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Positive solutions of higher-order Sturm-Liouville boundary value problems with derivative-dependent nonlinear terms

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Abstract

We consider the Sturm-Liouville boundary value problem

$$\begin{cases} y^{(m)}(t) + F(t, y(t), y'(t), \dots, y^{(q)}(t)) = 0, & t \in [0,1], \\ y^{(k)}(0) = 0, \quad 0 \le k \le m-3, \\ \zeta y^{(m-2)}(0) - \theta y^{(m-1)}(0) = 0, & \rho y^{(m-2)}(1) + \delta y^{(m-1)}(1) = 0, \end{cases}$$

where $m \ge 3$ and $1 \le q \le m-2$. We note that the nonlinear term F involves derivatives. This makes the problem challenging, and such cases are seldom investigated in the literature. In this paper we develop a *new* technique to obtain existence criteria for one or multiple positive solutions of the boundary value problem. Several examples with *known* positive solutions are presented to dwell upon the usefulness of the results obtained.

MSC: 34B15

Keywords: positive solutions; Sturm-Liouville boundary value problems; derivative-dependent

1 Introduction

In this paper we consider the higher-order Sturm-Liouville boundary value problem

$$\begin{cases} y^{(m)}(t) + F(t, y(t), y'(t), \dots, y^{(q)}(t)) = 0, & t \in [0,1], \\ y^{(k)}(0) = 0, \quad 0 \le k \le m-3, \\ \zeta y^{(m-2)}(0) - \theta y^{(m-1)}(0) = 0, & \rho y^{(m-2)}(1) + \delta y^{(m-1)}(1) = 0, \end{cases} \tag{1.1}$$

where $m \ge 3$, $1 \le q \le m-2$, and F is continuous at least in the domain of interest. The constants ζ, θ, ρ, and δ are such that

$$\theta \ge 0, \qquad \delta \ge 0, \qquad \theta + \zeta > 0, \qquad \delta + \rho > 0, \qquad \kappa \equiv \zeta\rho + \zeta\delta + \theta\rho > 0. \tag{1.2}$$

These assumptions allow ζ and ρ to be negative.

There is a vast amount of research done on the existence of positive solutions of Sturm-Liouville boundary value problems. The many interests in (1.1) may stem from the fact that boundary value problems of type (1.1) model various dynamic systems with m degrees of freedom in which m states are observed at m times; see Meyer [1]. For example, when $m = 2$, the boundary value problem (1.1) describes a vast spectrum of physical phenomena such as gas diffusion through porous media, diffusion of heat generated by positive temperature-dependent sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactors, fluid dynamics, electrical potential theory, combustion theory, steady-state of oxygen diffusion in a cell with Michaelis-Menten kinetics, cell membrane, and heat conduction in the human brain; see [2–8]. Singular boundary value problems of particular and related cases of (1.1) have also been the subject matter of many papers; see [9–15]. For recent developments in (1.1) and other types of boundary value problems, the reader is referred to the monographs [16, 17] and the hundreds of references cited therein. Note that in most of these investigations the nonlinear terms considered do *not* involve derivatives of the dependent variable, and only a relatively small number of papers tackle nonlinear terms that involve derivatives, of which we mention some below.

Fink [18] has studied the radial symmetric form of the semilinear elliptic equation $\Delta y + \lambda q(|x|)f(y) = 0$ in $\mathbb{R}^N$, which turns out to be a particular second-order Sturm Liouville eigenvalue problem that has y' in the nonlinear term, *viz.*,

$$\begin{cases} y'' + \frac{N-1}{t}y' + \lambda q(t)f(y) = 0, & t \in (0,1), \\ y'(0) = y(1) = 0. \end{cases}$$

Later, Wong [19] has considered (1.1) when $q = m - 2$ and obtained the existence of a solution (not necessarily positive) by assuming that (1.1) has lower and upper solutions v and w such that $v^{(m-2)}(t) \le w^{(m-2)}(t)$ on $[0,1]$,

$$F\bigl(t, v(t), \ldots, v^{(m-3)}(t), u_{m-1}\bigr) \le F(t, u_1, \ldots, u_{m-2}, u_{m-1}) \le F\bigl(t, w(t), \ldots, w^{(m-3)}(t), u_{m-1}\bigr)$$

for $t \in [0,1]$, and $(v(t), \ldots, v^{(m-3)}(t)) \le (u_1, \ldots, u_{m-2}) \le (w(t), \ldots, w^{(m-3)}(t))$. A few years later, Grossinho and Minhós [20] established the existence of a solution to a related problem of (1.1) when $q = m - 1$; their method requires again the assumption of lower and upper solutions, and, in addition, F must satisfy the *Nagumo-type condition* on some set $A \subset [0,1] \times \mathbb{R}^m$, *viz.*,

$$\begin{cases} \text{there exists a continuous function } h : [0,\infty) \to (0,\infty) \text{ such that} \\ |F(t, u_1, \ldots, u_m)| \le h(|u_m|), \quad (t, u_1, \ldots, u_m) \in A; \\ \int_0^\infty \frac{s}{h(s)}\,ds = \infty. \end{cases}$$

For infinite interval problems, Lian *et al.* [21, 22] have investigated the following problem:

$$\begin{cases} -y^{(m)}(t) = h(t)f(t, y(t), y'(t), \ldots, y^{(m-1)}(t)), & t \in (0,\infty), \\ y^{(k)}(0) = A_k, \quad 0 \le k \le m-3, \\ y^{(m-2)}(0) - ay^{(m-1)}(0) = B, \qquad y^{(m-1)}(\infty) = C. \end{cases}$$

Here, once again, the method of lower and upper solutions is used, and a Nagumo-type condition plays an important role in handling the derivatives in the nonlinear term. A relatively small number of papers on problems involving derivative-dependent nonlinearities indicates that problems of this type are more difficult to tackle analytically; we note, however, that numerical methods are more developed for this type of problems; see, for example, [23–28].

Motivated by the research mentioned, in the current work we develop a different and new technique to tackle the boundary value problem (1.1). Note that our technique requires *neither* the existence of lower and upper solutions *nor* a Nagumo-type condition; both of these conditions are not easy to check in practical applications.

The focus of this paper is on the existence of one or more positive solutions of (1.1). By a *positive solution* y of (1.1) we mean $y \in C^{(m)}[0,1]$ satisfying (1.1) and $y(t) \geq 0$ for $t \in [0,1]$. By using a variety of fixed point theorems we begin with the establishment of the existence of a solution (not necessary positive) and proceed to the existence of a nontrivial positive solution, two nontrivial positive solutions, and multiple nontrivial positive solutions. Due to the presence of derivatives in the nonlinear term, our work naturally generalizes and extends the known results for Sturm-Liouville boundary value problems [18, 29–36] and complements the work of many authors [19, 20, 37–46]. We remark that our conditions/assumptions, which do not involve lower and upper solutions and a Nagumo-type condition, are comparatively easy to check. We illustrate this practical usefulness by examples with known positive solutions.

The paper is organized as follows. In Section 2 we state the fixed point theorems and present some properties of a certain Green's function. The new technique and various existence criteria are developed in Section 3. Finally, in Section 4 we illustrate the usefulness of the results obtained by some examples. We remark that in all the examples, *known* positive solutions are given to validate the conclusions derived from the theorems.

2 Preliminaries

In this section, we state the fixed point theorems and some inequalities for certain Green's function. The first theorem is known as the *Leray-Schauder alternative*, and the second is usually called *Krasnosel'skii's fixed point theorem in a cone.*

Theorem 2.1 (Leray-Schauder alternative) [16] *Let B be a Banach space with $E \subseteq B$ closed and convex. Let U be a relatively open subset of E with $0 \in U$, and $S : \overline{U} \to E$ be a continuous and compact map. Then either*

(a) *S has a fixed point in $\overline{U}$, or*

(b) *there exist $x \in \partial U$ and $\lambda \in (0,1)$ such that $x = \lambda Sx$.*

Theorem 2.2 (Krasnosel'skii's fixed point theorem in a cone) [47] *Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C \subset B$ be a cone in B. Let Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$ be a completely continuous operator such that either*

(a) *$\|Sx\| \leq \|x\|$, $x \in C \cap \partial\Omega_1$, and $\|Sx\| \geq \|x\|$, $x \in C \cap \partial\Omega_2$, or*

(b) *$\|Sx\| \geq \|x\|$, $x \in C \cap \partial\Omega_1$, and $\|Sx\| \leq \|x\|$, $x \in C \cap \partial\Omega_2$.*

Then S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let $G(t,s)$ be the Green's function of the *second-order* Sturm-Liouville boundary value problem

$$\begin{cases} -w''(t) = 0, \quad t \in (0,1), \\ \zeta w(0) - \theta w'(0) = 0, \qquad \rho w(1) + \delta w'(1) = 0. \end{cases} \tag{2.1}$$

It is known that [33, 35, 36]

$$G(t,s) = \frac{1}{\kappa}\begin{cases}(\theta+\zeta s)[\delta+\rho(1-t)], & 0 \le s \le t \le 1,\\ (\theta+\zeta t)[\delta+\rho(1-s)], & 0 \le t \le s \le 1.\end{cases} \tag{2.2}$$

Lemma 2.3 [33, 35, 36] *The Green's function $G(t,s)$ has the following properties*:

(a) $G(t,s) \ge 0$ *for* $(t,s) \in [0,1] \times [0,1]$ *and* $G(t,s) > 0$ *for* $(t,s) \in (0,1) \times (0,1)$.

