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# Inverse nodal problems for the $p$-Laplacian with eigenparameter dependent energy functions

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**Abstract**

We study the inverse nodal problems for the $p$-Laplacian with two energy functions

$$\begin{cases} -(|y'|^{p-2}y')' = (p-1)(\kappa^2 - \kappa p q_1(x) - q_0(x))|y|^{p-2}y, \\ y(0)\sin_p'(\alpha) + y'(0)\sin_p(\alpha) = 0, \\ y(1)\sin_p'(\beta) + y'(1)\sin_p(\beta) = 0, \end{cases}$$

where $p > 1$, $\kappa$ is a spectral parameter, $\alpha, \beta \in [0, \pi_p]$, $\sin_p(x)$ is the *generalized sine function* and $\pi_p$ is the *generalized* $\pi$ *constant*. We use a Prüfer substitution derived by $\sin_p(x)$ to find the asymptotic expansions of the eigenvalues and nodal lengths. Furthermore, we consider the inverse nodal problem and give the reconstruction formulas for the boundary conditions $\alpha$, $\beta$, and the energy functions $q_1$, $q_0$ by only using the information of nodal data.

**Keywords:** inverse nodal problems; eigenparameter dependent energy functions

## 1 Introduction

In this paper, we investigate the $p$-Laplacian with two energy functions

$$\begin{cases} -(|y'|^{p-2}y')' = (p-1)(\kappa^2 - \kappa p q_1(x) - q_0(x))|y|^{p-2}y, \\ y(0)\sin_p'(\alpha) + y'(0)\sin_p(\alpha) = 0, \\ y(1)\sin_p'(\beta) + y'(1)\sin_p(\beta) = 0. \end{cases} \tag{1}$$

Here, $p > 1$, $\kappa$ is a spectral parameter and $\alpha, \beta \in [0, \pi_p]$ where $\pi_p$ and $\sin_p(x)$ are defined below. The nodal points are unchanged when the spectral parameter $\kappa$ shifts to $\kappa + \int_0^1 q_1$, so we assume that $q_1 \not\equiv 0$ and $\int_0^1 q_1 = 0$ without loss of generality. In this paper, we also assume $q_0 \in C[0,1]$ and $q_1 \in C^1[0,1]$. The $p$-Laplacian problems have attracted lots of interests recently (*cf.* [1–4]).

In 1979, Elbert [5] (see also [6]) showed that the solution of the initial value problem

$$\begin{cases} -(|y'|^{p-2}y')' = (p-1)|y|^{p-2}y, \\ y(0) = 0, \qquad y'(0) = 1, \end{cases}$$

can be defined by the inverse of the integral

$$x = \int_0^{\omega} \frac{1}{(1-t^p)^{1/p}}\,dt, \quad \text{for } 0 \le \omega \le 1.$$

Denote by $\sin_p(x) \equiv \omega(x)$. Then the first non-vanishing zero $\pi_p$ of $\sin_p$ is defined by

$$\pi_p \equiv 2\int_0^1 \frac{1}{(1-t^p)^{1/p}}\,dt = \frac{2\pi/p}{\sin(\pi/p)}.$$

Continuing $\sin_p(x)$ symmetrically over $x \in [\pi_p/2, \pi_p]$ and anti-symmetrically outside $[0, \pi_p]$, one obtains a $\sin_p$ graph analogous to a sine wave. Also, the function $\sin_p(x)$ satisfies

$$\left|\sin_p(x)\right|^p + \left|\sin_p'(x)\right|^p = 1,$$

which is an analog of the Pythagorean trigonometric identity $\sin^2 x + \cos^2 x = 1$. Specially, we call $\sin_p(x)$ the *generalized sine function*. Furthermore, we may define the $p$-version trigonometric functions by *generalized cosine function* $\sin_p'(x)$, *generalized tangent function* $\tan_p(x) \equiv \sin_p(x)/\sin_p'(x)$, and *generalized cotangent function* $\cot_p(x) \equiv \sin_p'(x)/\sin_p(x)$. For more related properties and results before 2003, the reader can refer to a survey [7].

Differential equations with nonlinear dependence on the spectral parameter and with turning points arise in various problems of mathematics as well as in applications, for example, non-Newtonian fluids, reaction-diffusion problems, theory of superconductors, biology, and so forth (see [8–14] and the references therein). In particular, the system (1) with $p = 2$ is the problem of describing the interactions between colliding particles in physics [10]. One is interested in collisions of two spinless particles, and it is supposed that the $s$-wave scattering matrix and the $s$-wave binding energies are exactly known from collision experiments. With a radially static potential $Q(\lambda, x)$, the $s$-wave Schrödinger equation is written as

$$-y'' + Q(\lambda, x)y = \lambda y,$$

where $Q(\lambda, x)$ is the following form for the energy dependence:

$$Q(\lambda, x) = 2\sqrt{\lambda}q_1(x) + q_0(x).$$

The diffusion problem (1) with $p = 2$ has been widely studied [15–21]. In particular, Buterin and Shieh [20, 22] derived the asymptotic expansion of nodal points of problem (1) with $p = 2$ and showed that $\alpha$, $\beta$, $q_0$, and $q_1$ can be uniquely determined and reconstructed by any dense set of nodal points. Moreover, they also gave the reconstruction formulas of $\alpha$, $\beta$, $q_0$, and $q_1$. The work of Koyunbakan [23] should also be mentioned. He considered the inverse problem on the problem (1) with Dirichlet boundary conditions. But he only gave the reconstruction formula for the energy $q_0$ and the formula depends on the eigenvalues and nodal lengths. In this paper, we reconstruct the energy functions $q_0$, $q_1$ and the boundary conditions $\alpha$, $\beta$. Moreover, the reconstruction formulas only depend on the nodal points.

In this paper, we study the direct problem and inverse nodal problem of the $p$-Laplacian operator with potentials depending on energy (1). Using the modified Prüfer substitution derived by generalized sine function $\sin_p$, we will show that, for sufficiently large $N \in \mathbb{N}$, the problem (1) has an eigenpair $(\kappa_n, y(x,\kappa_n))$ for $|n| > N$ which owns the Sturm-Liouville property. That is, the corresponding eigenfunction $y_n(x) \equiv y(x,\kappa_n)$ has $|n| - 1$ zeros $\{x_i^{(n)}\}$ in $(0,1)$ for $|n| > N$, which satisfies

$$0 < x_1^{(n)} < x_2^{(n)} < \cdots < x_{n-2}^{(n)} < x_{n-1}^{(n)} < 1$$

or

$$0 < x_{-1}^{(-n)} < x_{-2}^{(-n)} < \cdots < x_{-n+2}^{(-n)} < x_{-n+1}^{(-n)} < 1.$$

We will also derive the asymptotic expansion of eigenvalues and nodal points. Furthermore, we will solve the inverse nodal problem. That is, we will give the formulas of $\alpha$, $\beta$, $q_0(x)$, and $q_1(x)$ by using the information of nodal points. For $x \in [0,1]$, denote $j = j_n(x) = \max\{i : x_i^{(n)} \le x\}$. The following are our main results.

