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Local existence and blow-up criterion of the ideal density-dependent flows

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Abstract
In this paper, we consider two ideal density-dependent flows in a bounded domain, the Euler and magnetohydrodynamics equations. We prove the local existence and a blow-up criterion for each system.

MSC: 35Q35; 76D03

Keywords: Euler; ideal MHD; local existence; blow-up criterion

1 Introduction

First, we consider the following 3D density-dependent Euler system:

$$\partial_t\rho + \mathbf{u}\cdot\nabla\rho = 0, \tag{1.1}$$

$$\rho\partial_t\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\pi = 0, \tag{1.2}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{1.3}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.4}$$

$$(\rho,\mathbf{u})(\cdot,0) = (\rho_0,\mathbf{u}_0) \quad \text{in } \Omega\subset\mathbb{R}^3. \tag{1.5}$$

Here Ω is a bounded domain with smooth boundary $\partial\Omega\in C^\infty$, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$; the unknowns are the fluid velocity field $\mathbf{u} = \mathbf{u}(x,t)$, the pressure $\pi = \pi(x,t)$, and the density $\rho = \rho(x,t)$.

Beirão da Veiga and Valli [1, 2] and Valli and Zajaczkowski [3] proved the unique solvability, local in time, in some supercritical Sobolev spaces and Hölder spaces in bounded domains. It is worth pointing out that in 1995 Berselli [4] discussed the standard ideal flow.

When $\Omega := \mathbb{R}^3$, Danchin [5] and Danchin and Fanelli [6] (see also [7, 8]) proved the unique solvability, local in time, in some critical Besov spaces.

The first aim of this paper is to prove the local existence and a blow-up criterion of problem (1.1)-(1.5) in the L^p frame work. We will prove the following:

Theorem 1.1 *Let $0 < \inf\rho_0 \le \sup\rho_0 < \infty$, $\rho_0,\mathbf{u}_0 \in W^{s,p}(\Omega)$ with integer $s\ge 3$, $s > 1 + \frac{3}{p}$, and $2 < p < \infty$, and $\operatorname{div}\mathbf{u}_0 = 0$ and $\mathbf{u}_0\cdot\mathbf{n} = 0$ on $\partial\Omega$. Then there exists a positive time $T^* > 0$ such that problem* (1.1)-(1.5) *has a unique solution $(\rho,\mathbf{u})$ satisfying*

$$0 < \inf\rho_0 \le \rho \le \sup\rho_0 < \infty, \quad \rho,\mathbf{u}\in L^\infty\bigl(0,T^*;W^{s,p}\bigr). \tag{1.6}$$

Furthermore, if u satisfies

$$\nabla \mathbf{u} \in L^\infty\big(0,T;L^\infty\big) \tag{1.7}$$

with $0 < T < \infty$, *then the solution* $(\rho, \mathbf{u}, \pi)$ *can be extended beyond* $T > 0$.

Remark 1.1 When $1 < p \leq 2$, we can prove a similar result.

We also consider the following ideal density-dependent MHD system:

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \tag{1.8}$$

$$\rho \partial_t \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla\left(\pi + \frac{1}{2}|\mathbf{b}|^2\right) = (\mathbf{b} \cdot \nabla)\mathbf{b}, \tag{1.9}$$

$$\partial_t \mathbf{b} + (\mathbf{u} \cdot \nabla)\mathbf{b} = (\mathbf{b} \cdot \nabla)\mathbf{u}, \tag{1.10}$$

$$\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{b} = 0, \tag{1.11}$$

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.12}$$

$$(\rho, \mathbf{u}, \mathbf{b})(\cdot, 0) = (\rho_0, \mathbf{u}_0, \mathbf{b}_0) \quad \text{in } \Omega \subset \mathbb{R}^3. \tag{1.13}$$

Here Ω is a bounded domain with smooth boundary $\partial\Omega \in C^\infty$, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, and the unknowns are the plasma velocity $\mathbf{u} = \mathbf{u}(x,t)$, the magnetic field $\mathbf{b} = \mathbf{b}(x,t)$, the pressure $\pi = \pi(x,t)$, and the density $\rho = \rho(x,t)$. When $\mathbf{b} = 0$, system (1.8)-(1.13) reduces to the density-dependent Euler equations (1.1)-(1.5). When $\Omega := \mathbb{R}^3$, Zhou and Fan [9] proved the local well-posedness of problem (1.8)-(1.13). For other related works, we refer to [10–14] and references therein.

In 1993, Secchi [15] was the first one to consider problem (1.8)-(1.13) and proved the local unique solvability with the main condition that

$$\|\nabla \rho_0\|_{H^{s-1}} \text{ is small enough with integer } s \geq 3. \tag{1.14}$$

The second aim of this paper is to prove the local well-posedness of problem (1.8)-(1.13) without any smallness condition; furthermore, we will also prove a regularity criterion. We will prove the following:

Theorem 1.2 *Let* $0 < \inf \rho_0 \leq \sup \rho_0 < \infty$, $\rho_0, \mathbf{u}_0, \mathbf{b}_0 \in H^s$ *with integer* $s \geq 3$, $\operatorname{div} \mathbf{u}_0 = \operatorname{div} \mathbf{b}_0 = 0$ *in* Ω, *and* $\mathbf{u}_0 \cdot \mathbf{n} = \mathbf{b}_0 \cdot \mathbf{n} = 0$ *on* $\partial\Omega$.