(b) $G(t,s) \le LG(s,s)$ *for* $(t,s) \in [0,1] \times [0,1]$, *where*

$$L = \max\left\{1, \frac{\theta}{\theta+\zeta}, \frac{\delta}{\delta+\rho}\right\}.$$

(c) $G(t,s) \ge K_\eta G(s,s)$ *for* $(t,s) \in [\eta, 1-\eta] \times [0,1]$, *where* $\eta \in (0, \frac{1}{2})$ *is fixed, and*

$$K_\eta = \min\left\{\frac{\delta+\rho\eta}{\delta+\rho}, \frac{\delta+\rho(1-\eta)}{\delta+\rho\eta}, \frac{\theta+\zeta\eta}{\theta+\zeta}, \frac{\theta+\zeta(1-\eta)}{\theta+\zeta\eta}\right\}.$$

(d) $g_n(t,s)$, *defined by the relation* $\frac{\partial^{n-2}}{\partial t^{n-2}} g_n(t,s) = G(t,s)$, *is the Green's function of the nth-order Sturm-Liouville boundary value problem*

$$\begin{cases}-w^{(n)}(t) = 0, & t \in (0,1),\\ w^{(k)}(0) = 0, & 0 \le k \le n-3,\\ \zeta w^{(n-2)}(0) - \theta w^{(n-1)}(0) = 0, & \rho w^{(n-2)}(1) + \delta w^{(n-1)}(1) = 0.\end{cases} \tag{2.3$_n$}$$

(e) $0 \le g_n(t,s) \le \frac{L}{(n-2)!} G(s,s)$ *for* $(t,s) \in [0,1] \times [0,1]$.

3 Positive solutions of (1.1)

In this section, we establish criteria for the existence of one, two, or multiple nontrivial positive solutions of (1.1).

We rewrite (1.1) in a form suitable for investigation. To begin, we consider the initial value problem

$$\begin{cases}y^{(q)}(t) = x(t), & t \in [0,1],\\ y(0) = y'(0) = y''(0) = \cdots = y^{(q-1)}(0) = 0.\end{cases} \tag{3.1}$$

Due to the initial conditions in (3.1), it is clear that

$$y^{(k)}(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{q-k-1}} x(s_{q-k})\, ds_{q-k} \cdots ds_1, \quad 0 \le k \le q-1. \tag{3.2}$$

We introduce the notation of the k-tuple integral

$$J^k x(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{k-1}} x(s_k)\, ds_k \cdots ds_1, \quad k \ge 1.$$

Then, it follows from (3.1) and (3.2) that

$$y^{(k)}(t) = J^{q-k} x(t), \quad 0 \le k \le q, \tag{3.3}$$

where $J^0 x(t) \equiv x(t)$.

Denote $\tilde{J}x(t) = (J^q x(t), J^{q-1}x(t), \dots, Jx(t), x(t))$. Noting (3.1) and (3.3), we rewrite (1.1) as the following $(m-q)$th-order Sturm-Liouville boundary value problem:

$$\begin{cases} x^{(m-q)}(t) + F(t, \tilde{J}x(t)) = 0, \quad t \in [0,1], \\ x^{(k)}(0) = 0, \quad 0 \le k \le m-q-3, \\ \zeta x^{(m-q-2)}(0) - \theta x^{(m-q-1)}(0) = 0, \qquad \rho x^{(m-q-2)}(1) + \delta x^{(m-q-1)}(1) = 0. \end{cases} \tag{3.4}$$

If (3.4) has a solution x^*, then the boundary value problem (1.1) has a solution y^* given by

$$y^{*(k)}(t) = J^{q-k} x^*(t), \quad 0 \le k \le q, \tag{3.5}$$

and, in particular,

$$y^*(t) = J^q x^*(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{q-1}} x^*(s_q)\, ds_q \cdots ds_1. \tag{3.6}$$

Hence, the existence of a solution of (1.1) follows from the existence of a solution of (3.4). Further, it is obvious from (3.5) that for $0 \le k \le q$, $y^{*(k)}$ is positive if x^* is, and $y^{*(k)}$ is nontrivial if x^* is. We study (1.1) via (3.4) and employ a new technique to tackle the nonlinear term F.

Let the Banach space

$$B = \left\{ x \in C^{(m-q)}[0,1] \mid x^{(k)}(0) = 0, 0 \le k \le m-q-3 \right\}$$

be equipped with the norm

$$\|x\| = \sup_{t\in[0,1]} \left| x^{(m-q-2)}(t) \right|.$$

Throughout the paper, let $\eta \in (0, \frac{1}{2})$ be fixed. Define the cone C in B by

$$C = \left\{ x \in B \;\middle|\; x^{(m-q-2)}(t) \ge 0, t \in [0,1]; \min_{t\in[\eta,1-\eta]} x^{(m-q-2)}(t) \ge \gamma \|x\| \right\}, \tag{3.7}$$

where $\gamma = K_\eta / L$ (L and K_η are defined in Lemma 2.3).

Lemma 3.1 [35, 36] *Let $x \in B$. For $0 \le i \le m-q-2$, we have*

$$\left| x^{(i)}(t) \right| \le \frac{t^{m-q-2-i}}{(m-q-2-i)!} \|x\|, \quad t \in [0,1]. \tag{3.8}$$

In particular,

$$\left| x(t) \right| \le \frac{1}{(m-q-2)!} \|x\|, \quad t \in [0,1]. \tag{3.9}$$

Lemma 3.2 [35, 36] *Let $x \in C$. For $0 \le i \le m-q-2$, we have*

$$x^{(i)}(t) \ge 0, \quad t \in [0,1], \tag{3.10}$$

and

$$x^{(i)}(t) \geq (t-\eta)^{m-q-2-i} \frac{\gamma}{(m-q-2-i)!} \|x\|, \quad t \in [\eta, 1-\eta]. \tag{3.11}$$

In particular, for fixed $z \in (\eta, 1-\eta)$, we have

$$x(t) \geq (z-\eta)^{m-q-2} \frac{\gamma}{(m-q-2)!} \|x\|, \quad t \in [z, 1-\eta]. \tag{3.12}$$

Remark 3.1

(a) A solution y^* of (1.1) can be obtained via (3.6), where x^* is a solution of (3.4). In view of (3.5), if x^* is nontrivial/positive, then so is $y^{*(k)}$, $0 \leq k \leq q$.

(b) If $x^* \in C$ is a solution of (3.4), then (3.10) implies that x^* is a positive solution of (3.4).

The next result is useful in handling the nonlinear term F.

Lemma 3.3

(a) *Let $x \in B$. For $1 \leq k \leq q$, we have*

$$\left|J^k x(t)\right| \leq \frac{t^{m-q-2+k}}{(m-q-2+k)!} \|x\| \leq \frac{1}{(m-q-2+k)!} \|x\|, \quad t \in [0,1]. \tag{3.13}$$

(b) *Let $x \in C$ and $z \in (\eta, 1-\eta)$ be fixed. For $1 \leq k \leq q$, we have*

$$J^k x(t) \geq (z-\eta)^{m-q-2+k} \frac{\gamma}{(m-q-2+k)!} \|x\|, \quad t \in [z, 1-\eta]. \tag{3.14}$$

Proof (a) Since $x \in B$, using $(3.8)|_{i=0}$, we obtain that, for $1 \leq k \leq q$ and $t \in [0,1]$,

$$\begin{aligned} \left|J^k x(t)\right| &\leq \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{k-1}} \left|x(s_k)\right| ds_k \cdots ds_1 \\ &\leq \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{k-1}} \frac{s_k^{m-q-2} \|x\|}{(m-q-2)!} ds_k \cdots ds_1 \\ &= \frac{t^{m-q-2+k} \|x\|}{(m-q-2+k)!} \leq \frac{\|x\|}{(m-q-2+k)!}. \end{aligned}$$

(b) Since $x \in C$, using $(3.11)|_{i=0}$, we find that, for $1 \leq k \leq q$ and $t \in [z, 1-\eta]$,

$$\begin{aligned} J^k x(t) &= \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{k-1}} x(s_k)\, ds_k \cdots ds_1 \\ &\geq \int_\eta^z \int_\eta^{s_1} \int_\eta^{s_2} \cdots \int_\eta^{s_{k-1}} x(s_k)\, ds_k \cdots ds_1 \\ &\geq \int_\eta^z \int_\eta^{s_1} \int_\eta^{s_2} \cdots \int_\eta^{s_{k-1}} (s_k-\eta)^{m-q-2} \frac{\gamma \|x\|}{(m-q-2)!} ds_k \cdots ds_1 \\ &= (z-\eta)^{m-q-2+k} \frac{\gamma \|x\|}{(m-q-2+k)!}. \end{aligned}$$

□

The next result gives the estimate of $y^* = J^q x^*$ in terms of $\|x^*\|$.

Lemma 3.4 *Let x^* and y^* be related by* (3.5) *and* (3.6).

(a) *Let $x^* \in B$. For $0 \le k \le m-2$, we have*

$$\left|y^{*(k)}(t)\right| \le \frac{t^{m-k-2}}{(m-k-2)!}\left\|x^*\right\| \le \frac{1}{(m-k-2)!}\left\|x^*\right\|, \quad t \in [0,1]. \tag{3.15}$$

(b) *Let $x^* \in C$. For $0 \le k \le m-2$, we have*

$$y^{*(k)}(t) \ge (t-\eta)^{m-k-2}\frac{\gamma}{(m-k-2)!}\left\|x^*\right\|, \quad t \in [\eta, 1-\eta]. \tag{3.16}$$

Proof (a) Since $x^* \in B$, using (3.5) and (3.13), for $0 \le k \le q-1$, we obtain

$$\left|y^{*(k)}(t)\right| = \left|J^{q-k}x^*(t)\right| \le \frac{t^{m-k-2}\left\|x^*\right\|}{(m-k-2)!} \le \frac{\|x^*\|}{(m-k-2)!}, \quad t \in [0,1].$$

Further, since $y^{*(q)}(t) = x^*(t)$, we have $y^{*(q+i)}(t) = x^{*(i)}(t)$ for $0 \le i \le m-q-2$, and so from (3.8) it follows that

$$\begin{aligned}&\left|y^{*(q+i)}(t)\right| = \left|x^{*(i)}(t)\right| \le \frac{t^{m-q-2-i}\left\|x^*\right\|}{(m-q-2-i)!} \le \frac{\|x^*\|}{(m-q-2-i)!},\\ &\quad t \in [0,1], 0 \le i \le m-q-2,\end{aligned}$$

which is the same as

$$\left|y^{*(k)}(t)\right| \le \frac{t^{m-k-2}\|x^*\|}{(m-k-2)!} \le \frac{\|x^*\|}{(m-k-2)!}, \quad t \in [0,1], q \le k \le m-2.$$

Combining this with the inequality obtained earlier, we get (3.15).