**Theorem 1.1** *Let* $1 < p < 2$. *The problem* (1) *can be reconstructed by the following formulas*:

(i) *Either* $\alpha = 0$, *or with* $i/n$ *tending to* 0,

$$\left|\cot_p(\alpha)\right|_p^{p-2}\cot(\alpha) = \begin{cases} -\lim_{x_i^{(\pm n)}\to 0}(p-1)((n-\frac{1}{2})\pi_p)^p(x_i^{(\pm n)} - \frac{i-\frac{1}{2}}{n-\frac{1}{2}}), \\ -\lim_{x_i^{(\pm n)}\to 0}(p-1)((n-1)\pi_p)^p(x_i^{(\pm n)} - \frac{i-\frac{1}{2}}{n-1}), \end{cases}$$

*if the limit exists.*

(ii) *Either* $\beta = 0$, *or with* $i/n$ *tending to* 0,

$$\left|\cot_p(\beta)\right|_p^{p-2}\cot(\beta) = \begin{cases} \lim_{x_{\pm n-i}^{(\pm n)}\to 1}(p-1)((n-\frac{1}{2})\pi_p)^p(1 - x_{\pm n-i}^{(\pm n)} - \frac{i-\frac{1}{2}}{n-\frac{1}{2}}), \\ \lim_{x_{\pm n-i}^{(\pm n)}\to 1}(p-1)((n-1)\pi_p)^p(1 - x_{\pm n-i}^{(\pm n)} - \frac{i-\frac{1}{2}}{n-1}), \end{cases}$$

*if the limit exists.*

(iii) *The functions* $q_1(x)$ *and* $q_0(x)$ *can be reconstructed by the following limits*:

$$q_1(x) = \lim_{n\to\infty} n_{\alpha\beta}(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}\left(\ell_j^{(\pm n)} - \frac{1}{n_{\alpha\beta}}\right)$$

*and*

$$q_0(x) = \lim_{n\to\infty} p n_{\alpha\beta}(n_{\alpha\beta}\pi_p)^p\left(\ell_j^{(\pm n)} - \frac{1}{n_{\alpha\beta}} - \frac{1}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\int_{x_j^{(\pm n)}}^{x_{j+1}^{(\pm n)}} q_1(x)\,dx\right)$$
$$+ \frac{p}{p-1}\left(\operatorname{scot}_p(\beta) - \operatorname{scot}_p(\alpha)\right) + \int_0^1 q_0(x)\,dx,$$

*where*

$$\operatorname{scot}_p(x) = \begin{cases} 0 & \text{if } x = 0, \\ |\cot_p(x)|^{p-2}\cot_p(x) & \text{if } x \neq 0, \end{cases} \tag{2}$$

*and*

$$n_{\alpha\beta} = \begin{cases} n & \textit{for } \alpha = 0, \beta = 0; \\ n - \frac{1}{2} & \textit{for } \alpha = 0, \beta \neq 0, \textit{ or } \alpha \neq 0, \beta = 0; \\ n - 1 & \textit{for } \alpha \neq 0, \beta \neq 0. \end{cases} \tag{3}$$

In the following theorem, we consider the case of $p \geq 2$. For $p = 2$, we obtain a similar result of an inverse nodal problem for differential pencils [22].

**Theorem 1.2** *Let $p \geq 2$. Consider the problem* (1) *with Dirichlet boundary conditions. Then* (1) *can be reconstructed by the following formulas*:

$$q_1(x) = \lim_{n\to\infty} n(n\pi_p)^{\frac{p}{2}} \left( \ell_j^{(n)} - \frac{1}{n} \right)$$

*and*

$$\begin{aligned} & q_0(x) - \int_0^1 q_0(x)\,dx \\ &\quad = \lim_{n\to\infty} pn(n\pi_p)^p \Bigg\{ \ell_j^{(n)} - \frac{1}{n} - \frac{1}{(n\pi_p)^{\frac{p}{2}}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_1(x)\,dx \\ &\qquad - \frac{p}{(n\pi_p)^{\frac{p}{2}}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_1(x) \left( \left|\sin_p(n\pi_p x)\right|^p - \frac{1}{p} \right) dx \\ &\qquad + \frac{\pi_p}{(n\pi_p)^{\frac{p}{2}+2}} \int_0^1 q_1'(x) \sin_p(n\pi_p x) \left|\sin_p'(n\pi_p x)\right|^{p-2} \sin_p'(n\pi_p x)\,dx \Bigg\}. \end{aligned}$$

The paper is organized as follows. The asymptotic expansions of eigenvalues and nodal points are given in Section 2 and the proofs of our main results are divided into Sections 3 and 4, respectively.

## 2 The asymptotic expansions of eigenvalues and nodal points

In this section, we will investigate the asymptotics of the eigenvalues and nodal points of the problem (1). We will show that the problem (1) has infinitely many eigenpairs $(\kappa_n, y(x,\kappa_n))$. In particular, $\lim_{n\to\infty} \kappa_n = \infty$, $\lim_{n\to-\infty} \kappa_n = -\infty$, and $y(x,\kappa_n)$ has $|n| - 1$ nodal points in $(0,1)$ for sufficiently large $|n|$. Then, by a modified Prüfer substitution, we will derive the asymptotics of the eigenvalues and nodal points.

To achieve our goals, we first introduce the Prüfer transformation derived by generalized sine function $\sin_p(x)$:

$$y(x) = r(x) \sin_p\bigl(\theta(x)\bigr), \qquad y'(x) = r(x) \sin_p'\bigl(\theta(x)\bigr).$$

Here, $r(x) > 0$ is called the Prüfer radius and $\theta(x)$ is called the Prüfer angle. By directed calculation, we find

$$\theta'(x) = \bigl(\kappa^2 - \kappa p q_1(x) - q_0(x)\bigr) \left|\sin_p\bigl(\theta(x)\bigr)\right|^p + \left|\sin_p'\bigl(\theta(x)\bigr)\right|^p, \tag{4}$$

$$\frac{r'(x)}{r(x)} = \bigl[1 - \kappa^2 + \kappa p q_1(x) + q_0(x)\bigr] \left|\sin_p\bigl(\theta(x)\bigr)\right|^{p-2} \sin_p\bigl(\theta(x)\bigr) \sin_p'\bigl(\theta(x)\bigr). \tag{5}$$

Compared with the boundary conditions in (1), $\theta$ satisfies the conditions

$$\theta(0) = \begin{cases} m\pi_p & \text{if } \alpha = 0, \\ m\pi_p - \alpha & \text{if } \alpha > 0, \end{cases} \quad \text{for some } m \in \mathbb{Z}$$

and

$$\theta(1) = \begin{cases} n\pi_p & \text{if } \beta = 0, \\ n\pi_p - \beta & \text{if } \beta > 0, \end{cases} \quad \text{for some } n \in \mathbb{Z}.$$