Then there exists a positive time $T^* > 0$ *such that problem* (1.8)-(1.13) *has a unique solution* $(\rho, \mathbf{u}, \mathbf{b})$ *satisfying*

$$0 < \inf \rho_0 \leq \rho \leq \sup \rho_0 < \infty, \quad \rho, \mathbf{u}, \mathbf{b} \in L^\infty\big(0,T^*;H^s\big). \tag{1.15}$$

Furthermore, if u and $\mathbf{b}$ *satisfy*

$$\nabla \mathbf{u}, \nabla \mathbf{b} \in L^\infty\big(0,T;L^\infty\big) \tag{1.16}$$

with $0 < T < \infty$, *then the solution* $(\rho, \mathbf{u}, \mathbf{b}, \pi)$ *can be extended beyond* $T > 0$.

Remark 1.2 We are unable to prove Theorem 1.1 for the ideal density-dependent MHD system.

We will use the following well-known Osgood lemma in [16].

Lemma 1.3 (Osgood lemma) *Let y be a measurable positive function, f a positive, locally integrable function, and g a continuous increasing function. Assume that, for a positive real number a, the function y satisfies*

$$y(t) \leq a + \int_{t_0}^{t} f(s) g\big(y(s)\big)\, ds.$$

If a is different from zero, then we have

$$-G\big(y(t)\big) + G(a) \leq \int_{t_0}^{t} f(s)\, ds, \quad \text{where } G(s) := \int_s^1 \frac{dr}{g(r)}.$$

If a is zero and $g(s)$ satisfies $\int_0^1 \frac{dr}{g(r)} = +\infty$, then the function y is identically zero.

We will also use the following bilinear commutator and the product estimate:

(i) If $f \in W^{s,p}(\Omega) \cap C^1(\Omega)$ and $g \in W^{s-1,p}(\Omega) \cap C(\Omega)$, then, for $|\alpha| \leq s$,

$$\big\| D^\alpha (fg) - f D^\alpha g \big\|_{L^p(\Omega)} \leq C\big(\|f\|_{W^{s,p_1}(\Omega)} \|g\|_{L^{q_1}(\Omega)} + \|\nabla f\|_{L^{p_2}(\Omega)} \|g\|_{W^{s-1,q_2}(\Omega)} \big). \quad (1.17)$$

(ii) If $f, g \in W^{s,p}(\Omega) \cap C(\Omega)$, then, for $|\alpha| \leq s$,

$$\big\| D^\alpha (fg) \big\|_{L^p(\Omega)} \leq C\big(\|f\|_{W^{s,p_1}(\Omega)} \|g\|_{L^{q_1}(\Omega)} + \|f\|_{L^{p_2}(\Omega)} \|g\|_{W^{s,q_2}(\Omega)} \big) \quad (1.18)$$

with integer $s > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, and $1 < p < \infty$.

The case with $p = 2$, $p_1 = q_2 = p$, $q_1 = p_2 = \infty$ has been proved in [17]. Since the proof of (1.18) is similar to that of (1.17), we will prove (1.17) only in the Appendix.

2 Local existence of the Euler system

This section is devoted to the proof of local existence for the Euler system. We only need to prove a priori estimates (1.6).

First, by the maximum principle, we have the well-known estimates

$$0 < \inf \rho_0 \leq \rho \leq \sup \rho_0 < \infty. \quad (2.1)$$

Testing (1.2) by u and using (1.1), (1.3), and (1.4), we see that

$$\int_\Omega \rho |\mathbf{u}|^2\, dx = \int_\Omega \rho_0 |\mathbf{u}_0|^2\, dx. \quad (2.2)$$

Applying D^s to (1.1), testing by $|D^s \rho|^{p-2} D^s \rho$, and using (1.3), (1.4), and (1.17), we derive

$$\frac{1}{p} \frac{d}{dt} \int_\Omega \big| D^s \rho \big|^p\, dx$$
$$= - \int_\Omega \big(D^s (\mathbf{u} \cdot \nabla \rho) - \mathbf{u} \cdot \nabla D^s \rho \big) \big| D^s \rho \big|^{p-2} D^s \rho\, dx$$

$$
\begin{aligned}
&\leq \left\|D^s(\mathbf{u}\cdot\nabla\rho)-\mathbf{u}\cdot\nabla D^s\rho\right\|_{L^p}\left\|D^s\rho\right\|_{L^p}^{p-1}\\
&\leq C\big(\|\nabla\mathbf{u}\|_{L^\infty}\|\rho\|_{W^{s,p}}+\|\nabla\rho\|_{L^\infty}\|\mathbf{u}\|_{W^{s,p}}\big)\|\rho\|_{W^{s,p}}^{p-1}\\
&\leq C\|\mathbf{u}\|_{W^{s,p}}\|\rho\|_{W^{s,p}}^{p},\\
&\leq C\|\mathbf{u}\|_{W^{s,p}}^{p+1}+C\|\rho\|_{W^{s,p}}^{p+1}.
\end{aligned}
\tag{2.3}
$$

Using (1.1), we rewrite (1.2) as follows:

$$
\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{\rho}\nabla\pi=0.
\tag{2.4}
$$

