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The strong solutions for a class of fluid-particle interaction non-Newtonian models

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Abstract

The aim of this paper is to discuss the existence and uniqueness for a class of fluid-particle interaction non-Newtonian models which describe the evolution of particles dispersed in a viscous compressible non-Newtonian fluid. The strong nonlinearity of the system and the singularity of the viscosity term bring about difficulties. Also, we admit an initial vacuum.

Keywords: strong solution; fluid-particle interaction model; non-Newtonian fluid; vacuum

1 Introduction

In this paper, we consider the following non-Newtonian fluids system:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x - \lambda(|u_x|^{p-2}u_x)_x + (P+\eta)_x = -(\eta + \beta\rho)\Phi_x, \quad (x,t) \in \Omega_T, \\ \eta_t + (\eta(u-\Phi_x))_x = \eta_{xx}, \end{cases} \tag{1.1}$$

with the initial conditions

$$(\rho, u, \eta)|_{t=0} = (\rho_0, u_0, \eta_0), \quad x \in \Omega, \tag{1.2}$$

together with the no-slip boundary conditions for the velocity and the no-flux condition for the density of particles,

$$u|_{\partial\Omega} = (\eta_x + \eta\Phi_x)|_{\partial\Omega} = 0, \quad t \in [0,T], \tag{1.3}$$

where $\rho, u, \eta, P(\rho) = a\rho^\gamma, \Phi(x)$ denote the fluid density, velocity, the density of particles in the mixture, the pressure, and the external potential, respectively, $a > 0$, $\gamma > 1$, $\frac{4}{3} < p < 2$. $\lambda > 0$ is the viscosity coefficient and $\beta \neq 0$ is a constant, Ω is a one-dimensional bounded interval, and for simplicity we only consider $\Omega = (0,1)$, $\Omega_T = \Omega \times [0,T]$.

Fluid-particle interaction models arise in a lot of industrial procedures such as the analysis of the sedimentation phenomenon. These procedures find their applications in

biotechnology, medicine, mineral processing, and chemical engineering [1–4]. Such interaction systems are also used in combustion theory, when modeling Diesel engines or rocket propulsors [5, 6].

The coupled microscopic/macroscopic models describe the evolution of particles dispersed in a fluid. The system consists in a Vlasov-Fokker-Planck equation to describe the microscopic motion of the particles coupled to the equations for the fluid. Generally speaking, at the microscopic scale, the cloud of particles is described by its distribution function $f(x,\xi,t)$, a solution to a Vlasov-Fokker-Planck equation. The fluid, on the other hand, is modeled by macroscopic quantities, namely its density $\rho(x,t) \geq 0$ and its velocity field $u(x,t)$. If the fluid is compressible and isentropic, then (ρ,u) solves the compressible Euler (inviscid case) or Navier-Stokes system (viscous case) of equations. With the dynamic viscosity terms taken into consideration, Carrillo *et al.* [7] discussed the following system:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla(P(\rho) + \eta) - \mu\Delta u - \lambda\nabla \operatorname{div} u = -(\eta + \beta\rho)\nabla\Phi, \\ \eta_t + \operatorname{div}(\eta(u - \nabla\Phi)) - \Delta\eta = 0. \end{cases} \tag{1.4}$$

They obtained the global existence and asymptotic behavior of the weak solutions to (1.4) following the framework of Lions [8] and Feireisl *et al.* [9, 10]. In addition, Mellet and Vasseur [11] proved the global existence of weak solutions of equations by using the entropy method on the asymptotic regime corresponding to a strong drag force and strong Brownian motion.

In recent years, there has been an increasing recognition of the importance of non-Newtonian flow characteristics displayed by most materials encountered in everyday life, both in nature (gums, proteins, biological fluids such as blood, synovial fluid, *etc.*) and in technology (polymers and plastics, emulsions, slurries, *etc.*) (see [12]). Since there has been much research in the field of non-Newtonian flows, both theoretically and experimentally, let us briefly recall the related results in the literature. Bellout *et al.* [13] studied the non-Newtonian fluids for space periodic problems and showed the Young measure-valued solutions. In [14], Guo and Zhu investigated the partial regularity of the generalized solutions to the modified Navier-Stokes equations which describes the dynamics of the incompressible monopolar non-Newtonian fluids. Zhao *et al.* [15] constructed the trajectory attractor and global attractor for an autonomous two-dimensional non-Newtonian fluid. Yuan and Xu [16] obtained the existence and uniqueness of solutions for a class of non-Newtonian fluids with singularity and vacuum. For other results we may refer to [17–23].