Now, if $(\kappa, y)$ is an eigenpair of (1), then, by integration by parts, we have

$$\begin{aligned}(p-1)\kappa^2 \|y\|_p^p &= \int_0^1 (p-1)\kappa^2 |y|^{p-2} y\bar{y} \\ &= \int_0^1 \bigl[-\bigl(|y'|^{p-2}y'\bigr)' + \kappa(p-1)pq_1(x)|y|^{p-2}y + (p-1)q_0(x)|y|^{p-2}y\bigr]\bar{y} \\ &= \bigl|y'(1)\bigr|^{p-2}\bigl[\bar{y}'(1)y(1) - y'(1)\bar{y}(1)\bigr] + \bigl|y'(0)\bigr|^{p-2}\bigl[y'(0)\bar{y}(0) - \bar{y}'(0)y(0)\bigr] \\ &\quad + \int_0^1 (p-1)\bar{\kappa}^2 |\bar{y}|^{p-2}\bar{y}y + (p-1)p(\kappa - \bar{\kappa}) \int_0^1 q_1(x)|y|^p \\ &= (p-1)\bar{\kappa}^2 \|y\|_p^p + (p-1)p(\kappa - \bar{\kappa}) \int_0^1 q_1(x)|y|^p.\end{aligned}$$

The last equality is because

$$\bar{y}'(1)y(1) - y'(1)\bar{y}(1) = y'(0)\bar{y}(0) - \bar{y}'(0)y(0) = 0.$$

Hence, we have

$$\bigl(\kappa^2 - \bar{\kappa}^2\bigr)\|y\|_p^p = p(\kappa - \bar{\kappa}) \int_0^1 q_1(x)|y|^p.$$

Note that for $\kappa \in \mathbb{C}\backslash\mathbb{R}$, above equation implies

$$p \min_{x\in[0,1]} q_1(x) \int_0^1 |y|^p \le (\kappa + \bar{\kappa})\|y\|_p^p = p \int_0^1 q_1(x)|y|^p \le p \max_{x\in[0,1]} q_1(x) \int_0^1 |y|^p.$$

Hence, $p \min q_1(x) \le \kappa + \bar{\kappa} \le p \max q_1(x)$ if $\kappa \in \mathbb{C}\backslash\mathbb{R}$. In the following, we only consider $\kappa > \frac{p}{2} \max_{x\in[0,1]} q_1(x)$ or $\kappa < \frac{p}{2} \min_{x\in[0,1]} q_1(x)$. This condition leads to that all eigenvalues are real. By the following lemma, we can show that (1) has a sequence of eigenvalues $\{\kappa_n\}_{|n|>N}$ for some sufficiently large $N \in \mathbb{N}$ with $\lim_{n\to\infty} \kappa_n = \infty$, $\lim_{n\to-\infty} \kappa_n = -\infty$, and

$$\cdots < \kappa_{-n} < \cdots < \kappa_{-N-1} < \kappa_{-N} < 0 < \kappa_N < \kappa_{N+1} < \cdots < \kappa_n < \cdots,$$

and the corresponding eigenfunction $y_n(x) \equiv y(x, \kappa_n)$ has $|n| - 1$ zeros $\{x_i^{(n)}\}$ in $(0,1)$ for $|n| > N$, which satisfies

$$0 < x_1^{(n)} < x_2^{(n)} < \cdots < x_{n-2}^{(n)} < x_{n-1}^{(n)} < 1$$

or

$$0 < x_{-1}^{(-n)} < x_{-2}^{(-n)} < \cdots < x_{-n+2}^{(-n)} < x_{-n+1}^{(-n)} < 1.$$

The argument of the proof of the following lemma is similar to [7, 24], so we omit the proof.

**Lemma 2.1** *Assume $q_0, q_1 \in C[0,1]$. Then we have the following results.*

(a) *The function $\theta(\cdot,\kappa)$ satisfying $\theta(0,\kappa) = \delta_1 \in [0,\pi_p)$ is continuous. For $\kappa > \frac{p}{2}\max_{x\in[0,1]} q_1(x)$, we find $\theta(\cdot,\kappa)$ is strictly increasing in $\kappa$. Moreover, for every fixed $\kappa \in \mathbb{R}$, if $\theta(x_i,\kappa) = i\pi_p$ for $i \in \mathbb{N}$, then $\theta(x,\kappa) > i\pi_p$ for all $x > x_i$.*

(b) *The function $\theta(\cdot,\kappa)$ satisfying $\theta(1,\kappa) = \delta_2 \in (-\pi_p, 0]$ is continuous. For $\kappa < \frac{p}{2}\min_{x\in[0,1]} q_1(x)$, we find $\theta(\cdot,\kappa)$ is strictly increasing in $\kappa$. Moreover, for every fixed $\kappa \in \mathbb{R}$, if $\theta(x_i,\kappa) = i\pi_p$ for $-i \in \mathbb{N}$, then $\theta(x,\kappa) < i\pi_p$ for all $x < x_i$.*

(c) *For any $a \in (0,1]$, $\lim_{\kappa\to\infty}\theta(a,\kappa) = \infty$ and $\lim_{\kappa\to-\infty}\theta(a,\kappa) = -\infty$.*

Nest, we will use a modified Prüfer substitution to derive the asymptotic expansion of eigenvalues and nodal points:

$$y(x) = R(x)\sin_p\bigl(|\kappa|^{\frac{2}{p}}\vartheta(x,\kappa)\bigr) \quad \text{and} \quad y'(x) = |\kappa|^{\frac{2}{p}}R(x)\sin_p'\bigl(|\kappa|^{\frac{2}{p}}\vartheta(x,\kappa)\bigr). \tag{6}$$

By direct computation, we find

$$\begin{aligned}\vartheta'(x,\kappa) &= \frac{\kappa^2 - \kappa p q_1(x) - q_0(x)}{|\kappa|^2}\bigl|\sin_p\bigl(|\kappa|^{\frac{2}{p}}\vartheta(x,\kappa)\bigr)\bigr|^p + \bigl|\sin_p'\bigl(|\kappa|^{\frac{2}{p}}\vartheta(x,\kappa)\bigr)\bigr|^p \\ &= 1 - \frac{\kappa p q_1(x) + q_0(x)}{\kappa^2}\bigl|\sin_p\bigl(|\kappa|^{\frac{2}{p}}\vartheta(x,\kappa)\bigr)\bigr|^p. \end{aligned}\tag{7}$$

The following lemma can be referred to [25], so we omit the proof.