Applying D^s to (2.4), testing by $|D^su|^{p-2}D^su$, and using (1.3), (1.4), (1.17), (1.18), and (2.1), we deduce that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega\left|D^s\mathbf{u}\right|^p dx\\
&\quad\leq -\int_\Omega\big(D^s(\mathbf{u}\cdot\nabla\mathbf{u})-\mathbf{u}\cdot\nabla D^s\mathbf{u}\big)\left|D^s\mathbf{u}\right|^{p-2}D^s\mathbf{u}\,dx-\int_\Omega D^s\left(\frac{1}{\rho}\nabla\pi\right)\left|D^s\mathbf{u}\right|^{p-2}D^s\mathbf{u}\,dx\\
&\quad\leq \left\|D^s(\mathbf{u}\cdot\nabla\mathbf{u})-\mathbf{u}\cdot\nabla D^s\mathbf{u}\right\|_{L^p}\left\|D^s\mathbf{u}\right\|_{L^p}^{p-1}+\left\|D^s\left(\frac{1}{\rho}\nabla\pi\right)\right\|_{L^p}\left\|D^s\mathbf{u}\right\|_{L^p}^{p-1}\\
&\quad\leq C\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{W^{s,p}}^p+C\big(\|\nabla\pi\|_{W^{s,p}}+\|\nabla\pi\|_{L^\infty}\|\rho\|_{W^{s,p}}\big)\|\mathbf{u}\|_{W^{s,p}}^{p-1}.
\end{aligned}
\tag{2.5}
$$

Testing (2.4) by $\nabla\pi$ and using (1.3), (1.4), (2.1), and (2.2), we infer that

$$
\|\nabla\pi\|_{L^2}\leq C\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}\leq C\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^\infty}\leq C\|\nabla\mathbf{u}\|_{L^\infty}.
\tag{2.6}
$$

Taking div to (2.4), we observe that

$$
-\Delta\pi=f:=\rho\sum_i\nabla\mathbf{u}_i\partial_i\mathbf{u}-\frac{1}{\rho}\nabla\rho\cdot\nabla\pi.
\tag{2.7}
$$

Using (1.1), (1.2), and (1.4), we deduce that

$$
\frac{\partial\pi}{\partial\mathbf{n}}=g:=\rho\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}\quad\text{on }\partial\Omega.
\tag{2.8}
$$

Using (1.18) and the well-known $W^{s,p}$-estimates of problem (2.7)-(2.8) [18], we have

$$
\begin{aligned}
&\|\nabla\pi\|_{W^{s,p}(\Omega)}\\
&\quad\leq C\|f\|_{W^{s-1,p}(\Omega)}+C\|g\|_{W^{s-\frac{1}{p},p}(\partial\Omega)}\\
&\quad\leq C\left\|\rho\sum_i\nabla\mathbf{u}_i\partial_i\mathbf{u}\right\|_{W^{s-1,p}(\Omega)}+C\left\|\nabla\frac{1}{\rho}\nabla\pi\right\|_{W^{s-1,p}(\Omega)}+C\|\rho\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}\|_{W^{s-\frac{1}{p},p}(\partial\Omega)}\\
&\quad\leq C\big[\|\rho\|_{W^{s-1,p}}\|\nabla\mathbf{u}\|_{L^\infty}^2+\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{W^{s,p}}\big]\\
&\qquad+C\big[\|\nabla\rho\|_{L^\infty}\|\nabla\pi\|_{W^{s-1,p}}+\|\nabla\pi\|_{L^\infty}\|\rho\|_{W^{s,p}}\big]+C\|\rho\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}\|_{W^{s,p}(\Omega)}\\
&\quad\leq C\|\rho\|_{W^{s,p}}\|\mathbf{u}\|_{W^{s,p}}^2+C\|\mathbf{u}\|_{W^{s,p}}^2+C\|\rho\|_{W^{s,p}}\|\nabla\pi\|_{W^{s-1,p}}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\|\rho\mathbf{u}\cdot\mathbf{u}\|_{W^{s,p}} + C\|\rho\mathbf{u}^2\|_{L^\infty} \\
&\leq C\|\rho\|_{W^{s,p}}\|\mathbf{u}\|^2_{W^{s,p}} + C\|\mathbf{u}\|^2_{W^{s,p}} + C\|\rho\|_{W^{s,p}}\|\nabla\pi\|_{W^{s-1,p}},
\end{aligned} \tag{2.9}
$$

where we used the estimate [18]

$$
\left\|\nabla\frac{1}{\rho}\right\|_{W^{s-1,p}} \leq C\|\rho\|_{W^{s,p}}.
$$

By the Gagliardo-Nirenberg inequality

$$
\|\nabla\pi\|_{W^{s-1,p}} \leq C\|\nabla\pi\|^{1-\alpha}_{L^2}\|\nabla\pi\|^{\alpha}_{W^{s,p}}, \quad 1-\alpha = \frac{1}{s+\frac{3}{2}-\frac{3}{p}}, \tag{2.10}
$$

it follows from (2.6), (2.9), and (2.10) that

$$
\|\nabla\pi\|_{W^{s,p}} \leq C\|\rho\|_{W^{s,p}}\|\mathbf{u}\|^2_{W^{s,p}} + C\|\mathbf{u}\|^2_{W^{s,p}} + C\|\rho\|^{s+\frac{3}{2}-\frac{3}{p}}_{W^{s,p}}\|\nabla\mathbf{u}\|_{L^\infty}. \tag{2.11}
$$

Combining (2.3), (2.5), and (2.11) and using Osgood's lemma (for some T) and the inequalities

$$
\begin{aligned}
&\|\nabla\pi\|_{L^\infty} \leq C\|\nabla\pi\|_{W^{s,p}}, \qquad \|\nabla\mathbf{u}\|_{L^\infty} \leq C\|\mathbf{u}\|_{W^{s,p}}, \\
&\|\rho\|_{W^{s,p}} \leq C\big(\|\rho\|_{L^p} + \|D^s\rho\|_{L^p}\big) \\
&\qquad\quad \leq C + C\|D^s\rho\|_{L^p}, \\
&\|\mathbf{u}\|_{W^{s,p}} \leq C\big(\|\mathbf{u}\|_{L^p} + \|D^s\mathbf{u}\|_{L^p}\big) \\
&\qquad\quad \leq C + C\|D^s\mathbf{u}\|_{L^p},
\end{aligned}
$$

we arrive at

$$
\|\rho\|_{L^\infty(0,T;W^{s,p})} + \|\mathbf{u}\|_{L^\infty(0,T;W^{s,p})} \leq C. \tag{2.12}
$$

This completes the proof.