It is worth mentioning that most recently, Fang *et al.* [24] got the global existence of classical solutions of (1.4) in dimension one, namely $p = 2$ in (1.1). Compared with the work of [24], the strong nonlinearity of (1.1) brings us new difficulties in getting the upper bound of ρ, which plays an important role throughout their proof. The second equation of (1.1) is always with singularity and brings us another difficulty. Motivated by Cho *et al.*'s [25, 26] work on the Navier-Stokes equations, we establish local existence and uniqueness of strong solutions of (1.1).

Throughout the paper we assume that $a = \lambda = 1$ for simplicity. In the following sections, we will use simplified notations for standard Sobolev spaces and Bochner spaces, such as $L^p = L^p(\Omega)$, $H_0^1 = H_0^1(\Omega)$, $C([0,T];H^1) = C([0,T];H^1(\Omega))$.

1.1 Main results

Theorem 1.1 *Let* $\Phi \in C^2(\Omega)$ *and assume that the initial data* (ρ_0, u_0, η_0) *satisfy the following conditions*:

$$0 \le \rho_0 \in H^1(\Omega), \qquad u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \qquad \eta_0 \in H^2(\Omega),$$

and the compatibility condition

$$\big(|u_{0x}|^{p-2}u_{0x}\big)_x - \big(P(\rho_0) + \eta_0\big)_x - \eta_0\Phi_x = \rho_0(g + \beta\Phi_x), \tag{1.5}$$

for some $g \in L^2(\Omega)$. *Then there exist a* $T_* \in (0, +\infty)$ *and a unique strong solution* (ρ, u, η) *to* (1.1)-(1.3) *such that*

$$\begin{aligned}
&\rho \in L^\infty\big(0, T_*; H^1(\Omega)\big), \qquad \rho_t \in L^\infty\big(0, T_*; L^2(\Omega)\big),\\
&u \in L^\infty\big(0, T_*; W_0^{1,p}(\Omega) \cap H^2(\Omega)\big), \qquad u_t \in L^2\big(0, T_*; H_0^1(\Omega)\big),\\
&\eta \in L^\infty\big(0, T_*; H^2(\Omega)\big), \qquad \eta_t \in L^\infty\big(0, T_*; H^1(\Omega)\big),\\
&\sqrt{\rho}u_t \in L^\infty\big(0, T_*; L^2(\Omega)\big), \qquad \big(|u_x|^{p-2}u_x\big)_x \in L^2\big(0, T_*; L^2(\Omega)\big).
\end{aligned}$$

2 *A priori* estimates for smooth solutions

In this section, we will prove the local existence of strong solutions. Because equation $(1.1)_2$ always possesses a singularity, we overcome this difficulty by a regularized process, then taking the limiting process back to the original problem. First of all, we consider the following system:

$$\rho_t + (\rho u)_x = 0, \tag{2.1}$$

$$(\rho u)_t + \big(\rho u^2\big)_x - \left[\left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon}\right)^{\frac{2-p}{2}} u_x\right]_x + (P + \eta)_x = -(\eta + \beta\rho)\Phi_x, \tag{2.2}$$

$$\eta_t + \big(\eta(u - \Phi_x)\big)_x = \eta_{xx}, \tag{2.3}$$

with the initial and boundary conditions,

$$(\rho, u, \eta)|_{t=0} = (\rho_0, u_0, \eta_0), \quad x \in \Omega, \tag{2.4}$$

$$u|_{\partial\Omega} = (\eta_x + \eta\Phi_x)|_{\partial\Omega} = 0, \quad t \in [0, T], \tag{2.5}$$

and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ is the smooth solution of the boundary value problem

$$\begin{cases} [(\frac{\varepsilon u_{0x}^2 + 1}{u_{0x}^2 + \varepsilon})^{\frac{2-p}{2}} u_{0x}]_x - (P(\rho_0) + \eta_0)_x - \eta_0\Phi_x = \rho_0(g + \beta\Phi_x),\\ u_0(0) = u_0(1) = 0. \end{cases} \tag{2.6}$$

By using the iterative method step by step, the nonlinear coupled system admits a smooth solution (see Section 3). Provided that (ρ, u, η) is a smooth solution of (2.1)-(2.5) and $\rho_0 \ge \delta$, where $0 < \delta \ll 1$ is a positive number. We denote $M_0 = 1 + \mu_0 + \mu_0^{-1} + |\rho_0|_{H^1} + |g|_{L^2}$.