**Lemma 2.2** ([25], p.1314) *Define $\cot_p(x) \equiv \sin_p'(x)/\sin_p(x)$. Then we have*

(a) *$\cot_p$ is a decreasing function on $((n-1)\pi_p, n\pi_p)$ for all $n \in \mathbb{Z}$ satisfying $\cot_p((n-1/2)\pi_p) = 0$ and*

$$\lim_{x\to((n-1)\pi_p)^+}\cot_p(x) = \infty, \qquad \lim_{x\to(n\pi_p)^-}\cot_p(x) = -\infty.$$

(b) *Denote the inverse of $\cot_p$ by $\cot_p^{-1} : \mathbb{R} \to (0,\pi_p)$. Then the expansion of $\cot_p^{-1}$ at $x = 0$ satisfies*

$$\cot_p^{-1}(x) = \frac{\pi_p}{2} - \frac{1}{p-1}|x|^{p-2}x + \frac{1}{2p-1}|x|^{2p-2}x + O\bigl(x^{3p-1}\bigr).$$

By the boundary conditions in (1), the modified Prüfer substitution (6) and Lemma 2.2, we find that, for sufficiently large $n \in \mathbb{N}$,

(i) for $\alpha = 0$, $\vartheta(0,\kappa_n) = 0$, and $\vartheta(0,\kappa_{-n}) = -n\pi_p/|\kappa_{-n}|^{\frac{2}{p}}$; for $\alpha \neq 0$,

$$\vartheta(0,\kappa_n) = \frac{\pi_p}{2|\kappa_n|^{\frac{2}{p}}} + \frac{1}{(p-1)\kappa_n^2}\bigl|\cot_p(\alpha)\bigr|^{p-2}\cot_p(\alpha) + O\biggl(\frac{1}{\kappa_n^4}\biggr), \tag{8}$$

$$\vartheta(0,\kappa_{-n}) = \frac{-(n-\frac{1}{2})\pi_p}{|\kappa_{-n}|^{\frac{2}{p}}} + \frac{1}{(p-1)\kappa_{-n}^2}\left|\cot_p(\alpha)\right|^{p-2}\cot_p(\alpha) + O\left(\frac{1}{\kappa_{-n}^4}\right); \tag{9}$$

(ii) for $\beta = 0$, $\vartheta(1,\kappa_n) = n\pi_p/|\kappa_n|^{\frac{2}{p}}$, and $\vartheta(1,\kappa_{-n}) = 0$; for $\beta \neq 0$,

$$\vartheta(1,\kappa_n) = \frac{(n-\frac{1}{2})\pi_p}{|\kappa_n|^{\frac{2}{p}}} + \frac{1}{(p-1)\kappa_n^2}\left|\cot_p(\beta)\right|^{p-2}\cot_p(\beta) + O\left(\frac{1}{\kappa_n^4}\right), \tag{10}$$

$$\vartheta(1,\kappa_{-n}) = -\frac{\pi_p}{2|\kappa_{-n}|^{\frac{2}{p}}} + \frac{1}{(p-1)\kappa_{-n}^2}\left|\cot_p(\beta)\right|^{p-2}\cot_p(\beta) + O\left(\frac{1}{\kappa_{-n}^4}\right). \tag{11}$$

Furthermore, the nodal points $\{x_i^{(n)}\}$ of the eigenfunction $y_n(x) \equiv y(x,\kappa_n)$ satisfy

$$\vartheta\left(x_i^{(n)},\kappa_n\right) = \begin{cases} i\pi_p/|\kappa_n|^{\frac{2}{p}} & \text{for } i = 1,2,\dots,n-1; n > 0, \\ (n-i)\pi_p/|\kappa_n|^{\frac{2}{p}} & \text{for } i = -1,-2,\dots,n+1; n < 0. \end{cases}$$

By integrating (7) from 0 to 1 with respect to $x$, we find

$$\begin{aligned}\vartheta(1,\kappa_{\pm n}) - \vartheta(0,\kappa_{\pm n}) &= \int_0^1 \vartheta'(x,\kappa_{\pm n})\,dx \\ &= 1 - \int_0^1 \frac{\kappa_{\pm n} p q_1(x) + q_0(x)}{\kappa_{\pm n}^2}\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p dx. \end{aligned} \tag{12}$$

On the other hand, by (8)-(11), we have

$$\vartheta(1,\kappa_{\pm n}) - \vartheta(0,\kappa_{\pm n}) = \begin{cases} \frac{n\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} & \text{for } \alpha = 0, \beta = 0; \\ \frac{(n-\frac{1}{2})\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} + \frac{B}{(p-1)\kappa_{\pm n}^2} + O(\frac{1}{\kappa_{\pm n}^4}) & \text{for } \alpha = 0, \beta \neq 0; \\ \frac{(n-\frac{1}{2})\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} - \frac{A}{(p-1)\kappa_{\pm n}^2} + O(\frac{1}{\kappa_{\pm n}^4}) & \text{for } \alpha \neq 0, \beta = 0; \\ \frac{(n-1)\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} + \frac{B-A}{(p-1)\kappa_{\pm n}^2} + O(\frac{1}{\kappa_{\pm n}^4}) & \text{for } \alpha \neq 0, \beta \neq 0; \end{cases} \tag{13}$$

where $A \equiv |\cot_p(\alpha)|^{p-2}\cot_p(\alpha)$ and $B \equiv |\cot_p(\beta)|^{p-2}\cot_p(\beta)$. Hence, by (12) and (13), we find that

$$|\kappa_{\pm n}|^{\frac{2}{p}} = n_{\alpha\beta}\pi_p + \frac{\operatorname{scot}_p(\beta) - \operatorname{scot}_p(\alpha)}{(p-1)|\kappa_{\pm n}|^{2-\frac{2}{p}}} \pm \frac{p}{|\kappa_{\pm n}|^{1-\frac{2}{p}}}Q_1 + \frac{1}{|\kappa_{\pm n}|^{2-\frac{2}{p}}}Q_0 + O\left(\frac{1}{|\kappa_{\pm n}|^{4-\frac{2}{p}}}\right), \tag{14}$$

where $\operatorname{scot}_p(x)$ is defined as (2) while $n_{\alpha\beta}$ is defined as (3), and

$$\begin{aligned} Q_1 \equiv Q_1(\kappa_{\pm n}) &= \int_0^1 q_1(x)\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p x, \\ Q_0 \equiv Q_0(\kappa_{\pm n}) &= \int_0^1 q_0(x)\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p dx. \end{aligned}$$