3 A blow-up criterion for the Euler system

This section is devoted to the proof of regularity criterion for the Euler system. We only need to establish a priori estimates.

First, we still have (2.1) and (2.2).

Taking ∇ to (1.1), testing by $|\nabla\rho|^{p-2}\nabla\rho$. and using (1.3) and (1.4), we derive

$$
\frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla\rho|^p\,dx \leq \|\nabla\mathbf{u}\|_{L^\infty}\int_\Omega |\nabla\rho|^p\,dx,
$$

whence

$$
\frac{d}{dt}\|\nabla\rho\|_{L^p} \leq \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^p}.
$$

Integrating this inequality and taking the limit as $p \to +\infty$, we have

$$\|\nabla\rho\|_{L^\infty(0,T;L^\infty)} \le C. \tag{3.1}$$

It follows from (2.6) that

$$\|\nabla\pi\|_{L^\infty(0,T;L^2)} \le C. \tag{3.2}$$

It follows from (2.7), (2.8), (1.7), (3.1), (3.2), and the $W^{2,p}$-estimates of problem (2.7)-(2.8) that

$$\begin{aligned}
\|\nabla\pi\|_{W^{1,p}} &\le C\|f\|_{L^p} + C\|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \\
&\le C\left\|\rho\sum_i \nabla\mathbf{u}_i\partial_i\mathbf{u}\right\|_{L^p} + C\left\|\nabla\frac{1}{\rho}\nabla\pi\right\|_{L^p} + C\|\rho\mathbf{u}\cdot\nabla n\cdot\mathbf{u}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \\
&\le C + C\|\nabla\pi\|_{L^p} + C\|\rho\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}\|_{W^{1,p}} \\
&\le C + C\|\nabla\pi\|_{L^2}^{1-\tilde{\alpha}}\|\nabla\pi\|_{W^{1,p}}^{\tilde{\alpha}} \\
&\le \frac{1}{2}\|\nabla\pi\|_{W^{1,p}} + C
\end{aligned}$$

for any $3 < p < \infty$, and thus

$$\|\nabla\pi\|_{L^\infty(0,T;L^\infty)} \le C. \tag{3.3}$$

Similarly to (2.9), we have

$$\begin{aligned}
\|\nabla\pi\|_{W^{s,p}} &\le C\|\rho\|_{W^{s,p}} + C\|\mathbf{u}\|_{W^{s,p}} + C\|\nabla\pi\|_{W^{s-1,p}} \\
&\le \frac{1}{2}\|\nabla\pi\|_{W^{s,p}} + C + C\|\rho\|_{W^{s,p}} + C\|\mathbf{u}\|_{W^{s,p}},
\end{aligned}$$

and thus

$$\|\nabla\pi\|_{W^{s,p}} \le C + C\|\rho\|_{W^{s,p}} + C\|\mathbf{u}\|_{W^{s,p}}. \tag{3.4}$$

Combining (2.3), (2.5), (3.4), (1.7), (3.3), and (3.1) and using the Gronwall inequality, we arrive at (2.12).

This completes the proof.

4 Local existence for the MHD system

This section is devoted to the proof of local existence for the MHD system. We only need to prove a priori estimates (1.15). Before going to detailed estimates, we write the case with $p = 2$, $p_1 = q_2 = p$, $q_1 = p_2 = \infty$ in (1.17) and (1.18) as follows:

(i) If $f, g \in H^s(\Omega) \cap C(\Omega)$, then

$$\|fg\|_{H^s(\Omega)} \le C\big(\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{H^s(\Omega)}\big). \tag{4.1}$$

(ii) If $f \in H^s(\Omega) \cap C^1(\Omega)$ and $g \in H^{s-1}(\Omega) \cap C(\Omega)$, then, for $|\alpha| \leq s$,

$$\left\| D^\alpha(fg) - fD^\alpha g \right\|_{L^2(\Omega)} \leq C\left(\|f\|_{H^s(\Omega)} \|g\|_{L^\infty(\Omega)} + \|f\|_{W^{1,\infty}(\Omega)} \|g\|_{H^{s-1}(\Omega)} \right). \tag{4.2}$$

First, by the maximum principle we have the well-known estimates

$$0 < \inf \rho_0 \leq \rho \leq \sup \rho_0 < \infty. \tag{4.3}$$

Testing (1.2) by $\mathbf{u}$ and using (1.8) and (1.11), we see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|\mathbf{u}|^2\,dx = \int_\Omega (\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\mathbf{u}\,dx. \tag{4.4}$$

Testing (1.10) by $\mathbf{b}$ and using (1.4), we find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{b}|^2\,dx = \int_\Omega (\mathbf{b}\cdot\nabla)\mathbf{u}\cdot\mathbf{b}\,dx. \tag{4.5}$$

Summing up (4.4) and (4.5) and noting the cancellation of the terms on the right-hand sides of (4.4) and (4.5), we get

$$\int_\Omega \left(\rho|\mathbf{u}|^2 + |\mathbf{b}|^2\right) dx = \int_\Omega \left(\rho|\mathbf{u}_0|^2 + |\mathbf{b}_0|^2\right) dx. \tag{4.6}$$

Applying D^s to (1.8), testing by $D^s\rho$, and using (1.11) and (4.2), we derive