(b) Since $x^* \in C$, noting $(3.11)|_{i=0}$, we find that, for $0 \le k \le q-1$ and $t \in [\eta, 1-\eta]$,

$$\begin{aligned} y^{*(k)}(t) &= J^{q-k}x^*(t) = \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{q-k-1}} x^*(s_{q-k})\, ds_{q-k} \cdots ds_1 \\ &\ge \int_\eta^t \int_\eta^{s_1} \int_\eta^{s_2} \cdots \int_\eta^{s_{q-k-1}} x^*(s_{q-k})\, ds_{q-k} \cdots ds_1 \\ &\ge \int_\eta^t \int_\eta^{s_1} \int_\eta^{s_2} \cdots \int_\eta^{s_{q-k-1}} (s_{q-k}-\eta)^{m-q-2}\frac{\gamma\|x^*\|}{(m-q-2)!}\, ds_{q-k} \cdots ds_1 \\ &= (t-\eta)^{m-k-2}\frac{\gamma\|x^*\|}{(m-k-2)!}. \end{aligned}$$

Next, since $y^{*(q)}(t) = x^*(t)$, we have $y^{*(q+i)}(t) = x^{*(i)}(t)$ for $0 \le i \le m-q-2$, and so from (3.11) we have

$$y^{*(q+i)}(t) = x^{*(i)}(t) \ge \frac{(t-\eta)^{m-q-2-i}\gamma\|x^*\|}{(m-q-2-i)!}, \quad t \in [\eta, 1-\eta], 0 \le i \le m-q-2,$$

or, equivalently,

$$y^{*(k)}(t) \ge \frac{(t-\eta)^{m-k-2}\gamma\|x^*\|}{(m-k-2)!}, \quad t \in [\eta, 1-\eta], q \le k \le m-2.$$

A combination with the earlier inequality yields (3.16). □

Let the operator $S : B \to B$ be defined by

$$Sx(t) = \int_0^1 g_{m-q}(t,s)F\bigl(s,\tilde{J}x(s)\bigr)\,ds, \quad t \in [0,1]. \tag{3.17}$$

Noting that $g_{m-q}(t,s)$ is the Green's function of $(2.3)_{m-q}$ (see Lemma 2.3(d)), it is clear that a fixed point of S is a solution of (3.4). Moreover, (3.17) is equivalent to

$$(Sx)^{(m-q-2)}(t) = \int_0^1 G(t,s)F\bigl(s,\tilde{J}x(s)\bigr)\,ds, \quad t \in [0,1], \tag{3.18}$$

where $G(t,s)$ is the Green's function of (2.1). In view of Remark 3.1, to obtain a positive solution of (1.1), we shall seek a fixed point of the operator S in the cone C.

For easy reference, the conditions that will be used further are listed below. In these conditions, the sets K and $\tilde{K}$ are defined respectively by

$$\tilde{K} = \bigl\{u \in C[0,1] \mid u(t) \geq 0, t \in [0,1]\bigr\}$$

and

$$K = \bigl\{u \in \tilde{K} \mid u(t) > 0 \text{ on some subset of } [0,1] \text{ of positive measure}\bigr\}.$$

(A1) F is continuous on $[0,1] \times \tilde{K}^{q+1}$ with

$$F(t,u_1,\ldots,u_{q+1}) \geq 0, \quad (t,u_1,\ldots,u_{q+1}) \in [0,1] \times \tilde{K}^{q+1},$$

and

$$F(t,u_1,\ldots,u_{q+1}) > 0, \quad (t,u_1,\ldots,u_{q+1}) \in [0,1] \times K^{q+1}.$$

(A2) There exist continuous functions $\beta : [0,1] \to [0,\infty)$ and $f : [0,\infty)^{q+1} \to [0,\infty)$ such that f is nondecreasing in each of its arguments and

$$F(t,u_1,\ldots,u_{q+1}) \leq \beta(t)f(u_1,\ldots,u_{q+1}), \quad (t,u_1,\ldots,u_{q+1}) \in [0,1] \times \tilde{K}^{q+1}.$$

(A3) There exists $a > 0$ such that

$$a > Mf\left(\frac{a}{(m-2)!}, \frac{a}{(m-3)!}, \ldots, \frac{a}{(m-q-2)!}\right),$$

where $M = \sup_{t\in[0,1]} \int_0^1 G(t,s)\beta(s)\,ds$.

(A4) Let $z \in (\eta, 1-\eta)$ be fixed. There exists a continuous function $\alpha : [z, 1-\eta] \to (0,\infty)$ such that

$$F(t,u_1,\ldots,u_{q+1}) \geq \alpha(t)f(u_1,\ldots,u_{q+1}), \quad (t,u_1,\ldots,u_{q+1}) \in [z,1-\eta] \times K^{q+1}.$$

(A5) Let $z \in (\eta, 1-\eta)$ be fixed. There exists $b > 0$ such that

$$b \le Nf\left(\frac{(z-\eta)^{m-2}\gamma b}{(m-2)!}, \frac{(z-\eta)^{m-3}\gamma b}{(m-3)!}, \ldots, \frac{(z-\eta)^{m-q-2}\gamma b}{(m-q-2)!}\right),$$

where $N = \sup_{t\in[0,1]} \int_z^{1-\eta} G(t,s)\alpha(s)\,ds$ and $\gamma = K_\eta/L$.

Remark 3.2 The computation of the constants M and N in (A3) and (A5) can be avoided by using some upper bound of M and some lower bound of N. As a consequence, *stricter* inequalities are obtained. Indeed, using Lemma 2.3, we have

$$M = \sup_{t\in[0,1]} \int_0^1 G(t,s)\beta(s)\,ds \le \int_0^1 LG(s,s)\beta(s)\,ds \equiv M'$$

and

$$\begin{aligned} N &= \sup_{t\in[0,1]} \int_z^{1-\eta} G(t,s)\alpha(s)\,ds \ge \sup_{t\in[\eta,1-\eta]} \int_z^{1-\eta} G(t,s)\alpha(s)\,ds \\ &\ge \int_z^{1-\eta} K_\eta G(s,s)\alpha(s)\,ds \equiv N'. \end{aligned}$$

Let (A3)$'$ denote condition (A3) with M replaced by M', and (A5)$'$ denote condition (A5) with N replaced by N'. Obviously, (A3) is satisfied if the stronger condition (A3)$'$ is met; likewise, (A5) is satisfied if the stronger condition (A5)$'$ holds.

The first result below gives the existence of a solution, which may *not* be positive.

Theorem 3.5 *Let $F : [0,1] \times \mathbb{R}^{q+1} \to \mathbb{R}$ be continuous. Suppose that there exists a constant d, independent of λ, such that $\|x\| \ne d$ for any solution $x \in B$ of the equation*

$$x(t) = \lambda \int_0^1 g_{m-q}(t,s)F\big(s, \tilde{J}x(s)\big)\,ds, \quad t \in [0,1], \tag{3.19$_\lambda$}$$

where $0 < \lambda < 1$. Then, (1.1) has at least one solution $y^* \in C^{(m)}[0,1]$ such that, for $0 \le k \le m-2$,

$$\big|y^{*(k)}(t)\big| \le \frac{t^{m-k-2}}{(m-k-2)!}d \le \frac{d}{(m-k-2)!}, \quad t \in [0,1]. \tag{3.20}$$

Proof We recognize that a solution of (3.19)$_\lambda$ is a fixed point of the equation $x = \lambda Sx$, where S is defined in (3.17). Using the Arzelà-Ascoli theorem, we see that S is continuous and completely continuous. Now, in the context of Theorem 2.1, let $U = \{x \in B \mid \|x\| < d\}$. Noting that $\|x\| \ne d$, where x is any solution of (3.19)$_\lambda$, we see that $x \notin \partial U$, and so conclusion (b) of Theorem 2.1 is not valid. Hence, conclusion (a) of Theorem 2.1 must hold, that is, S has a fixed point in $\overline{U}$. Hence, (3.4) has a solution $x^* \in \overline{U}$ with $\|x^*\| \le d$.

By Remark 3.1(a), (1.1) has a solution $y^* = J^q x^*$. Noting that $\|x^*\| \le d$, (3.20) is immediate from (3.15). □

Using Theorem 3.5, the next result gives the existence of a *positive* solution.

Theorem 3.6 *Let* (A1)-(A3) *hold. Then,* (1.1) *has a positive solution* $y^* \in C^{(m)}[0,1]$ *such that, for* $0 \le k \le m-2$,

$$0 \le y^{*(k)}(t) < \frac{t^{m-k-2}}{(m-k-2)!}a \le \frac{a}{(m-k-2)!}, \quad t \in [0,1]. \tag{3.21}$$

Proof Let $\hat{F} : [0,1] \times \mathbb{R}^{q+1} \to \mathbb{R}$ be defined by

$$\hat{F}(t, u_1, \ldots, u_{q+1}) = F\bigl(t, |u_1|, \ldots, |u_{q+1}|\bigr). \tag{3.22}$$

Noting (A1), we see that the function $\hat{F}$ is well defined and continuous.

Since we plan to employ Theorem 3.5, we consider the equation

$$x(t) = \lambda \int_0^1 g_{m-q}(t,s)\hat{F}\bigl(s, \tilde{J}x(s)\bigr)\,ds, \quad t \in [0,1], \tag*{$(3.23)_\lambda$}$$

where $0 < \lambda < 1$, and prove that any solution $x \in B$ of $(3.23)_\lambda$ satisfies $\|x\| \ne a$.