Finally, for $n \in \mathbb{N}$ and $i = \pm 1, \pm 2, \ldots, \pm(n-1)$, we may integrate (7) from 0 to $x_i^{(n)}$ with respect to $x$ to obtain the asymptotic expansion of the nodal points. For $\alpha = 0$, we have

$$x_i^{(\pm n)} = \frac{|i|\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} + \frac{p}{\kappa_{\pm n}} \int_0^{x_i^{(\pm n)}} q_1(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx + \frac{1}{\kappa_{\pm n}^2} \int_0^{x_i^{(\pm n)}} q_0(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx$$

and, for $\alpha \neq 0$,

$$x_i^{(\pm n)} = \frac{(|i| - \frac{1}{2})\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} - \frac{A}{(p-1)\kappa_{\pm n}^2} + \frac{p}{\kappa_{\pm n}} \int_0^{x_i^{(\pm n)}} q_1(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx + \frac{1}{\kappa_{\pm n}^2} \int_0^{x_i^{(\pm n)}} q_0(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx + O\biggl(\frac{1}{\kappa_{\pm n}^4}\biggr).$$

Similarly, by integrating from $x_{n-i}^{(n)}$ to 1 with respect to $x$, we find that, for $\beta = 0$,

$$1 - x_{\pm n-i}^{(\pm n)} = \frac{|i|\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} + \frac{p}{\kappa_{\pm n}} \int_{x_{\pm n-i}^{(\pm n)}}^1 q_1(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_n)\bigr)\bigr|^p \, dx + \frac{1}{\kappa_{\pm n}^2} \int_{x_{\pm n-i}^{(\pm n)}}^1 q_0(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx$$

and, for $\beta \neq 0$,

$$1 - x_{\pm n-i}^{(\pm n)} = \frac{(|i| - \frac{1}{2})\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} + \frac{B}{(p-1)\kappa_{\pm n}^2} + \frac{p}{\kappa_{\pm n}} \int_{x_{\pm n-i}^{(\pm n)}}^1 q_1(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx + \frac{1}{\kappa_{\pm n}^2} \int_{x_{n-i}^{(\pm n)}}^1 q_0(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx + O\biggl(\frac{1}{\kappa_{\pm n}^4}\biggr).$$

In particular, we find that the nodal length $\ell_i^{(n)} \equiv x_{i+1}^{(n)} - x_i^{(n)}$ satisfies, for $n \in \mathbb{N}$ and $i = \pm 1, \pm 2, \ldots, \pm(n-1)$,

$$\ell_i^{(\pm n)} = \pm \frac{\pi_p}{|\kappa_{\pm n}|^{\frac{2}{p}}} + \frac{p}{\kappa_{\pm n}} \int_{x_i^{(\pm n)}}^{x_{i+1}^{(\pm n)}} q_1(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx + \frac{1}{\kappa_{\pm n}^2} \int_{x_i^{(\pm n)}}^{x_{i+1}^{(\pm n)}} q_0(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx + O\biggl(\frac{1}{\kappa_{\pm n}^4}\biggr).$$

The next lemma is used to estimate order of the asymptotic expansion of nodal points.

**Lemma 2.3** *For $p > 1$, we have*

1.

$$Q_1 = \int_0^1 q_1(x) \bigl|\sin_p\bigl(|\kappa_{\pm n}|^{\frac{2}{p}} \vartheta(x, \kappa_{\pm n})\bigr)\bigr|^p \, dx = O\biggl(\frac{1}{n_{\alpha\beta}}\biggr) + o\biggl(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}}}\biggr),$$

$$
\begin{aligned}
Q_0 &= \int_0^1 q_0(x)\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p dx \\
&= \frac{1}{p}\int_0^1 q_0(x)\,dx + O\left(\frac{1}{n_{\alpha\beta}}\right) + o\left(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}}}\right).
\end{aligned}
$$

2. *For* $j = 0, 1$,

$$
\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_j(x)\left[\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p - \frac{1}{p}\right] dx = O\left(\frac{1}{n_{\alpha\beta}^2}\right) + o\left(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}+1}}\right).
$$

*Proof*

1. First, we recall the identities

$$
\left(\sin_p(x)\left|\sin_p'(x)\right|^{p-2}\sin_p'(x)\right)' = 1 - p\left|\sin_p(x)\right|^p \tag{15}
$$

and

$$
\left|\sin_p(x)\right|^p + \left|\sin_p'(x)\right|^p = 1. \tag{16}
$$

Then by (7), (15)-(16), and the general Riemann-Lebesgue lemma [25], Lemma 3.1, we find that

$$
\begin{aligned}
Q_1 &= \int_0^1 q_1(x)\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p dx \\
&= \frac{1}{p}\int_0^1 q_1(x)\,dx + \int_0^1 q_1(x)\left[\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p - \frac{1}{p}\right] dx \\
&= \int_0^1 q_1(x)\left[\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p - \frac{1}{p}\right] \\
&\quad \times \left[\vartheta'(x,\kappa_{\pm n}) + \frac{\kappa_{\pm n} p q_1(x) + q_0(x)}{\kappa_{\pm n}^2}\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p\right] dx \\
&= \frac{1}{|\kappa_{\pm n}|^{\frac{2}{p}}}\int_0^1 q_1(x)\left[\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p - \frac{1}{p}\right]|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta'(x,\kappa_{\pm n})\,dx + o\left(\frac{1}{|\kappa_{\pm n}|}\right) \\
&= \left[-\frac{q_1(x)}{p|\kappa_{\pm n}|^{\frac{2}{p}}}\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\left|\sin_p'\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^{p-2}\right. \\
&\quad \left. \times \sin_p'\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right]_0^1 \\
&\quad + \int_0^1 \frac{q_1'(x)}{p|\kappa_{\pm n}|^{\frac{2}{p}}}\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\left|\sin_p'\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^{p-2} \\
&\quad \times \sin_p'\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right) dx + o\left(\frac{1}{|\kappa_{\pm n}|}\right) \\
&= O\left(\frac{1}{|\kappa_{\pm n}|^{\frac{2}{p}}}\right) + o\left(\frac{1}{|\kappa_{\pm n}|}\right) \\
&= O\left(\frac{1}{n_{\alpha\beta}}\right) + o\left(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}}}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
Q_0 &= \int_0^1 q_0(x)\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p dx \\
&= \frac{1}{p}\int_0^1 q_0(x)\,dx + \int_0^1 q_0(x)\left[\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p - \frac{1}{p}\right] dx \\
&= \frac{1}{p}\int_0^1 q_0(x)\,dx + O\left(\frac{1}{n_{\alpha\beta}}\right) + o\left(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}}}\right).
\end{aligned}
$$

2. Since $\sin_p(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x_{i+1}^{(n)},\kappa_{\pm n})) = 0$, we find that, by a similar argument to part 1,

$$
\begin{aligned}
&\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_j(x)\left[\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p - \frac{1}{p}\right] dx \\
&= \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_j(x)\left[\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p - \frac{1}{p}\right] \\
&\quad \times \left[\vartheta'(x,\kappa_{\pm n}) + \frac{\kappa_{\pm n}pq_1(x)+q_0(x)}{\kappa_{\pm n}^2}\left|\sin_p\left(|\kappa_{\pm n}|^{\frac{2}{p}}\vartheta(x,\kappa_{\pm n})\right)\right|^p\right] dx \\
&= \frac{1}{|\kappa_{\pm n}|^{\frac{2}{p}}} O\left(\ell_i^{(n)}\right) + \frac{1}{\kappa_{\pm n}} o\left(\ell_i^{(n)}\right) = O\left(\frac{1}{n_{\alpha\beta}^2}\right) + o\left(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}+1}}\right), \quad \text{for } j = 0,1.
\end{aligned}
$$

□

By Lemma 2.3, we can reformulate the asymptotic expansion of nodal points whose leading term does not depend on $\kappa_n$.