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \left|D^s\rho\right|^2 dx &= -\int_\Omega \left(D^s(\mathbf{u}\cdot\nabla\rho) - \mathbf{u}\cdot\nabla D^s\rho\right)D^s\rho\,dx \\
&\leq \left\|D^s(\mathbf{u}\cdot\nabla\rho) - \mathbf{u}\cdot\nabla D^s\rho\right\|_{L^2}\left\|D^s\rho\right\|_{L^2} \\
&\leq C\left(\|\nabla\rho\|_{L^\infty}\|\mathbf{u}\|_{H^s} + \|\mathbf{u}\|_{W^{1,\infty}}\|\nabla\rho\|_{H^{s-1}}\right)\left\|D^s\rho\right\|_{L^2} \\
&\leq C\|\rho\|_{H^s}^3 + C\|\mathbf{u}\|_{H^s}^3.
\end{aligned} \tag{4.7}$$

Applying D^s to (1.9), testing by $D^s\mathbf{u}$, and using (1.11), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho\left|D^s\mathbf{u}\right|^2 dx \\
&\quad= \int_\Omega \left(D^s(\mathbf{b}\cdot\nabla\mathbf{b}) - b\cdot\nabla D^s\mathbf{b}\right)D^s\mathbf{u}\,dx + \int_\Omega \mathbf{b}\cdot\nabla D^s\mathbf{b}\cdot D^s\mathbf{u}\,dx \\
&\qquad- \int_\Omega \left(D^s(\rho\partial_t\mathbf{u}) - \rho D^s\partial_t\mathbf{u}\right)D^s\mathbf{u}\,dx - \int_\Omega \left(D^s(\rho\mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}\cdot\nabla D^s\mathbf{u}\right)D^s\mathbf{u}\,dx \\
&\qquad- \int_\Omega D^s\nabla\left(\pi + \frac{1}{2}\mathbf{b}^2\right)\cdot D^s\mathbf{u}\,dx =: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{4.8}$$

Applying D^s to (1.10), testing by $D^s\mathbf{b}$, and using (1.11), we deduce

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \left|D^s\mathbf{b}\right|^2 dx &= \int_\Omega \left(D^s(\mathbf{b}\cdot\nabla\mathbf{u}) - \mathbf{b}\cdot\nabla D^s\mathbf{u}\right)D^s\mathbf{b}\,dx + \int_\Omega \mathbf{b}\cdot\nabla D^s\mathbf{u}\cdot D^s\mathbf{b}\,dx \\
&\quad- \int_\Omega \left(D^s(\mathbf{u}\cdot\nabla\mathbf{b}) - \mathbf{u}\cdot\nabla D^s\mathbf{b}\right)D^s\mathbf{b}\,dx =: I_6 + I_7 + I_8.
\end{aligned} \tag{4.9}$$

Summing up (4.8) and (4.9) and noting that $I_2 + I_7 = 0$, we find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\rho\left|D^s\mathbf{u}\right|^2 + \left|D^s\mathbf{b}\right|^2\right)dx = I_1 + I_3 + I_4 + I_5 + I_6 + I_8. \tag{4.10}$$

Using (4.2) and (4.1), we bound I_1, I_3, I_4, I_5, I_6, and I_8 as follows:

$$\begin{aligned}
I_1 &\le C\|\mathbf{b}\|_{W^{1,\infty}}\|\mathbf{b}\|_{H^s}\|\mathbf{u}\|_{H^s} \le C\|\mathbf{b}\|_{H^s}^2\|\mathbf{u}\|_{H^s},\\
I_3 &\le C\big(\|\rho\|_{H^s}\|\partial_t\mathbf{u}\|_{L^\infty} + \|\rho\|_{W^{1,\infty}}\|\partial_t\mathbf{u}\|_{H^{s-1}}\big)\left\|D^s\mathbf{u}\right\|_{L^2}\\
&\le C\|\rho\|_{H^s}\|\partial_t\mathbf{u}\|_{H^{s-1}}\|\mathbf{u}\|_{H^s},\\
I_4 &\le C\big(\|\rho\mathbf{u}\|_{H^s}\|\nabla\mathbf{u}\|_{L^\infty} + \|\rho\mathbf{u}\|_{W^{1,\infty}}\|\nabla\mathbf{u}\|_{H^{s-1}}\big)\left\|D^s\mathbf{u}\right\|_{L^2}\\
&\le C\big[\big(\|\rho\|_{L^\infty}\|\mathbf{u}\|_{H^s} + \|\mathbf{u}\|_{L^\infty}\|\rho\|_{H^s}\big)\|\nabla\mathbf{u}\|_{L^\infty} + \|\rho\|_{W^{1,\infty}}\|\mathbf{u}\|_{W^{1,\infty}}\|\nabla\mathbf{u}\|_{H^{s-1}}\big]\left\|D^s\mathbf{u}\right\|_{L^2}\\
&\le C\|\rho\|_{H^s}\|\mathbf{u}\|_{H^s}^3,\\
I_5 &\le \left\|D^s\nabla\left(\pi + \frac{1}{2}|\mathbf{b}|^2\right)\right\|_{L^2}\left\|D^s\mathbf{u}\right\|_{L^2},\\
I_6 &\le C\big(\|\mathbf{b}\|_{H^s}\|\nabla\mathbf{u}\|_{L^\infty} + \|\mathbf{b}\|_{W^{1,\infty}}\|\nabla\mathbf{u}\|_{H^{s-1}}\big)\left\|D^s\mathbf{b}\right\|_{L^2} \le C\|\mathbf{b}\|_{H^s}^2\|\mathbf{u}\|_{H^s},\\
I_8 &\le C\big(\|\mathbf{u}\|_{H^s}\|\nabla\mathbf{b}\|_{L^\infty} + \|\mathbf{u}\|_{W^{1,\infty}}\|\nabla\mathbf{b}\|_{H^{s-1}}\big)\left\|D^s b\right\|_{L^2} \le C\|\mathbf{b}\|_{H^s}^s\|\mathbf{u}\|_{H^s}.
\end{aligned}$$