To proceed, let $x \in B$ be any solution of $(3.23)_\lambda$. Using (3.22), Lemma 2.3(e), and (A1), we get

$$\begin{aligned} x(t) &= \lambda \int_0^1 g_{m-q}(t,s)\hat{F}\bigl(s, \tilde{J}x(s)\bigr)\,ds \\ &= \lambda \int_0^1 g_{m-q}(t,s)F\bigl(s, \bigl|J^q x(s)\bigr|, \ldots, \bigl|Jx(s)\bigr|, \bigl|x(s)\bigr|\bigr)\,ds \ge 0, \quad t \in [0,1]. \end{aligned}$$

Thus, x is a *positive* solution.

Similarly, it is easily seen that

$$x^{(m-q-2)}(t) = \lambda \int_0^1 G(t,s)\hat{F}\bigl(s, \tilde{J}x(s)\bigr)\,ds \ge 0, \quad t \in [0,1].$$

Then, applying (A2), (3.13), and (3.9), we find that, for $t \in [0,1]$,

$$\begin{aligned} \bigl|x^{(m-q-2)}(t)\bigr| &= x^{(m-q-2)}(t) \le \int_0^1 G(t,s)F\bigl(s, \bigl|J^q x(s)\bigr|, \ldots, \bigl|Jx(s)\bigr|, \bigl|x(s)\bigr|\bigr)\,ds \\ &\le \int_0^1 G(t,s)\beta(s)f\bigl(\bigl|J^q x(s)\bigr|, \ldots, \bigl|Jx(s)\bigr|, \bigl|x(s)\bigr|\bigr)\,ds \\ &\le \int_0^1 G(t,s)\beta(s)f\biggl(\frac{\|x\|}{(m-2)!}, \frac{\|x\|}{(m-3)!}, \ldots, \frac{\|x\|}{(m-q-2)!}\biggr)\,ds. \end{aligned}$$

Taking the suprema of both sides yields

$$\|x\| \le Mf\biggl(\frac{\|x\|}{(m-2)!}, \frac{\|x\|}{(m-3)!}, \ldots, \frac{\|x\|}{(m-q-2)!}\biggr). \tag{3.24}$$

Comparing (3.24) and (A3), it is clear that $\|x\| \ne a$.

It now follows from the proof of Theorem 3.5 that $(3.23)|_{\lambda=1}$ has a solution $x^* \in B$ with $\|x^*\| \le a$. Using a similar argument as before, it can be easily seen that x^* is a *positive* solution and $\|x^*\| \neq a$. Thus, $\|x^*\| < a$. Moreover, since x^* is *positive*, we have $|J^k x^*(s)| = J^k x^*(s)$ for $0 \le k \le q$ and $s \in [0,1]$. Using this we find that, for $t \in [0,1]$,

$$\begin{aligned} x^*(t) &= \int_0^1 g_{m-q}(t,s)\hat{F}\big(s,\tilde{J}x^*(s)\big)\,ds \\ &= \int_0^1 g_{m-q}(t,s)F\big(s,\big|J^q x^*(s)\big|,\ldots,\big|Jx^*(s)\big|,\big|x^*(s)\big|\big)\,ds \\ &= \int_0^1 g_{m-q}(t,s)F\big(s,J^q x^*(s),\ldots,Jx^*(s),x^*(s)\big)\,ds. \end{aligned}$$

Hence, x^* is actually a positive solution of (3.4) with $\|x^*\| < a$. By Remark 3.1(a), $y^* = J^q x^*$ is a positive solution of (1.1) satisfying (3.15), which, in view of $\|x^*\| < a$, leads to (3.21) immediately. □

Remark 3.3 Note that the last inequality in (A1),

$$F(t,u_1,\ldots,u_{q+1}) > 0, \quad (t,u_1,\ldots,u_{q+1}) \in [0,1] \times K^{q+1},$$

is *not* needed in Theorem 3.6.

The positive solution guaranteed in Theorem 3.6 may be trivial. Our next result gives the existence of a *nontrivial positive* solution.

Theorem 3.7 *Let* (A1)-(A5) *hold. Then,* (1.1) *has a nontrivial positive solution* $y^* \in C^{(m)}[0,1]$ *such that, for* $0 \le k \le m-2$,

$$0 \le y^{*(k)}(t) \begin{cases} < \frac{t^{m-k-2}}{(m-k-2)!}a \le \frac{a}{(m-k-2)!}, & t \in [0,1], \quad \text{if } a > b, \\ \le \frac{t^{m-k-2}}{(m-k-2)!}b \le \frac{b}{(m-k-2)!}, & t \in [0,1], \quad \text{if } a < b, \end{cases} \tag{3.25}$$

and

$$y^{*(k)}(t) \begin{cases} \ge \frac{(t-\eta)^{m-k-2}}{(m-k-2)!}\gamma b, & t \in [\eta,1-\eta], \quad \text{if } a > b, \\ > \frac{(t-\eta)^{m-k-2}}{(m-k-2)!}\gamma a, & t \in [\eta,1-\eta], \quad \text{if } a < b. \end{cases} \tag{3.26}$$

Proof We apply Theorem 2.2 with the operator S and the cone C defined respectively in (3.17) and (3.7). To begin, note that the operator $S : B \to B$ is continuous and completely continuous. Further, from (3.10) we see that if $x \in C$, then x is nonnegative, and so $J^k x \in \tilde{K}$ (or $J^k x \in K$ if x is nontrivial) for $0 \le k \le q$.

First, we show that S maps C into C. Let $x \in C$. Noting (3.18), Lemma 2.3(a), and (A1), it is clear that

$$(Sx)^{(m-q-2)}(t) = \int_0^1 G(t,s)F\big(s,\tilde{J}x(s)\big)\,ds \ge 0, \quad t \in [0,1]. \tag{3.27}$$

Using Lemma 2.3(b), we have that, for $t \in [0,1]$,

$$\big|(Sx)^{(m-q-2)}(t)\big| = (Sx)^{(m-q-2)}(t) \le \int_0^1 LG(s,s)F\big(s,\tilde{J}x(s)\big)\,ds,$$

which immediately implies

$$\|Sx\| \le \int_0^1 LG(s,s)F\big(s,\tilde{J}x(s)\big)\,ds. \tag{3.28}$$

Now, using Lemma 2.3(c) and (3.28), we find that, for $t \in [\eta, 1-\eta]$,

$$(Sx)^{(m-q-2)}(t) \ge \int_0^1 K_\eta G(s,s)F\big(s,\tilde{J}x(s)\big)\,ds \ge \frac{K_\eta}{L}\|Sx\| = \gamma\|Sx\|.$$

It follows that

$$\min_{t\in[\eta,1-\eta]} Sx(t) \ge \gamma\|Sx\|. \tag{3.29}$$

Inequalities (3.27) and (3.29) imply that $S(C) \subseteq C$.

Next, let $\Omega_a = \{x \in B \mid \|x\| < a\}$. Let $x \in C \cap \partial\Omega_a$, so $\|x\| = a$. Applying (A2), (3.13), and (3.9), we have, for $t \in [0,1]$,

$$\begin{aligned}\big|(Sx)^{(m-q-2)}(t)\big| &= (Sx)^{(m-q-2)}(t)\\ &\le \int_0^1 G(t,s)\beta(s)f\big(\tilde{J}x(s)\big)\,ds\\ &\le \int_0^1 G(t,s)\beta(s)f\left(\frac{a}{(m-2)!}, \frac{a}{(m-3)!}, \dots, \frac{a}{(m-q-2)!}\right)ds.\end{aligned}$$

Taking the suprema and using (A3), we get

$$\|Sx\| \le Mf\left(\frac{a}{(m-2)!}, \frac{a}{(m-3)!}, \dots, \frac{a}{(m-q-2)!}\right) < a = \|x\|. \tag{3.30}$$

Hence, we have shown that $\|Sx\| \le \|x\|$ for $x \in C \cap \partial\Omega_a$.

Next, let $\Omega_b = \{x \in B \mid \|x\| < b\}$. Let $x \in C \cap \partial\Omega_b$, so that $\|x\| = b$. Noting (A4), we find that, for $t \in [0,1]$,

$$\begin{aligned}\big|(Sx)^{(m-q-2)}(t)\big| &\ge \int_z^{1-\eta} G(t,s)F\big(s,\tilde{J}x(s)\big)\,ds\\ &\ge \int_z^{1-\eta} G(t,s)\alpha(s)f\big(\tilde{J}x(s)\big)\,ds\\ &\ge \int_z^{1-\eta} G(t,s)\alpha(s)f\left(\frac{(z-\eta)^{m-2}\gamma b}{(m-2)!}, \frac{(z-\eta)^{m-3}\gamma b}{(m-3)!}, \dots,\right.\\ &\qquad \left.\frac{(z-\eta)^{m-q-2}\gamma b}{(m-q-2)!}\right)ds,\end{aligned}$$

where we have used (3.14) and (3.12) in the last inequality. Taking the suprema and using (A5) lead to

$$\|Sx\| \ge Nf\left(\frac{(z-\eta)^{m-2}\gamma b}{(m-2)!}, \frac{(z-\eta)^{m-3}\gamma b}{(m-3)!}, \dots, \frac{(z-\eta)^{m-q-2}\gamma b}{(m-q-2)!}\right) \ge b = \|x\|. \tag{3.31}$$

Hence, we have $\|Sx\| \ge \|x\|$ for $x \in C \cap \partial\Omega_b$.

In view of (3.30) and (3.31), we conclude from Theorem 2.2 that S has a fixed point $x^* \in C \cap (\overline{\Omega}_{\max\{a,b\}} \backslash \Omega_{\min\{a,b\}})$. Thus, $\min\{a,b\} \le \|x^*\| \le \max\{a,b\}$. We further note that $\|x^*\| \ne a$ follows from a similar argument as in the first part of the proof of Theorem 3.6. Hence, we obtain

$$a < \|x^*\| \le b \quad \text{if } a < b \quad \text{and} \quad b \le \|x^*\| < a \quad \text{if } a > b. \tag{3.32}$$

By Remark 3.1, (1.1) has a nontrivial positive solution $y^* = J^q x^*$. Since $x^* \in B$, y^* satisfies (3.15) which, in view of (3.32), gives (3.25). Further, since $x^* \in C$, using (3.32) in (3.16) leads to (3.26) immediately. □

The next result gives the existence of *two positive* solutions.