**Theorem 2.4** *The nodal point satisfies one of the following formulas.*

1. *For* $\alpha = 0$,

$$
\begin{aligned}
x_i^{(n)} &= \frac{i}{n_{\alpha\beta}} \mp \frac{ip\pi_p}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}+1}} Q_1 - \frac{i\pi_p}{(n_{\alpha\beta}\pi_p)^{p+1}}\left(\frac{\operatorname{scot}_p(\beta)}{(p-1)} + Q_0\right) \\
&\quad + \frac{1}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\left[\int_0^{x_i^{(n)}} q_1(x)\,dx + p\int_0^{x_i^{(n)}} q_1(x)\left(\left|\sin_p\left(\kappa_n^{\frac{2}{p}}\vartheta(x,\kappa_n)\right)\right|^p - \frac{1}{p}\right) dx\right] \\
&\quad + \int_0^{x_i^{(n)}} q_1(x)\,dx\left[-\frac{p}{2(p-1)(n_{\alpha\beta}\pi_p)^{\frac{3}{2}p}}\left(\operatorname{scot}_p(\beta) + (p-1)Q_0\right) \mp \frac{p^2}{2(n_{\alpha\beta}\pi_p)^p} Q_1\right] \\
&\quad + \frac{1}{p(n_{\alpha\beta}\pi_p)^p}\int_0^{x_i^{(n)}} q_0(x)\,dx + \frac{1}{(n_{\alpha\beta}\pi_p)^p}\int_0^{x_i^{(n)}} q_0(x)\left(\left|\sin_p\left(\kappa_n^{\frac{2}{p}}\vartheta(x,\kappa_n)\right)\right|^p - \frac{1}{p}\right) dx \\
&\quad + O\left(\frac{1}{n_{\alpha\beta}^{2p}} + \frac{1}{n_{\alpha\beta}^{p+2}}\right).
\end{aligned}
$$

2. *For* $\alpha \neq 0$,

$$
\begin{aligned}
x_i^{(n)} &= \frac{i-\frac{1}{2}}{n_{\alpha\beta}} \mp \frac{p(i-\frac{1}{2})\pi_p}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}+1}} Q_1 - \frac{(i-\frac{1}{2})\pi_p}{(n_{\alpha\beta}\pi_p)^{p+1}}\left(\frac{\operatorname{scot}_p(\beta) - A}{(p-1)} + Q_0\right) - \frac{A}{(p-1)(n_{\alpha\beta}\pi_p)^p} \\
&\quad + \frac{1}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\left[\int_0^{x_i^{(n)}} q_1(x)\,dx + p\int_0^{x_i^{(n)}} q_1(x)\left(\left|\sin_p\left(\kappa_n^{\frac{2}{p}}\vartheta(x,\kappa_n)\right)\right|^p - \frac{1}{p}\right) dx\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^{x_i^{(n)}} q_1(x)\,dx\left[-\frac{p}{2(p-1)(n_{\alpha\beta}\pi_p)^{\frac{3}{2}p}}\bigl(\operatorname{scot}_p(\beta) - A + (p-1)Q_0\bigr)\right. \\
&\left.\mp \frac{p^2}{2(n_{\alpha\beta}\pi_p)^p}Q_1\right] + \frac{1}{p(n_{\alpha\beta}\pi_p)^p}\int_0^{x_i^{(n)}} q_0(x)\,dx \\
&+ \frac{1}{(n_{\alpha\beta}\pi_p)^p}\int_0^{x_i^{(n)}} q_0(x)\left(\bigl|\sin_p\bigl(\kappa_n^{\frac{2}{p}}\vartheta(x,\kappa_n)\bigr)\bigr|^p - \frac{1}{p}\right)dx + O\left(\frac{1}{n_{\alpha\beta}^{2p}} + \frac{1}{n_{\alpha\beta}^{p+2}}\right).
\end{aligned}
$$

## 3 Reconstruction formulas for $1 < p < 2$

In this section, we will derive the reconstruction formula for $\alpha$, $\beta$, $q_1$, and $q_0$. Here, we only consider the case for $\alpha \neq 0$ on the nodal set $\{x_i^{(n)}\}_{n>0}$. The other cases are similar, so we omit them. Note that Lemma 2.3 implies that

$$
Q_1 = o\left(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}}}\right) \quad \text{and} \quad Q_0 = \frac{1}{p}\int_0^1 q_0(x)\,dx + o\left(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}}}\right).
$$

Hence, for $\alpha \neq 0$, the nodal point satisfies

$$
\begin{aligned}
x_i^{(n)} &= \frac{i-\frac{1}{2}}{n_{\alpha\beta}} - \frac{(i-\frac{1}{2})\pi_p}{(n_{\alpha\beta}\pi_p)^{p+1}}\left(\frac{\operatorname{scot}_p(\beta) - A}{(p-1)} + \frac{1}{p}\int_0^1 q_0(x)\,dx\right) - \frac{A}{(p-1)(n_{\alpha\beta}\pi_p)^p} \\
&\quad + \frac{1}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\int_0^{x_i^{(n)}} q_1(x)\,dx + \frac{1}{p(n_{\alpha\beta}\pi_p)^p}\int_0^{x_i^{(n)}} q_0(x)\,dx + o\left(\frac{1}{n_{\alpha\beta}^p}\right).
\end{aligned}
$$

Let $i$ be fixed. Then for $\alpha \neq 0$, $\alpha$ satisfies $A = |\cot_p(\alpha)|^{p-2}\cot_p(\alpha)$ where

$$
A = \begin{cases}
-\lim_{x_i^{(n)}\to 0}(p-1)((n-\frac{1}{2})\pi_p)^p(x_i^{(n)} - \frac{i-\frac{1}{2}}{n-\frac{1}{2}}) & \text{if the limit exists,} \\
-\lim_{x_i^{(n)}\to 0}(p-1)((n-1)\pi_p)^p(x_i^{(n)} - \frac{i-\frac{1}{2}}{n-1}) & \text{if the limit exists.}
\end{cases}
$$