Inserting these estimates into (4.10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\rho\left|D^s\mathbf{u}\right|^2 + \left|D^s\mathbf{b}\right|^2\right)dx\\
&\quad\le C\|\mathbf{b}\|_{H^s}^2\|\mathbf{u}\|_{H^s} + C\|\rho\|_{H^s}\|\partial_t\mathbf{u}\|_{H^{s-1}}\|\mathbf{u}\|_{H^s}\\
&\qquad + C\|\rho\|_{H^s}\|\mathbf{u}\|_{H^s}^3 + \left\|D^s\nabla\left(\pi + \frac{1}{2}|\mathbf{b}|^2\right)\right\|_{L^2}\left\|D^s\mathbf{u}\right\|_{L^2}.
\end{aligned} \tag{4.11}$$

Testing (1.9) by $\partial_t\mathbf{u}$ and using (1.11), we find that

$$\int_\Omega \rho|\partial_t\mathbf{u}|^2\,dx \le C\big(\left\|(\mathbf{b}\cdot\nabla)\mathbf{b}\right\|_{L^2} + \left\|\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\right\|_{L^2}\big)\|\partial_t\mathbf{u}\|_{L^2},$$

whence

$$\begin{aligned}
\|\partial_t\mathbf{u}\|_{L^2} &\le C\big(\|\nabla\mathbf{b}\|_{L^\infty} + \|\nabla\mathbf{u}\|_{L^\infty}\big)\\
&\le C\|\mathbf{b}\|_{H^s} + C\|\mathbf{u}\|_{H^s}.
\end{aligned} \tag{4.12}$$

Applying D^{s-1} to (1.9), testing by $D^{s-1}\partial_t\mathbf{u}$, and using (1.8), we have

$$\begin{aligned}
\int_\Omega \rho\left|D^{s-1}\partial_t\mathbf{u}\right|^2 dx &= \int_\Omega D^{s-1}(\mathbf{b}\cdot\nabla\mathbf{b})D^{s-1}\partial_t\mathbf{u}\,dx - \int_\Omega D^{s-1}(\rho\mathbf{u}\cdot\nabla\mathbf{u})D^{s-1}\partial_t\mathbf{u}\,dx\\
&\quad - \int_\Omega \big(D^{s-1}(\rho\partial_t\mathbf{u}) - \rho D^{s-1}\partial_t\mathbf{u}\big)D^{s-1}\partial_t\mathbf{u}\,dx\\
&\quad - \int_\Omega D^{s-1}\nabla\left(\pi + \frac{1}{2}\mathbf{b}^2\right)\cdot D^{s-1}\partial_t\mathbf{u}\,dx,
\end{aligned}$$

whence

$$\begin{aligned}\|D^{s-1}\partial_t\mathbf{u}\|_{L^2} &\le C\|D^{s-1}(\mathbf{b}\cdot\nabla\mathbf{b})\|_{L^2} + C\|D^{s-1}(\rho\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2} \\ &\quad + C\|D^{s-1}(\rho\partial_t\mathbf{u}) - \rho D^{s-1}\partial_t\mathbf{u}\|_{L^2} + C\left\|D^{s-1}\nabla\left(\pi + \frac{1}{2}|\mathbf{b}|^2\right)\right\|_{L^2} \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{4.13}$$

Using (4.1) and (4.2) again, we bound J_1, J_2, and J_3 as follows:

$$\begin{aligned} J_1 &\le C\|\mathbf{b}\|_{L^\infty}\|\mathbf{b}\|_{H^s} \le C\|\mathbf{b}\|_{H^s}^2, \\ J_2 &\le C\big(\|\rho\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{H^{s-1}} + \|\rho\mathbf{u}\|_{H^{s-1}}\|\nabla\mathbf{u}\|_{L^\infty}\big) \\ &\le C\|\rho\|_{H^{s-1}}\|\mathbf{u}\|_{H^{s-1}}\|\mathbf{u}\|_{H^s} \le C\|\rho\|_{H^s}\|\mathbf{u}\|_{H^s}^2, \\ J_3 &\le C\big(\|\rho\|_{H^{s-1}}\|\partial_t\mathbf{u}\|_{L^\infty} + \|\rho\|_{W^{1,\infty}}\|\partial_t\mathbf{u}\|_{H^{s-2}}\big) \\ &\le C\|\rho\|_{H^s}\big(\|\partial_t\mathbf{u}\|_{L^\infty} + \|\partial_t\mathbf{u}\|_{H^{s-2}}\big) \\ &\le C\|\rho\|_{H^s}\big(\|\partial_t\mathbf{u}\|_{L^2}^{\frac{s-5/2}{s-1}}\|\partial_t\mathbf{u}\|_{H^{s-1}}^{\frac{3/2}{s-1}} + \|\partial_t\mathbf{u}\|_{L^2}^{\frac{1}{s-1}}\|\partial_t\mathbf{u}\|_{H^{s-1}}^{\frac{s-2}{s-1}}\big) \\ &\le \epsilon\|\partial_t\mathbf{u}\|_{H^{s-1}} + C\big(\|\rho\|_{H^s}^{s-1} + \|\rho\|_{H^s}^{\frac{s-1}{s-5/2}}\big)\|\partial_t\mathbf{u}\|_{L^2} \end{aligned}$$

for any $0 < \epsilon < 1$.