Theorem 3.8 *Let* (A1)-(A5) *hold with* $a < b$. *Then*, (1.1) *has* (*at least*) *two positive solutions* $y_1, y_2 \in C^{(m)}[0,1]$ *such that*, *for* $0 \le k \le m-2$,

$$\begin{cases} 0 \le y_1^{(k)}(t) < \frac{t^{m-k-2}}{(m-k-2)!} a \le \frac{a}{(m-k-2)!}, & t \in [0,1], \\ 0 \le y_2^{(k)}(t) \le \frac{t^{m-k-2}}{(m-k-2)!} b \le \frac{b}{(m-k-2)!}, & t \in [0,1], \\ y_2^{(k)}(t) > \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma a, & t \in [\eta, 1-\eta]. \end{cases} \tag{3.33}$$

Proof From the proofs of Theorems 3.6 and 3.7 we see that (3.4) has two positive solutions $x_1 \in B$ and $x_2 \in C$ (x_2 is nontrivial) such that

$$0 \le \|x_1\| < a < \|x_2\| \le b. \tag{3.34}$$

By Remark 3.1, (1.1) has two positive solutions $y_1 = J^q x_1$ and $y_2 = J^q x_2$ (y_2 is nontrivial). Using (3.34) in (3.15) and (3.16) gives (3.33) immediately. □

One of the solutions (y_1) may be trivial in Theorem 3.8. Our next result guarantees the existence of *two nontrivial positive* solutions.

Theorem 3.9 *Let* (A1)-(A5) *and* $(\mathrm{A5})|_{b=b'}$ *hold, where* $0 < b' < a < b$. *Then*, (1.1) *has* (*at least*) *two nontrivial positive solutions* $y_1, y_2 \in C^{(m)}[0,1]$ *such that*, *for* $0 \le k \le m-2$,

$$\begin{cases} 0 \le y_1^{(k)}(t) < \frac{t^{m-k-2}}{(m-k-2)!} a \le \frac{a}{(m-k-2)!}, & t \in [0,1], \\ y_1^{(k)}(t) \ge \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma b', & t \in [\eta, 1-\eta], \\ 0 \le y_2^{(k)}(t) \le \frac{t^{m-k-2}}{(m-k-2)!} b \le \frac{b}{(m-k-2)!}, & t \in [0,1], \\ y_2^{(k)}(t) > \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma a, & t \in [\eta, 1-\eta]. \end{cases} \tag{3.35}$$

Proof From the proof of Theorem 3.7 (see (3.32)) we derive that (3.4) has two nontrivial positive solutions $x_1, x_2 \in C$ such that

$$0 < b' \le \|x_1\| < a < \|x_2\| \le b. \tag{3.36}$$

By Remark 3.1, (1.1) has two nontrivial positive solutions $y_1 = J^q x_1$ and $y_2 = J^q x_2$. Using (3.36) in (3.15) and (3.16) gives (3.35) immediately. □

Note that in Theorem 3.9, *both* (A3) and (A5) are required to obtain the existence of *two nontrivial positive* solutions. In the next two theorems, only *one* of (A3) and (A5) is used to ensure the existence of *two nontrivial positive* solutions. Define

$$f_0 = \lim_{u_i \to 0+, 1 \le i \le q+1} \frac{f(u_1, \ldots, u_{q+1})}{u_{q+1}} \quad \text{and}$$
$$f_\infty = \lim_{u_i \to \infty, 1 \le i \le q+1} \frac{f(u_1, \ldots, u_{q+1})}{u_{q+1}}.$$

Theorem 3.10 *Let* (A1)-(A4) *hold and* $0 < \int_z^{1-\eta} G(s,s)\alpha(s)\,ds < \infty$.

(a) *If* $f_0 = \infty$, *then* (1.1) *has a nontrivial positive solution* $y_1 \in C^{(m)}[0,1]$ *such that, for* $0 \le k \le m-2$,

$$0 \le y_1^{(k)}(t) < \frac{t^{m-k-2}}{(m-k-2)!} a \le \frac{a}{(m-k-2)!}, \quad t \in [0,1]. \tag{3.37}$$

(b) *If* $f_\infty = \infty$, *then* (1.1) *has a nontrivial positive solution* $y_2 \in C^{(m)}[0,1]$ *such that, for* $0 \le k \le m-2$,

$$y_2^{(k)}(t) > \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma a, \quad t \in [\eta, 1-\eta]. \tag{3.38}$$

(c) *If* $f_0 = f_\infty = \infty$, *then* (1.1) *has* (*at least*) *two nontrivial positive solutions* $y_1, y_2 \in C^{(m)}[0,1]$ *such that* (3.37) *and* (3.38) *hold for* $0 \le k \le m-2$.

Proof We apply Theorem 2.2 with the operator S and the cone C defined respectively in (3.17) and (3.7). As seen in the proof of Theorem 3.7, S maps C into C. Let $\Omega_a = \{x \in B \mid \|x\| < a\}$. Using (A2) and (A3) as in the proof of Theorem 3.7, we obtain (3.30), and hence

$$\|Sx\| \le \|x\|, \quad x \in C \cap \partial\Omega_a. \tag{3.39}$$

(a) Define

$$P = \left[\frac{(z-\eta)^{m-q-2}\gamma K_\eta}{(m-q-2)!} \int_z^{1-\eta} G(s,s)\alpha(s)\,ds \right]^{-1}. \tag{3.40}$$

Since $f_0 = \infty$, there exists $0 < r < a$ such that

$$f(u_1, \ldots, u_{q+1}) \ge P u_{q+1}, \quad 0 < u_i \le r, 1 \le i \le q+1. \tag{3.41}$$

Let $\Omega_r = \{x \in B \mid \|x\| < r\}$. Let $x \in C \cap \partial\Omega_r$, so $\|x\| = r$. Note that from (3.13) and (3.9) we have

$$J^k x(s) \le \frac{\|x\|}{(m-q-2+k)!} = \frac{r}{(m-q-2+k)!} < r, \quad s \in [0,1], 0 \le k \le q. \tag{3.42}$$

For $t \in [\eta, 1-\eta]$, we use (A4), Lemma 2.3(c), (3.42), (3.41), (3.12), and (3.40) successively to get

$$\begin{aligned}\left|(Sx)^{(m-q-2)}(t)\right| &\geq \int_z^{1-\eta} G(t,s)F\bigl(s,\tilde{J}x(s)\bigr)\,ds \\ &\geq \int_z^{1-\eta} K_\eta G(s,s)\alpha(s)f\bigl(\tilde{J}x(s)\bigr)\,ds \\ &\geq \int_z^{1-\eta} K_\eta G(s,s)\alpha(s)Px(s)\,ds \\ &\geq \int_z^{1-\eta} K_\eta G(s,s)\alpha(s)P\frac{(z-\eta)^{m-q-2}\gamma\|x\|}{(m-q-2)!}\,ds = \|x\|.\end{aligned}$$

Hence, we have

$$\|Sx\| \geq \|x\|, \quad x \in C \cap \partial\Omega_r. \tag{3.43}$$

Having established (3.39) and (3.43), by Theorem 2.2 we conclude that S has a fixed point $x_1 \in C \cap (\overline{\Omega}_a \backslash \Omega_r)$ such that $r \leq \|x_1\| \leq a$. Using a similar argument as in the first part of the proof of Theorem 3.6, we see that $\|x_1\| \neq a$. Hence, we get $r \leq \|x_1\| < a$ (x_1 is nontrivial). By Remark 3.1, (1.1) has a nontrivial positive solution $y_1 = J^q x_1$. Since $\|x_1\| < a$, (3.37) is immediate from (3.15).

(b) Since $f_\infty = \infty$, we may choose $w > a$ such that

$$f(u_1, \ldots, u_{q+1}) \geq P u_{q+1}, \quad u_i \geq w, 1 \leq i \leq q+1, \tag{3.44}$$

where P is defined in (3.40). Let

$$w_0 = \max\left\{ w\left[\frac{(z-\eta)^{m-q-2}\gamma}{(m-q-2)!}\right]^{-1}, w\left[\frac{(z-\eta)^{m-q-2+k}\gamma}{(m-q-2+k)!}\right]^{-1}, 1 \leq k \leq q\right\} = \frac{w(m-2)!}{\gamma(z-\eta)^{m-2}}.$$

Clearly, $w_0 > w > a$. Let $\Omega_{w_0} = \{x \in B \mid \|x\| < w_0\}$. Let $x \in C \cap \partial\Omega_{w_0}$, so that $\|x\| = w_0$. Note that from (3.12), (3.14), and the definition of w_0 we have that, for $s \in [z, 1-\eta]$,

$$\begin{cases} x(s) \geq \frac{(z-\eta)^{m-q-2}\gamma}{(m-q-2)!}\|x\| = \frac{(z-\eta)^{m-q-2}\gamma}{(m-q-2)!} w_0 \geq w, \\ J^k x(s) \geq \frac{(z-\eta)^{m-q-2+k}\gamma}{(m-q-2+k)!}\|x\| = \frac{(z-\eta)^{m-q-2+k}\gamma}{(m-q-2+k)!} w_0 \geq w, \quad 1 \leq k \leq q. \end{cases} \tag{3.45}$$

Using (A4), Lemma 2.3(c), (3.45), (3.44), (3.12), and (3.40) successively, we get that, for $t \in [\eta, 1-\eta]$,

$$\begin{aligned}\left|(Sx)^{(m-q-2)}(t)\right| &\geq \int_z^{1-\eta} K_\eta G(s,s)\alpha(s)f\bigl(\tilde{J}x(s)\bigr)\,ds \\ &\geq \int_z^{1-\eta} K_\eta G(s,s)\alpha(s)Px(s)\,ds \\ &\geq \int_z^{1-\eta} K_\eta G(s,s)\alpha(s)P\frac{(z-\eta)^{m-q-2}\gamma\|x\|}{(m-q-2)!}\,ds = \|x\|.\end{aligned}$$

It follows that

$$\|Sx\| \geq \|x\|, \quad x \in C \cap \partial\Omega_{w_0}. \tag{3.46}$$

With (3.39) and (3.46), by Theorem 2.2 we conclude that S has a fixed point $x_2 \in C \cap (\overline{\Omega}_{w_0} \backslash \Omega_a)$ such that $a \leq \|x_2\| \leq w_0$. Once again, as seen earlier, $\|x_2\| \neq a$, so that $a < \|x_2\| \leq w_0$ (x_2 is nontrivial). By Remark 3.1, (1.1) has a nontrivial positive solution $y_2 = J^q x_2$. Since $\|x_2\| > a$, (3.38) is immediate from (3.16).