Similarly, for $x_{n-i}^{(n)} \to 1$, we have $\frac{i}{n} \to 0$ and hence

$$
\begin{aligned}
1 - x_{n-i}^{(n)} &= \frac{i-\frac{1}{2}}{n_{\alpha\beta}} + \frac{\operatorname{scot}_p(\beta)}{(p-1)(n_{\alpha\beta}\pi_p)^p} \\
&\quad + \frac{p}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\int_{x_{n-i}^{(n)}}^1 q_1(x)\bigl|\sin_p\bigl(\kappa_n^{\frac{2}{p}}\vartheta(x,\kappa_n)\bigr)\bigr|^p\,dx + o\left(\frac{1}{n_{\alpha\beta}^p}\right),
\end{aligned}
$$

then, for fixed $i$ and $\beta \neq 0$, $\beta$ satisfies $B = |\cot_p(\beta)|^{p-2}\cot_p(\beta)$ where

$$
B = \begin{cases}
\lim_{x_{n-i}^{(n)}\to 1}(p-1)((n-\frac{1}{2})\pi_p)^p(1 - x_{n-i}^{(n)} - \frac{i-\frac{1}{2}}{n-\frac{1}{2}}) & \text{if the limit exists,} \\
\lim_{x_{n-i}^{(n)}\to 1}(p-1)((n-1)\pi_p)^p(1 - x_{n-i}^{(n)} - \frac{i-\frac{1}{2}}{n-1}) & \text{if the limit exists.}
\end{cases}
$$

It should be mentioned that, once the values $\alpha$, $\beta$ are determined, the value $n_{\alpha\beta}$ can be defined by (3).

Next, to reconstruct the functions $q_1$ and $q_0$, we first derive the nodal length, which satisfies the following estimate:

$$\ell_i^{(n)} = \frac{1}{n_{\alpha\beta}} - \frac{\pi_p}{(n_{\alpha\beta}\pi_p)^{p+1}}\left(\frac{\operatorname{scot}_p(\beta)-\operatorname{scot}_p(\alpha)}{(p-1)} + \frac{1}{p}\int_0^1 q_0(x)\,dx\right) + \frac{1}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_1(x)\,dx + \frac{1}{p(n_{\alpha\beta}\pi_p)^p}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_0(x)\,dx + o\left(\frac{1}{n_{\alpha\beta}^{p+1}}\right).$$

Then

$$n_{\alpha\beta}(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}\left(\ell_i^{(n)} - \frac{1}{n_{\alpha\beta}}\right) = n_{\alpha\beta}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_1(x)\,dx + o(1).$$

Because the sequence of intervals $\{[x_j^{(n)}, x_{j+1}^{(n)}) : n \text{ is sufficiently large}\}$ shrinks to $x$ nicely (*cf.* Rudin [26], p.140), we conclude that $n_{\alpha\beta}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_1(t)\,dt \to q_1(x)$ a.e. $x \in (0,1)$. In particular, applying the Lebesgue dominated convergence theorem, $n_{\alpha\beta}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_1(t)\,dt \to q_1(x)$ in $L^1(0,1)$. Hence

$$q_1(x) = \lim_{n\to\infty} n_{\alpha\beta}(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}\left(\ell_j^{(n)} - \frac{1}{n_{\alpha\beta}}\right).$$

Note that the function $q_1$ also can be reconstructed from the nodal point by the following procedure:

$$\begin{aligned}
&(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}\left(x_i^{(n)} - \frac{i-\frac{1}{2}}{n_{\alpha\beta}}\right) \\
&\quad = -\frac{1}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\frac{i-\frac{1}{2}}{n_{\alpha\beta}}\left(\frac{\operatorname{scot}_p(\beta)-\operatorname{scot}_p(\alpha)}{(p-1)} + \frac{1}{p}\int_0^1 q_0(x)\,dx\right) - \frac{A}{(p-1)(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}} \\
&\qquad + \int_0^{x_i^{(n)}} q_1(x)\,dx + \frac{1}{p(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\int_0^{x_i^{(n)}} q_0(x)\,dx + o\left(\frac{1}{n_{\alpha\beta}^{\frac{p}{2}}}\right).
\end{aligned}$$

Define

$$f(x) \equiv \lim_{x_i^{(n)}\to x} (n_{\alpha\beta}\pi_p)^{\frac{p}{2}}\left(x_i^{(n)} - \frac{i-\frac{1}{2}}{n_{\alpha\beta}}\right) = \int_0^x q_1(t)\,dt.$$

Then $q_1(x) = f'(x)$.

Finally, to reconstruct the function $q_0$, we observe that the asymptotic expansion of nodal length implies

$$\begin{aligned}
&pn_{\alpha\beta}(n_{\alpha\beta}\pi_p)^p\left(\ell_i^{(n)} - \frac{1}{n_{\alpha\beta}} - \frac{1}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_1(x)\,dx\right) \\
&\quad = -p\left(\frac{\operatorname{scot}_p(\beta)-\operatorname{scot}_p(\alpha)}{(p-1)} + \frac{1}{p}\int_0^1 q_0(x)\,dx\right) + n_{\alpha\beta}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_0(x)\,dx + o(1).
\end{aligned}$$

Hence, we find

$$q_0(x) = \lim_{n\to\infty} pn_{\alpha\beta}(n_{\alpha\beta}\pi_p)^p\left(\ell_j^{(n)} - \frac{1}{n_{\alpha\beta}} - \frac{1}{(n_{\alpha\beta}\pi_p)^{\frac{p}{2}}}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_1(x)\,dx\right)$$
$$+ \frac{p}{p-1}\big(\operatorname{scot}_p(\beta) - \operatorname{scot}_p(\alpha)\big) + \int_0^1 q_0(x)\,dx.$$

## 4 Reconstruction formulas for $p \geq 2$

In this section, we only consider the Dirichlet case. First, we can obtain the result that the nodal length satisfies the following estimate:

$$\ell_i^{(n)} = \frac{1}{n} - \frac{p\pi_p}{(n\pi_p)^{\frac{p}{2}+1}}Q_1 - \frac{\pi_p}{(n\pi_p)^{p+1}}Q_0 + \frac{1}{(n\pi_p)^{\frac{p}{2}}}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_1(x)\,dx$$
$$+ \frac{p}{(n\pi_p)^{\frac{p}{2}}}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_1(x)\left(\left|\sin_p\left(\kappa_n^{\frac{2}{p}}\vartheta(x,\kappa_n)\right)\right|^p - \frac{1}{p}\right)dx$$
$$+ \frac{1}{p(n\pi_p)^p}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_0(x)\,dx + o\left(\frac{1}{n^{p+1}}\right).$$

Then the function $q_1(x)$ can be reconstructed from the nodal length immediately:

$$q_1(x) = \lim_{n\to\infty} n(n\pi_p)^{\frac{p}{2}}\left(\ell_j^{(n)} - \frac{1}{n}\right).$$

