue of $H(c_i)$ for all $c_i < \delta$, say $\lambda_1(c_i), \ldots, \lambda_{2j-1}(c_i)$. Since $\overline{\lambda_k(c_i)}$ is also an eigenvalue of $H(c_i)$, then by the simpleness $\lambda_1(c_i), \ldots, \lambda_{2j-1}(c_i)$ are all real.

If $\lambda_1(c_i),\ldots,\lambda_{2j-1}(c_i)$ are all the negative eigenvalues of $H(c_i)$, then we must have $D(c_i) < 0$ since the number is odd and non-real eigenvalues appear in complex conjugate pairs. Suppose that for any integer $N > 0$ there exists a c_i^N such that

$$c_i^N < 1/N, \qquad \lambda\bigl(c_i^N\bigr) < 0, \qquad \lambda\bigl(c_i^N\bigr) \neq \lambda_k\bigl(c_i^N\bigr)$$

for $1 \leq k \leq 2j-1$. This also means that

$$\lambda\bigl(c_i^N\bigr) \notin O_k, \quad 1 \leq k \leq 2j-1,$$

by the definition of $\lambda_k(c_i^N)$. By Lemma 3.6, $\lambda(c_i^N)$ is bounded for all $N \geq 1$. Then there exists a convergent subsequence, say $\{\lambda(c_i^N)\}$. Let λ_0 be the limit of $\{\lambda(c_i^N)\}$ as $N \to \infty$, then Lemma 3.5 implies that λ_0 is an eigenvalue of $H(0)$. Since $\lambda(c_i^N) < 0$ and $\lambda(c_i^N) \notin O_k$, we see that $\lambda_0 \leq 0$ and $\lambda_0 \neq \lambda_k(0)$ for $1 \leq k \leq 2j-1$. But $H(0)$ has no other negative eigenvalue except for $\lambda_1(c_i),\ldots,\lambda_{2j-1}(c_i)$, we must have $\lambda_0 = 0$. This is also a contradiction since $H(0)$ has no zero eigenvalue.

Now we prove the continuity of $D(c_i)$ on $(0,\infty)$ provided that $D(c_i) \neq 0$. Let $c_i^0 \in (0,\infty)$ and $\varepsilon > 0$ be given. Suppose that $\mu_1,\ldots,\mu_m$ are the first m distinct eigenvalues of $H(c_i^0)$ and that each μ_j has a multiplicity n_j and $n_1 + \cdots + n_m = M_m$. Then $\{M_m\}$ is increasing and the continuity of $D(c_i)$ is equivalent to the continuity of the partial product $D_m(c_i) := \prod_{k=1}^{M_m}(1 - e^{-\lambda_k(c_i)})^{n_k(c_i)}$ for every m because of the uniform convergence of the infinite product.

Choose an $\eta > 0$ so small that, for $1 \leq j \leq m$, μ_j is the only eigenvalue of $H(c_i^0)$ in the disc $O_j = \{\lambda : |\lambda - \mu_j| < \eta\}$ and $\partial O_j \cap \sigma(H(c_i^0)) = \emptyset$. Furthermore, η can be chosen sufficiently small such that, for $\lambda \in O_j$, $1 \leq j \leq m$,

$$\bigl|\ln\bigl|1 - e^{-\lambda}\bigr| - \ln\bigl|1 - e^{-\mu_j}\bigr|\bigr| < \varepsilon/M_m. \tag{3.22}$$

Then by Lemma 3.4, there exists $\delta > 0$ such that O_j contains exactly n_j eigenvalues of $H(c_i)$ for c_i: $|c_i - c_i^0| < \delta$. Therefore, we have from (3.22), for $|c_i - c_i^0| < \delta$,

$$\bigl|\ln\bigl|D_m(c_i)\bigr| - \ln\bigl|D_m\bigl(c_i^0\bigr)\bigr|\bigr| < \varepsilon.$$

This gives the continuity of $D_m(c_i)$ and completes the proof of Theorem 1.3. □

Example 3.7 Take $U(y) = y^\rho(2 - y^n)$ with $\rho = 3/5$ and n an even number. Then $H(0)$ is given by

$$\begin{aligned} &-\varphi'' - \biggl(\frac{n(n+2\rho-1)}{2-y^n}y^{n-2} + \rho(1-\rho)y^{-2}\biggr)\varphi = \lambda\varphi, \quad y \in (0,1], \\ &\bigl[y^\rho\varphi'(y) - \rho y^{\rho-1}\varphi(y)\bigr]\big|_{y=0} = 0, \qquad \varphi(1) = 0. \end{aligned} \tag{3.23}$$

Since $H(0)$ has at least m negative eigenvalues for $n > 2(m+2)^2\pi^2$, we know from Theorem 1.3 that the Rayleigh problem has at least k unstable modes if $\alpha < \sqrt{|\lambda_k|}$, $1 \leq k \leq m$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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