Inserting these estimates into (4.12) and (4.13) and taking ϵ small enough, we have

$$\begin{aligned}\|\partial_t\mathbf{u}\|_{H^{s-1}} &\le C\|\mathbf{b}\|_{H^s}^2 + C\|\rho\|_{H^s}\|\mathbf{u}\|_{H^s}^2 + C\|\mathbf{b}\|_{H^s} + C\|\mathbf{u}\|_{H^s} \\ &\quad + C\big(\|\rho\|_{H^s}^{s-1} + \|\rho\|_{H^s}^{\frac{s-1}{s-5/2}}\big)\big(\|\mathbf{b}\|_{H^s} + \|\mathbf{u}\|_{H^s}\big) \\ &\quad + C\left\|D^{s-1}\nabla\left(\pi + \frac{1}{2}\mathbf{b}^2\right)\right\|_{L^2}. \end{aligned} \tag{4.14}$$

Using (1.8) and (1.11) and setting $\tilde{\pi} := \pi + \frac{1}{2}\mathbf{b}^2$, we rewrite (1.9) as

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla\tilde{\pi} = \frac{1}{\rho}\mathbf{b}\cdot\nabla b. \tag{4.15}$$

Testing (4.15) by $\nabla\tilde{\pi}$ and using (1.11) and (4.3), we infer that

$$\|\nabla\tilde{\pi}\|_{L^2} \le C\|\mathbf{b}\cdot\nabla\mathbf{b}\|_{L^2} + C\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2} \le C\|\mathbf{b}\|_{H^s} + C\|\mathbf{u}\|_{H^s}. \tag{4.16}$$

Using (1.8), (1.9), and (1.12), we deduce that

$$\frac{\partial\tilde{\pi}}{\partial\mathbf{n}} = g := -\mathbf{b}\cdot\nabla\mathbf{n}\cdot\mathbf{b} + \rho\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u} \quad \text{on } \partial\Omega. \tag{4.17}$$

Taking div to (4.15), we observe that

$$-\Delta\tilde{\pi} = f := \rho\sum_i\nabla\mathbf{u}_i\partial_i\mathbf{u} - \frac{1}{\rho}(\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\nabla\rho - \sum_i\nabla\mathbf{b}_i\partial_i\mathbf{b} - \frac{1}{\rho}\nabla\rho\cdot\nabla\tilde{\pi}. \tag{4.18}$$

Using (4.1) and the well-known H^{s+1}-estimates of problems (4.18) and (4.17) [18], we have

$$\begin{aligned}\|\nabla\tilde{\pi}\|_{H^s} &\le C\|f\|_{H^{s-1}} + C\|g\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\\ &\le C\|f\|_{H^{s-1}} + C\|\mathbf{b}\cdot\nabla\mathbf{n}\cdot\mathbf{b}\|_{H^s} + C\|\rho\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}\|_{H^s}\\ &\le C\|\rho\|_{H^s}\|\mathbf{u}\|_{H^s}^2 + C\|\rho\|_{H^s}\|\mathbf{b}\|_{H^s}^2 + C\|\mathbf{b}\|_{H^s} + C\|\mathbf{u}\|_{H^s}\\ &\quad + C\|\mathbf{b}\|_{H^s}^2 + C\|\rho\|_{H^s}\|\nabla\tilde{\pi}\|_{\dot{H}^{s-1}},\end{aligned} \tag{4.19}$$

whence

$$\begin{aligned}\|\nabla\tilde{\pi}\|_{H^s} &\le C\|\rho\|_{H^s}\|\mathbf{u}\|_{H^s}^2 + C\|\rho\|_{H^s}\|\mathbf{b}\|_{H^s}^2 + C\|\mathbf{b}\|_{H^s}\\ &\quad + C\|\mathbf{u}\|_{H^s} + C\|\mathbf{b}\|_{H^s}^2 + C\|\rho\|_{H^s}^s\|\nabla\tilde{\pi}\|_{L^2},\end{aligned} \tag{4.20}$$

where we used the Gagliardo-Nirenberg inequality

$$\|\nabla\tilde{\pi}\|_{\dot{H}^{s-1}} \le C\|\nabla\tilde{\pi}\|_{L^2}^{\frac{1}{s}}\|\nabla\tilde{\pi}\|_{H^s}^{\frac{s-1}{s}}$$

and the well-known estimate [18]

$$\left\|D^s\left(\frac{1}{\rho}\right)\right\|_{L^2} \le C\|\rho\|_{H^s}.$$

Combining (4.7), (4.11), (4.14), and (4.20) and using the Osgood lemma, we arrive at (1.15).

This completes the proof.

5 A blow-up criterion for the MHD system

This section is devoted to the proof of regularity criterion for the MHD system. We only need to establish a priori estimates.

First, we still have (4.3) and (4.6).