To reconstruct the function $q_0(x)$, we shall first find $Q_1$. So, we need to refine the estimate of $Q_1$ mentioned in Lemma 2.3. Since

$$\vartheta(x,\kappa_n) - \vartheta(0,\kappa_n) = \int_0^x \vartheta'(x,\kappa_n)\,dx = x - \int_0^x \frac{\kappa_n p q_1(x) + q_0(x)}{\kappa_n^2}\left|\sin_p\left(|\kappa_n|^{\frac{2}{p}}\vartheta(x,\kappa_n)\right)\right|^p dx,$$

we find

$$\kappa_n^{\frac{2}{p}}\vartheta(x,\kappa_n) = \kappa_n^{\frac{2}{p}}x - \int_0^x \frac{\kappa_n p q_1(x) + q_0(x)}{\kappa_n^{2-\frac{2}{p}}}\left|\sin_p\left(|\kappa_n|^{\frac{2}{p}}\vartheta(x,\kappa_n)\right)\right|^p dx$$
$$= n\pi_p x + \frac{p}{\kappa_n^{1-\frac{2}{p}}}Q_1 x + \frac{1}{\kappa_n^{2-\frac{2}{p}}}Q_0 x$$
$$- \int_0^x \frac{\kappa_n p q_1(x) + q_0(x)}{\kappa_n^{2-\frac{2}{p}}}\left|\sin_p\left(|\kappa_n|^{\frac{2}{p}}\vartheta(x,\kappa_n)\right)\right|^p dx + O\left(\frac{1}{\kappa_n^{4-\frac{2}{p}}}\right).$$

Applying the Taylor expansion, we have

$$\sin_p\left(|\kappa_n|^{\frac{2}{p}}\vartheta(x,\kappa_n)\right)$$
$$= \sin_p(n\pi_p x) + \sin_p'(n\pi_p x)\left[|\kappa_n|^{\frac{2}{p}}\vartheta(x,\kappa_n) - n\pi_p x\right] + O\left(|\kappa_n|^{\frac{2}{p}}\vartheta(x,\kappa_n) - n\pi_p x\right)^2$$
$$= \sin_p(n\pi_p x) + \sin_p'(n\pi_p x)\Bigg[\frac{pQ_1 x}{\kappa_n^{1-\frac{2}{p}}} + \frac{Q_0 x}{\kappa_n^{2-\frac{2}{p}}}$$

$$
\begin{aligned}
&- \int_0^x \frac{\kappa_n p q_1(x) + q_0(x)}{\kappa_n^{2-\frac{2}{p}}} \left|\sin_p\left(|\kappa_n|^{\frac{2}{p}} \vartheta(x,\kappa_n)\right)\right|^p dx\Bigg] + O\left(\frac{1}{n^p}\right) \\
&= \sin_p(n\pi_p x) + \frac{p \sin_p'(n\pi_p x)}{(n\pi_p)^{\frac{p}{2}-1}} \left[ Q_1 x - \int_0^x q_1(x) \left|\sin_p\left(|\kappa_n|^{\frac{2}{p}} \vartheta(x,\kappa_n)\right)\right|^p dx\right] + o\left(\frac{1}{n^{\frac{p}{2}-1}}\right) \\
&= \sin_p(n\pi_p x) + \frac{p \sin_p'(n\pi_p x)}{(n\pi_p)^{\frac{p}{2}-1}} \left[ x \int_0^1 q_1(x) \left|\sin_p(n\pi_p x)\right|^p dx \right.\\
&\quad \left. - \int_0^x q_1(x) \left|\sin_p(n\pi_p x)\right|^p dx\right] + o\left(\frac{1}{n^{\frac{p}{2}-1}}\right).
\end{aligned}
$$

Note that $\sin_p(|\kappa_n|^{\frac{2}{p}} \vartheta(0,\kappa_n)) = \sin_p(|\kappa_n|^{\frac{2}{p}} \vartheta(1,\kappa_n)) = 0$. Then we can refine the estimate of $Q_1$ mentioned in Lemma 2.3. More precisely, we have

$$
\begin{aligned}
Q_1 &= \frac{1}{pn\pi_p} \int_0^1 q_1'(x) \sin_p(n\pi_p x) \left|\sin_p'(n\pi_p x)\right|^{p-2} \sin_p'(n\pi_p x)\, dx \\
&\quad + \frac{1}{(n\pi_p)^{\frac{p}{2}}} \int_0^1 q_1'(x) \left[ x \int_0^1 q_1(t) \left|\sin_p(n\pi_p t)\right|^p dt \right.\\
&\quad \left. - \int_0^x q_1(t) \left|\sin_p(n\pi_p t)\right|^p dt\right] \left|\sin_p'(n\pi_p x)\right|^p dx + o\left(\frac{1}{n^{\frac{p}{2}}}\right).
\end{aligned}
$$

Moreover, the nodal length satisfies

$$
\begin{aligned}
\ell_i^{(n)} &= \frac{1}{n} + \frac{1}{(n\pi_p)^{\frac{p}{2}}} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_1(x)\, dx + \frac{p}{(n\pi_p)^{\frac{p}{2}}} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_1(x) \left( \left|\sin_p(n\pi_p x)\right|^p - \frac{1}{p} \right) dx \\
&\quad - \frac{\pi_p}{(n\pi_p)^{\frac{p}{2}+2}} \int_0^1 q_1'(x) \sin_p(n\pi_p x) \left|\sin_p'(n\pi_p x)\right|^{p-2} \sin_p'(n\pi_p x)\, dx \\
&\quad - \frac{\pi_p}{p(n_{\alpha\beta}\pi_p)^{p+1}} \int_0^1 q_0(x)\, dx + \frac{1}{p(n\pi_p)^p} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} q_0(x)\, dx + o\left(\frac{1}{n^{p+1}}\right).
\end{aligned}
$$

Hence, we can reconstruct $q_0$ by the following formula:

$$
\begin{aligned}
q_0(x) &= \lim_{n\to\infty} pn(n\pi_p)^p \left\{ \ell_j^{(n)} - \frac{1}{n} - \frac{1}{(n\pi_p)^{\frac{p}{2}}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_1(x)\, dx \right.\\
&\quad - \frac{p}{(n\pi_p)^{\frac{p}{2}}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_1(x) \left( \left|\sin_p(n\pi_p x)\right|^p - \frac{1}{p} \right) dx \\
&\quad \left. + \frac{\pi_p}{(n\pi_p)^{\frac{p}{2}+2}} \int_0^1 q_1'(x) \sin_p(n\pi_p x) \left|\sin_p'(n\pi_p x)\right|^{p-2} \sin_p'(n\pi_p x)\, dx \right\} + \int_0^1 q_0(x)\, dx.
\end{aligned}
$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The author, Y.H. Cheng, was a major contributor in writing the manuscript. The author, W.C. Wang, performed the literature review. Both of them performed the final editing of the manuscript. They also give final approval of the version to be submitted and any revised version.

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