(c) This follows from Cases (a) and (b). □

Theorem 3.11 *Let* (A1), (A2), (A4), (A5) *hold, and* $0 < \int_0^1 G(s,s)\beta(s)\,ds < \infty$.

(a) *If* $f_0 = 0$, *then* (1.1) *has a nontrivial positive solution* $y_1 \in C^{(m)}[0,1]$ *such that, for* $0 \leq k \leq m-2$,

$$0 \leq y_1^{(k)}(t) \leq \frac{t^{m-k-2}}{(m-k-2)!} b \leq \frac{b}{(m-k-2)!}, \quad t \in [0,1]. \tag{3.47}$$

(b) *If* $f_\infty = 0$, *then* (1.1) *has a nontrivial positive solution* $y_2 \in C^{(m)}[0,1]$ *such that, for* $0 \leq k \leq m-2$,

$$y_2^{(k)}(t) \geq \frac{(t-\eta)^{m-k-2}}{(m-k-2)!} \gamma b, \quad t \in [\eta, 1-\eta]. \tag{3.48}$$

(c) *If* $f_0 = f_\infty = 0$, *then* (1.1) *has* (*at least*) *two nontrivial positive solutions* $y_1, y_2 \in C^{(m)}[0,1]$ *such that* (3.47) *and* (3.48) *hold for* $0 \leq k \leq m-2$.

Proof Once again, we apply Theorem 2.2 with the operator S and the cone C defined respectively in (3.17) and (3.7). Let $\Omega_b = \{x \in B \mid \|x\| < b\}$. Using (A4) and (A5) as in the proof of Theorem 3.7, we obtain (3.31), and so

$$\|Sx\| \geq \|x\|, \quad x \in C \cap \partial\Omega_b. \tag{3.49}$$

(a) Let

$$T = \left[\frac{L}{(m-q-2)!} \int_0^1 G(s,s)\beta(s)\,ds \right]^{-1}. \tag{3.50}$$

Since $f_0 = 0$, there exists $0 < r < b$ such that

$$f(u_1, \ldots, u_{q+1}) \leq T u_{q+1}, \quad 0 < u_i \leq r, 1 \leq i \leq q+1. \tag{3.51}$$

Let $\Omega_r = \{x \in B \mid \|x\| < r\}$. Let $x \in C \cap \partial\Omega_r$, so $\|x\| = r$. Note that (3.42) holds. Using (A2), Lemma 2.3(b), (3.42), (3.51), (3.9), and (3.50) successively, we find that, for $t \in [0,1]$,

$$\begin{aligned}
\left|(Sx)^{(m-q-2)}(t)\right| &\leq \int_0^1 LG(s,s)\beta(s) f\big(\tilde{J}x(s)\big)\,ds \\
&\leq \int_0^1 LG(s,s)\beta(s) T x(s)\,ds \leq \int_0^1 LG(s,s)\beta(s) T \frac{\|x\|}{(m-q-2)!}\,ds = \|x\|.
\end{aligned}$$

Hence, we have

$$\|Sx\| \leq \|x\|, \quad x \in C \cap \partial\Omega_r. \tag{3.52}$$

Noting (3.49) and (3.52), it follows from Theorem 2.2 that S has a fixed point $x_1 \in C \cap (\overline{\Omega}_b \backslash \Omega_r)$ such that $r \leq \|x_1\| \leq b$ (x_1 is nontrivial). Hence, we see from Remark 3.1 that (1.1) has a nontrivial positive solution $y_1 = J^q x_1$. Using $\|x_1\| \leq b$ in (3.15) yields (3.47) immediately.

(b) Since $f_\infty = 0$, we may choose $w > b$ such that

$$f(u_1, \ldots, u_{q+1}) \leq T u_{q+1}, \quad u_i \geq w, 1 \leq i \leq q+1, \tag{3.53}$$

where T is defined in (3.50). To proceed, we consider two cases, when f is bounded and when f is unbounded.

Case 1. Suppose that f is bounded. Then, for some $A > 0$,

$$f(u_1, \ldots, u_{q+1}) \leq A, \quad u_i \in [0, \infty), 1 \leq i \leq q+1. \tag{3.54}$$

Let

$$w_0 = \max\left\{b+1, LA \int_0^1 G(s,s)\beta(s)\, ds\right\}.$$

Clearly, $w_0 > b$. Let $\Omega_{w_0} = \{x \in B \mid \|x\| < w_0\}$. Let $x \in C \cap \partial\Omega_{w_0}$, so $\|x\| = w_0$. Using (A2), Lemma 2.3(b), and (3.54) provides, for $t \in [0,1]$,

$$\begin{aligned} \left|(Sx)^{(m-q-2)}(t)\right| &\leq \int_0^1 LG(s,s)\beta(s) f\bigl(\tilde{J}x(s)\bigr)\, ds \\ &\leq \int_0^1 LG(s,s)\beta(s) A\, ds \leq w_0 = \|x\|. \end{aligned}$$

Hence, we have

$$\|Sx\| \leq \|x\|, \quad x \in C \cap \partial\Omega_{w_0}. \tag{3.55}$$

Case 2. Suppose that f is unbounded. Then, there exists $w_0 > w(m-2)!$ ($> b$) such that

$$\begin{aligned} &f(u_1, \ldots, u_{q+1}) \leq f\left(\frac{w_0}{(m-2)!}, \frac{w_0}{(m-3)!}, \ldots, \frac{w_0}{(m-q-2)!}\right), \\ &\quad 0 \leq u_i \leq w_0, 1 \leq i \leq q+1. \end{aligned} \tag{3.56}$$

Let $\Omega_{w_0} = \{x \in B \mid \|x\| < w_0\}$. Let $x \in C \cap \partial\Omega_{w_0}$, so $\|x\| = w_0$. It follows from (3.13) and (3.9) that

$$J^k x(s) \leq \frac{\|x\|}{(m-q-2+k)!} = \frac{w_0}{(m-q-2+k)!} < w_0, \quad s \in [0,1], 0 \leq k \leq q. \tag{3.57}$$

Now, we apply (A2), Lemma 2.3(b), (3.57), (3.56), (3.53), and (3.50) successively to obtain, for $t \in [0,1]$,

$$\begin{aligned}\bigl|(Sx)^{(m-q-2)}(t)\bigr| &\le \int_0^1 LG(s,s)\beta(s)f\bigl(\tilde{J}x(s)\bigr)\,ds\\ &\le \int_0^1 LG(s,s)\beta(s)f\biggl(\frac{w_0}{(m-2)!},\frac{w_0}{(m-3)!},\ldots,\frac{w_0}{(m-q-2)!}\biggr)\,ds\\ &\le \int_0^1 LG(s,s)\beta(s)T\frac{w_0}{(m-q-2)!}\,ds = w_0 = \|x\|.\end{aligned}$$

It follows that $\|Sx\| \le \|x\|$ for $x \in C \cap \partial\Omega_{w_0}$, that is, (3.55) holds.

Having established (3.49) and (3.55), by Theorem 2.2 we see that S has a fixed point $x_2 \in C \cap (\overline{\Omega}_{w_0} \backslash \Omega_b)$ such that $b \le \|x_2\| \le w_0$ (x_2 is nontrivial). It follows from Remark 3.1 that (1.1) has a nontrivial positive solution $y_2 = J^q x_2$. Using $\|x_2\| \ge b$ in (3.16) leads to (3.48) immediately.

(c) This follows from Cases (a) and (b). □

Remark 3.4 Comparing Theorem 3.9 with Theorems 3.10(c) and 3.11(c), we note that all of them guarantee the existence of *two nontrivial positive* solutions of (1.1); also, conclusion (3.35) in Theorem 3.9 gives more details than the conclusions in Theorems 3.10(c) and 3.11(c). This might be explained by the fact that condition (A5) is required in Theorem 3.9 *twice* but not at all in Theorems 3.10(c) and 3.11(c); further, more effort might be needed to check (A5). Therefore, the 'more' details in (3.35) require possibly greater efforts.

Using the earlier results, we now give the existence of *multiple positive* solutions of (1.1).

Theorem 3.12 *Let* (A1), (A2), *and* (A4) *hold. Suppose that* (A3) *is satisfied for* $a = a_\ell$, $\ell = 1,2,\ldots,k$, *and* (A5) *is satisfied for* $b = b_\ell$, $\ell = 1,2,\ldots,n$.