Taking ∇ to (1.8), testing by $|\nabla\rho|^{p-2}\nabla\rho$, and using (1.11) and (1.16), we derive

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla\rho|^p\,dx \le \|\nabla\mathbf{u}\|_{L^\infty}\int_\Omega |\nabla\rho|^p\,dx,$$

whence

$$\frac{d}{dt}\|\nabla\rho\|_{L^p} \le \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^p}.$$

Integrating this inequality and taking the limits as $p\to+\infty$, we have

$$\|\nabla\rho\|_{L^\infty(0,T;L^\infty)} \le C. \tag{5.1}$$

It follows from (4.6) and (1.16) that

$$\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty})} + \|\mathbf{b}\|_{L^\infty(0,T;W^{1,\infty})} \le C. \tag{5.2}$$

Similarly to (4.16), we find that

$$\|\nabla\tilde{\pi}\|_{L^2} \leq C. \tag{5.3}$$

It follows from (4.17), (4.18), (5.1), (5.2), (5.3), and the $W^{2,p}$-estimates of problem (4.17)-(4.18) that

$$\begin{aligned}\|\nabla\tilde{\pi}\|_{W^{1,p}(\Omega)} &\leq C\|f\|_{L^p(\Omega)} + C\|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \\ &\leq C + C\|\mathbf{b}\cdot\nabla\mathbf{n}\cdot\mathbf{b}\|_{W^{1,p}} + C\|\rho\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}\|_{W^{1,p}} \\ &\leq C\end{aligned}$$

for any $3 < p < \infty$, and thus

$$\|\tilde{\pi}\|_{L^\infty(0,T;W^{1,\infty})} \leq C. \tag{5.4}$$

It follows from (4.15), (4.3), and (5.4) that

$$\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^\infty)} \leq C.$$

Similarly to (4.19), we have

$$\|\nabla\tilde{\pi}\|_{H^s} \leq C\|\mathbf{u}\|_{H^s} + C\|\mathbf{b}\|_{H^s} + C\|\rho\|_{H^s} + C\|\nabla\tilde{\pi}\|_{\dot{H}^{s-1}},$$

whence

$$\|\nabla\tilde{\pi}\|_{H^s} \leq C\|\mathbf{u}\|_{H^s} + C\|\mathbf{b}\|_{H^s} + C\|\rho\|_{H^s} + C.$$

We still have (4.13), and similarly to (4.14), we have

$$\|\partial_t\mathbf{u}\|_{H^{s-1}} \leq C\|\mathbf{b}\|_{H^s} + C\|\rho\|_{H^s} + C\|\mathbf{u}\|_{H^s} + C\|\partial_t\mathbf{u}\|_{H^{s-2}} + C\left\|D^{s-1}\nabla\tilde{\pi}\right\|_{L^2},$$

which gives

$$\|\partial_t\mathbf{u}\|_{H^{s-1}} \leq C\|\rho\|_{H^s} + C\|\mathbf{u}\|_{H^s} + C\|\mathbf{b}\|_{H^s} + C.$$

Similarly to (4.7), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|D^s\rho\right|^2 dx \leq C\|\rho\|_{H^s}^2 + C\|\mathbf{u}\|_{H^s}^2. \tag{5.5}$$

We still have (4.10). We bound I_1, I_3, I_4, I_5, I_6, and I_8 as follows:

$$\begin{aligned}I_1 &\leq C\|\mathbf{u}\|_{H^s}^2 + C\|b\|_{H^s}^2, \\ I_3 &\leq C\|\mathbf{u}\|_{H^s}^2 + C\|\rho\|_{H^s}^2 + C\|\partial_t\mathbf{u}\|_{H^{s-1}}^2 \\ &\leq C\|\rho\|_{H^s}^2 + C\|\mathbf{u}\|_{H^s}^2 + C\|\mathbf{b}\|_{H^s}^2 + C,\end{aligned}$$

$$I_4 \leq C\|\rho\|_{H^s}^2 + C\|\mathbf{u}\|_{H^s}^2,$$
$$I_5 \leq C\|\rho\|_{H^s}^2 + C\|\mathbf{u}\|_{H^s}^2 + C\|\mathbf{b}\|_{H^s}^2 + C,$$
$$I_6 \leq C\|\mathbf{b}\|_{H^s}^2 + C\|\mathbf{u}\|_{H^s}^2,$$
$$I_8 \leq C\|\mathbf{b}\|_{H^s}^2 + C\|\mathbf{u}\|_{H^s}^2.$$

Inserting these estimates into (4.10) and using (5.5) and the Gronwall inequality, we conclude that

$$\|(\rho, \mathbf{u}, \mathbf{b})\|_{L^\infty(0,T;H^s)} \leq C.$$

This completes the proof.

Appendix: Proof of (1.17)

We only prove the case $|\alpha| = s$. We have

$$\begin{aligned} \|D^\alpha(fg) - fD^\alpha g\|_{L^p} &\leq \sum_{i=1}^{s} C_i \|D^i f D^{s-i} g\|_{L^p} \\ &\leq C\|\nabla f\|_{L^{p_2}} \|g\|_{W^{s-1,q_2}} + C\|f\|_{W^{s,p_1}} \|g\|_{L^{q_1}} \\ &\quad + \sum_{i=2}^{s-1} C_i \|D^i f\|_{L^{p_i}} \|D^{s-i} g\|_{L^{q_i}}. \end{aligned} \tag{A.1}$$

We will use the following two Gagliardo-Nirenberg inequalities:

$$\|D^i f\|_{L^{p_i}} \leq C\|\nabla f\|_{L^{p_2}}^{1-\alpha_i} \|f\|_{W^{s,p_1}}^{\alpha_i}, \tag{A.2}$$
$$\|D^{s-i} g\|_{L^{q_i}} \leq C\|g\|_{L^{q_1}}^{\alpha_i} \|g\|_{W^{s-1,q_2}}^{1-\alpha_i}, \tag{A.3}$$

with $i - \frac{d}{p_i} = (1-\alpha_i)(1-\frac{d}{p_2}) + \alpha_i(s - \frac{d}{p_1})$, where d is the dimension number.

Inserting (A.2) and (A.3) into (A.1) and using the Young inequality give (1.17).

This completes the proof.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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