(a) *If* $n = k+1$ *and* $0 < b_1 < a_1 < \cdots < b_k < a_k < b_{k+1}$, *then* (1.1) *has* (*at least*) $2k$ *nontrivial positive solutions* $y_1,\ldots,y_{2k} \in C^{(m)}[0,1]$ *such that, for* $0 \le i \le m-2$ *and* $\ell = 1,2,\ldots,k$,

$$\begin{cases} 0 \le y_{2\ell-1}^{(i)}(t) < \frac{t^{m-i-2}}{(m-i-2)!}a_\ell \le \frac{a_\ell}{(m-i-2)!}, & t \in [0,1],\\ y_{2\ell-1}^{(i)}(t) \ge \frac{(t-\eta)^{m-i-2}}{(m-i-2)!}\gamma b_\ell, & t \in [\eta,1-\eta],\\ 0 \le y_{2\ell}^{(i)}(t) \le \frac{t^{m-i-2}}{(m-i-2)!}b_{\ell+1} \le \frac{b_{\ell+1}}{(m-i-2)!}, & t \in [0,1],\\ y_{2\ell}^{(i)}(t) > \frac{(t-\eta)^{m-i-2}}{(m-i-2)!}\gamma a_\ell, & t \in [\eta,1-\eta]. \end{cases} \tag{3.58}$$

(b) *If* $n = k$ *and* $0 < b_1 < a_1 < \cdots < b_k < a_k$, *then* (1.1) *has* (*at least*) $2k-1$ *nontrivial positive solutions* $y_1,\ldots,y_{2k-1} \in C^{(m)}[0,1]$ *such that, for* $0 \le i \le m-2$, $\ell = 1,2,\ldots,k$, *and* $j = 1,2,\ldots,k-1$,

$$\begin{cases} 0 \le y_{2\ell-1}^{(i)}(t) < \frac{t^{m-i-2}}{(m-i-2)!}a_\ell \le \frac{a_\ell}{(m-i-2)!}, & t \in [0,1],\\ y_{2\ell-1}^{(i)}(t) \ge \frac{(t-\eta)^{m-i-2}}{(m-i-2)!}\gamma b_\ell, & t \in [\eta,1-\eta],\\ 0 \le y_{2j}^{(i)}(t) \le \frac{t^{m-i-2}}{(m-i-2)!}b_{j+1} \le \frac{b_{j+1}}{(m-i-2)!}, & t \in [0,1],\\ y_{2j}^{(i)}(t) > \frac{(t-\eta)^{m-i-2}}{(m-i-2)!}\gamma a_j, & t \in [\eta,1-\eta]. \end{cases} \tag{3.59}$$

(c) *If* $k = n+1$ *and* $0 < a_1 < b_1 < \cdots < a_n < b_n < a_{n+1}$, *then* (1.1) *has* (*at least*) $2n+1$ *positive solutions* $y_0, \ldots, y_{2n} \in C^{(m)}[0,1]$, *where* $y_1, \ldots, y_{2n}$ *are nontrivial, such that, for* $0 \le i \le m-2$ *and* $\ell = 1, 2, \ldots, n$,

$$\begin{cases} 0 \le y_0^{(i)}(t) < \frac{t^{m-i-2}}{(m-i-2)!} a_1 \le \frac{a_1}{(m-i-2)!}, & t \in [0,1], \\ 0 \le y_{2\ell-1}^{(i)}(t) \le \frac{t^{m-i-2}}{(m-i-2)!} b_\ell \le \frac{b_\ell}{(m-i-2)!}, & t \in [0,1], \\ y_{2\ell-1}^{(i)}(t) > \frac{(t-\eta)^{m-i-2}}{(m-i-2)!} \gamma a_\ell, & t \in [\eta, 1-\eta], \\ 0 \le y_{2\ell}^{(i)}(t) < \frac{t^{m-i-2}}{(m-i-2)!} a_{\ell+1} \le \frac{a_{\ell+1}}{(m-i-2)!}, & t \in [0,1], \\ y_{2\ell}^{(i)}(t) \ge \frac{(t-\eta)^{m-i-2}}{(m-i-2)!} \gamma b_\ell, & t \in [\eta, 1-\eta]. \end{cases} \tag{3.60}$$

(d) *If* $k = n$ *and* $0 < a_1 < b_1 < \cdots < a_k < b_k$, *then* (1.1) *has* (*at least*) $2k$ *positive solutions* $y_0, \ldots, y_{2k-1} \in C^{(m)}[0,1]$, *where* $y_1, \ldots, y_{2k-1}$ *are nontrivial, such that, for* $0 \le i \le m-2$, $\ell = 1, 2, \ldots, k$, *and* $j = 1, 2, \ldots, k-1$,

$$\begin{cases} 0 \le y_0^{(i)}(t) < \frac{t^{m-i-2}}{(m-i-2)!} a_1 \le \frac{a_1}{(m-i-2)!}, & t \in [0,1], \\ 0 \le y_{2\ell-1}^{(i)}(t) \le \frac{t^{m-i-2}}{(m-i-2)!} b_\ell \le \frac{b_\ell}{(m-i-2)!}, & t \in [0,1], \\ y_{2\ell-1}^{(i)}(t) > \frac{(t-\eta)^{m-i-2}}{(m-i-2)!} \gamma a_\ell, & t \in [\eta, 1-\eta], \\ 0 \le y_{2j}^{(i)}(t) < \frac{t^{m-i-2}}{(m-i-2)!} a_{j+1} \le \frac{a_{j+1}}{(m-i-2)!}, & t \in [0,1], \\ y_{2j}^{(i)}(t) \ge \frac{(t-\eta)^{m-i-2}}{(m-i-2)!} \gamma b_j, & t \in [\eta, 1-\eta]. \end{cases} \tag{3.61}$$

Proof The proof involves repeated usage of Theorems 3.6 and 3.7. In (a) and (b), we apply (3.32) repeatedly to get multiple positive solutions of (3.4) as follows.

(a) If $n = k+1$ and $0 < b_1 < a_1 < \cdots < b_k < a_k < b_{k+1}$, then (3.4) has (at least) $2k$ nontrivial positive solutions $x_1, \ldots, x_{2k} \in C$ such that

$$0 < b_1 \le \|x_1\| < a_1 < \|x_2\| \le b_2 \le \cdots < a_k < \|x_{2k}\| \le b_{k+1}. \tag{3.62}$$

(b) If $n = k$ and $0 < b_1 < a_1 < \cdots < b_k < a_k$, then (3.4) has (at least) $2k-1$ nontrivial positive solutions $x_1, \ldots, x_{2k-1} \in C$ such that

$$0 < b_1 \le \|x_1\| < a_1 < \|x_2\| \le b_2 \le \cdots \le b_k \le \|x_{2k-1}\| < a_k. \tag{3.63}$$

Hence, conclusions (a) and (b) follow from Remark 3.1. Inequalities (3.58) and (3.59) are obtained by using (3.62) and (3.63) in (3.15) and (3.16).

Next, in (c) and (d), from the proof of Theorem 3.6 we see that (3.4) has a positive solution $x_0 \in B$ with $0 \le \|x_0\| < a_1$. Applying (3.32) repeatedly again, we get more solutions as follows.

(c) If $k = n+1$ and $0 < a_1 < b_1 < \cdots < a_n < b_n < a_{n+1}$, then (3.4) has (at least) $2n+1$ positive solutions $x_0 \in B$, $x_1, \ldots, x_{2n} \in C$ such that

$$0 \le \|x_0\| < a_1 < \|x_1\| \le b_1 \le \|x_2\| < a_2 < \cdots \le b_n \le \|x_{2n}\| < a_{n+1}. \tag{3.64}$$

(d) If $k = n$ and $0 < a_1 < b_1 < \cdots < a_k < b_k$, then (3.4) has (at least) $2k$ positive solutions $x_0 \in B, x_1, \ldots, x_{2k-1} \in C$ such that

$$0 \leq \|x_0\| < a_1 < \|x_1\| \leq b_1 \leq \|x_2\| < a_2 < \cdots < a_k < \|x_{2k-1}\| \leq b_k. \tag{3.65}$$

Hence, conclusions (c) and (d) follow from Remark 3.1. Inequalities (3.60) and (3.61) are obtained by using (3.64) and (3.65) in (3.15) and (3.16). □

4 Examples

In this section, we illustrate the theorems obtained in Section 3 by some examples. We remark that in all the examples presented, explicit *known* solutions are given to validate the conclusions derived from the theorems.

Example 4.1 Consider the Sturm-Liouville boundary value problem

$$\begin{cases} y^{(5)}(t) + F(t, y(t), y'(t), y''(t), y'''(t)) = 0, \quad t \in [0,1], \\ y(0) = y'(0) = y''(0) = 0, \qquad 2y^{(3)}(0) - y^{(4)}(0) = 0, \qquad -y^{(3)}(1) + 3y^{(4)}(1) = 0, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} F(t, y, y', y'', y''') = {} & \frac{36}{5}\left(\frac{290 + 660t + 96t^2 + 14t^3 - t^4 - 6t^5}{10}\right)^{-3} \\ & \times \left(y + 2y' + 3y'' + 4y''' + 5\right)^3. \end{aligned} \tag{4.2}$$

Here, $m = 5$, $q = 3$, $\zeta = 2$, $\theta = 1$, $\rho = -1$ and $\delta = 3$. Let $\eta = \frac{1}{4}$ and $z = \frac{1}{2}$. A direct computation gives $L = \frac{3}{2}$, $K_{\frac{1}{4}} = \frac{1}{2}$, and $\gamma = \frac{1}{3}$.

Clearly, (A1), (A2), and (A4) are satisfied with

$$\alpha(t) = \beta(t) = \frac{36}{5}\left(\frac{290 + 660t + 96t^2 + 14t^3 - t^4 - 6t^5}{10}\right)^{-3}$$

and

$$f(u_1, u_2, u_3, u_4) = (u_1 + 2u_2 + 3u_3 + 4u_4 + 5)^3.$$

It is easy to check that $f_0 = f_\infty = \infty$. Next, let us check if (A3) is satisfied, and for this, using Remark 3.2, we shall check the easier but stricter (A3)′, *viz.*,

$$a > M' f\left(\frac{a}{3!}, \frac{a}{2!}, a, a\right), \tag{4.3}$$

where $M' = \int_0^1 LG(s,s)\beta(s)\,ds$. This inequality reduces to

$$a > M'\left(\frac{a}{6} + 2\frac{a}{2} + 3a + 4a + 5\right)^3,$$

which can be solved to get $a \in [0.012243, 3.5027]$. Hence, (A3)′ (and so (A3)) is satisfied if $a \in [0.012243, 3.5027]$.

In summary, (A1)-(A4) are met (with $a \in [0.012243, 3.5027]$), and also $f_0 = f_\infty = \infty$. By Theorem 3.10(c), (4.1)-(4.2) has (at least) two *nontrivial positive* solutions $y_1, y_2 \in C^{(5)}[0,1]$ such that, for $0 \le k \le 3$,

$$\begin{cases} 0 \le y_1^{(k)}(t) < \frac{t^{3-k}}{(3-k)!}a \le \frac{a}{(3-k)!}, & t \in [0,1], \\ y_2^{(k)}(t) > \frac{1}{(3-k)!}(t-\frac{1}{4})^{3-k}\gamma a, & t \in [\frac{1}{4}, \frac{3}{4}]. \end{cases} \tag{4.4}$$

Since $a \in [0.012243, 3.5027]$, it follows from (4.4) that, for $0 \le k \le 3$,

$$\begin{cases} 0 \le y_1^{(k)}(t) < \frac{t^{3-k}}{(3-k)!}(0.012243) \le \frac{0.012243}{(3-k)!}, & t \in [0,1], \\ y_2^{(k)}(t) > \frac{1}{(3-k)!}(t-\frac{1}{4})^{3-k}\gamma(3.5027), & t \in [\frac{1}{4}, \frac{3}{4}]. \end{cases} \tag{4.5}$$

In fact, a positive solution of (4.1), (4.2) is known to be

$$y^*(t) = \frac{50t^3 + 25t^4 - 3t^5}{50}. \tag{4.6}$$

By direct computation, we find that, for $0 \le k \le 3$,

$$y^{*(k)}(t) \le c_k, \quad t \in [0,1] \quad \text{and} \quad y^{*(k)}(t) \ge d_k\left(t - \frac{1}{4}\right)^{3-k}, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \tag{4.7}$$

where c_k and d_k are respectively the smallest and the largest constants for the inequalities to hold, and they are given as follows:

$$\begin{aligned} &c_0 = 1.44, \quad c_1 = 4.7, \quad c_2 = 10.8, \quad c_3 = 14.4, \\ &d_0 = 4.5267, \quad d_1 = 9.7453, \quad d_2 = 14.7375, \quad d_3 = 8.775. \end{aligned} \tag{4.8}$$

Since $d_k > \gamma(3.5027)/(3-k)!$, this y^* may be y_2 in (4.5). This y^* is certainly not y_1. Hence, conclusion (4.5) is somewhat validated.

Example 4.2 Consider the Sturm-Liouville boundary value problem (4.1)-(4.2) again. Let us check if (A5) is satisfied. For this, using Remark 3.2, we shall check the easier but stricter (A5)′, *viz.*,

$$b \le N' f\left(\frac{\gamma b}{4^3 3!}, \frac{\gamma b}{4^2 2!}, \frac{\gamma b}{4}, \gamma b\right), \tag{4.9}$$

where $N' = \int_{\frac{1}{2}}^{\frac{3}{4}} K_{\frac{1}{4}} G(s,s)\alpha(s)\,ds$. This inequality reduces to

$$b \le N'\left(\frac{\gamma b}{4^3 3!} + 2\frac{\gamma b}{4^2 2!} + 3\frac{\gamma b}{4} + 4\gamma b + 5\right)^3,$$

which we solve to get $b \in (0, 5.4735 \times 10^{-4}] \cup [230.39, \infty)$. Hence, (A5)′ (and so (A5)) is satisfied if $b \in (0, 5.4735 \times 10^{-4}] \cup [230.39, \infty)$.

Combining with the investigation in Example 4.1, we have that (A1)-(A5) is satisfied with $a \in [0.012243, 3.5027]$ and $b \in (0, 5.4735 \times 10^{-4}] \cup [230.39, \infty)$. Now, applying Theorem 3.9 with $a \in [0.012243, 3.5027]$, $b' \in (0, 5.4735 \times 10^{-4}]$, and $b \in [230.39, \infty)$ ($b' <$

$a < b$), we see that (4.1)-(4.2) has two *nontrivial positive* solutions $y_1, y_2 \in C^{(5)}[0,1]$ such that (3.35) holds. Noting the ranges of a, b', b, we further deduce from (3.35) the following for $0 \le k \le 3$:

$$\begin{cases} 0 \le y_1^{(k)}(t) < \frac{t^{3-k}}{(3-k)!}(0.012243) \le \frac{0.012243}{(3-k)!}, & t \in [0,1], \\ y_1^{(k)}(t) \ge \frac{1}{(3-k)!}(t-\frac{1}{4})^{3-k}\gamma(5.4735 \times 10^{-4}), & t \in [\frac{1}{4}, \frac{3}{4}], \\ 0 \le y_2^{(k)}(t) \le \frac{t^{3-k}}{(3-k)!}(230.39) \le \frac{230.39}{(3-k)!}, & t \in [0,1], \\ y_2^{(k)}(t) > \frac{1}{(3-k)!}(t-\frac{1}{4})^{3-k}\gamma(3.5027), & t \in [\frac{1}{4}, \frac{3}{4}]. \end{cases} \tag{4.10}$$

As seen in Example 4.1, the boundary value problem (4.1)-(4.2) has a known positive solution y^* given in (4.6), (4.8). Noting that $d_k > \gamma(3.5027)/(3-k)!$ and $c_k < (230.39)/(3-k)!$, this y^* may be y_2 in (4.10). This y^* is certainly not y_1. Hence, conclusion (4.10) is somewhat validated.

Further, it is obvious that (4.10) (obtained from Theorem 3.9) gives more details than (4.5) (obtained from Theorem 3.10(c)). As noted in Remark 3.4, more details come from (A5) being used twice in Theorem 3.9 but not at all in Theorem 3.10(c).

Example 4.3 Consider the Sturm-Liouville boundary value problem (4.1) with

$$\begin{aligned} F(t,y,y',y'',y''') = {} & \frac{36}{5}\left(\frac{35+90t+27t^2+9t^3+t^4-3t^5}{5}\right)^{-0.6} \\ & \times \left(y+y'+y''+y'''+1\right)^{0.6}. \end{aligned} \tag{4.11}$$

Clearly, (A1), (A2), and (A4) are satisfied with

$$\alpha(t) = \beta(t) = \frac{36}{5}\left(\frac{35+90t+27t^2+9t^3+t^4-3t^5}{5}\right)^{-0.6}$$

and

$$f(u_1,u_2,u_3,u_4) = (u_1+u_2+u_3+u_4+1)^{0.6}.$$

Note that Theorems 3.10(c) or 3.11(c) *cannot* be applied to this example because $f_0 = \infty$ and $f_\infty = 0$.

We proceed with checking (A3) and (A5). Similarly to Examples 4.1 and 4.2, solving the stricter inequalities (4.3) and (4.9), we obtain $a \in [80.313, \infty)$ and $b \in (0, 0.30913]$. Hence, (A3) and (A5) are satisfied if $a \in [80.313, \infty)$ and $b \in (0, 0.30913]$. Note that $a > b$.

Applying Theorem 3.7, we conclude that (4.1), (4.11) has a *nontrivial positive* solution $y_0 \in C^{(5)}[0,1]$ satisfying (3.25) and (3.26) for the case $a > b$. Noting that $a \in [80.313, \infty)$ and $b \in (0, 0.30913]$, we further obtain, for $0 \le k \le 3$,

$$\begin{cases} 0 \le y_0^{(k)}(t) < \frac{t^{3-k}}{(3-k)!}(80.313) \le \frac{80.313}{(3-k)!}, & t \in [0,1], \\ y_0^{(k)}(t) \ge \frac{1}{(3-k)!}(t-\frac{1}{4})^{3-k}\gamma(0.30913), & t \in [\frac{1}{4}, \frac{3}{4}]. \end{cases} \tag{4.12}$$

Now, it is known that (4.1), (4.11) has a positive solution y^* given in (4.6), (4.8). Noting that $c_k < (80.313)/(3-k)!$ and $d_k > \gamma(0.30913)/(3-k)!$, this y^* could just be y_0 in (4.12). Hence, conclusion (4.12) is somewhat validated.

Example 4.4 Consider the Sturm-Liouville boundary value problem (4.1) with

$$F(t,y,y',y'',y''') = \frac{36}{5}\left(\frac{530+90t+27t^2+9t^3+t^4-3t^5}{50}\right) \times \left(\frac{y+y'+y''+y'''+100}{10}\right). \tag{4.13}$$

Clearly, (A1), (A2), and (A4) are satisfied with

$$\alpha(t) = \beta(t) = \frac{36}{5}\left(\frac{530+90t+27t^2+9t^3+t^4-3t^5}{50}\right)$$

and

$$f(u_1,u_2,u_3,u_4) = \frac{u_1+u_2+u_3+u_4+100}{10}.$$

Once again, Theorems 3.10(c) or 3.11(c) *cannot* be applied to this example because $f_0 = \infty$ and $f_\infty = 0.4$.

Checking (A3) and (A5) as in Example 4.3, we solve (4.3) and (4.9) to get $a \in [26.577,\infty)$ and $b \in (0,1.4883]$. Hence, (A3) and (A5) are satisfied if $a \in [26.577,\infty)$ and $b \in (0,1.4883]$. Note that $a > b$.

An application of Theorem 3.7 gives a *nontrivial positive* solution $\bar{y} \in C^{(5)}[0,1]$ of (4.1), (4.13) satisfying (3.25) and (3.26) for the case $a > b$. Since $a \in [26.577,\infty)$ and $b \in (0,1.4883]$, we further obtain, for $0 \le k \le 3$,

$$\begin{cases} 0 \le \bar{y}^{(k)}(t) < \frac{t^{3-k}}{(3-k)!}(26.577) \le \frac{26.577}{(3-k)!}, & t \in [0,1], \\ \bar{y}^{(k)}(t) \ge \frac{1}{(3-k)!}(t-\frac{1}{4})^{3-k}\gamma(1.4883), & t \in [\frac{1}{4},\frac{3}{4}]. \end{cases} \tag{4.14}$$

In fact, (4.1), (4.13) has a positive solution y^* given in (4.6), (4.8). Since $c_k < (26.577)/(3-k)!$ and $d_k > \gamma(1.4883)/(3-k)!$, this y^* could be $\bar{y}$ in (4.14). Hence, conclusion (4.14) is somewhat validated.

Competing interests
None of the authors have any competing interests in the paper.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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