First we obtain the estimate of $|u_{0xx}|_{L^2}$. From (2.6), we have

$$u_{0xx} = \left(\frac{\varepsilon u_{0x}^2 + 1}{u_{0x}^2 + \varepsilon}\right)^{\frac{p}{2}} \frac{(u_{0x}^2 + \varepsilon)^2[(P(\rho_0) + \eta_0)_x + \eta_0\Phi_x + \rho_0(g + \beta\Phi_x)]}{(\varepsilon u_{0x}^2 + 1)(u_{0x}^2 + \varepsilon) - (2 - p)(1 - \varepsilon^2)u_{0x}^2}.$$

Then

$$\begin{aligned}|u_{0xx}|_{L^2} &\le \frac{1}{p-1}\left|\left(\frac{u_{0x}^2+\varepsilon}{\varepsilon u_{0x}^2+1}\right)^{1-\frac{p}{2}}\right|_{L^\infty}\left|\big(P(\rho_0)+\eta_0\big)_x+\eta_0\Phi_x+\rho_0(g+\beta\Phi_x)\right|_{L^2}\\ &\le \frac{1}{p-1}\big(|u_{0x}|_{L^\infty}^2+1\big)^{1-\frac{p}{2}}\left(\left|\big(P(\rho_0)+\eta_0\big)_x+\eta_0\Phi_x+\rho_0(g+\beta\Phi_x)\right|_{L^2}\right)\\ &\le \frac{1}{p-1}\big(|u_{0xx}|_{L^2}^2+1\big)^{1-\frac{p}{2}}\big(a\gamma|\rho_0|_{L^\infty}^{\gamma-1}|\rho_{0x}|_{L^2}+|\eta_{0x}|_{L^2}+|\eta_0|_{L^\infty}|\Phi_x|_{L^2}\\ &\quad+|\rho_0|_{L^\infty}|g|_{L^2}+\beta|\rho_0|_{L^\infty}|\Phi_x|_{L^2}\big).\end{aligned}$$

Using Young's inequality, we have

$$|u_{0xx}|_{L^2}\le C,$$

where C is a positive constant, depending only on M_0.

Next, we introduce an auxiliary function,

$$\Psi(t)=\sup_{0\le s\le t}\big(1+\big|\rho(s)\big|_{H^1}+\big|u(s)\big|_{W_0^{1,p}}+\big|\sqrt{\rho}u_t(s)\big|_{L^2}+\big|\eta_t(s)\big|_{L^2}+\big|\eta(s)\big|_{H^1}\big).$$

Then we estimate each term of $\Psi(t)$ in terms of some integrals of $\Psi(t)$, apply arguments of Gronwall-type, and thus prove that $\Psi(t)$ is locally bounded.

2.1 Estimate for $|\rho|_{H^1}$

First we need the following estimates for u and η. By virtue of (2.2)

$$\left[\left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x\right]_x=\rho u_t+\rho uu_x+(P+\eta)_x+(\eta+\beta\rho)\Phi_x.$$

Then, we have

$$|u_{xx}|\le\frac{1}{p-1}\big(u_x^2+\varepsilon\big)^{1-\frac{p}{2}}\big|\rho u_t+\rho uu_x+(P+\eta)_x+(\eta+\beta\rho)\Phi_x\big|.$$

Taking the L^2 norm and using Young's inequality, we get

$$\begin{aligned}|u_{xx}|_{L^2}^{p-1} &\le C\big(1+|\rho u_t|_{L^2}+|\rho uu_x|_{L^2}+\big|(P+\eta)_x\big|_{L^2}+\big|(\eta+\beta\rho)\Phi_x\big|_{L^2}\big)\\ &\le C\big(1+|\rho|_{L^\infty}^{\frac12}|\sqrt{\rho}u_t|_{L^2}+|\rho|_{L^\infty}|u|_{L^\infty}|u_x|_{L^p}^{\frac{p}{2}}|u_x|_{L^\infty}^{1-\frac{p}{2}}\\ &\quad+|\rho|_{L^\infty}^{\gamma-1}|\rho_x|_{L^2}+|\eta_x|_{L^2}+|\eta|_{L^\infty}|\Phi_x|_{L^2}+|\rho|_{L^\infty}|\Phi_x|_{L^2}\big)\\ &\le C\big[1+|\rho|_{L^\infty}^{\frac12}|\sqrt{\rho}u_t|_{L^2}+\big(|\rho|_{L^\infty}|u|_{L^\infty}|u_x|_{L^p}^{\frac{p}{2}}\big)^{\frac{2(p-1)}{3p-4}}+|\rho|_{L^\infty}^{\gamma-1}|\rho_x|_{L^2}\\ &\quad+|\eta_x|_{L^2}+|\eta|_{L^\infty}|\Phi_x|_{L^2}+|\rho|_{L^\infty}|\Phi_x|_{L^2}\big]+\frac12|u_{xx}|_{L^2}^{p-1}\\ &\le C\Psi^{\frac{(4+p)(p-1)}{3p-4}\gamma}(t)+\frac12|u_{xx}|_{L^2}^{p-1}.\end{aligned}\tag{2.7}$$

Hence, we deduce that

$$|u_{xx}|_{L^2}\le C\Psi^{\frac{6\gamma}{3p-4}}(t).\tag{2.8}$$

From (2.3), taking the L^2 norm, we get

$$\begin{aligned}|\eta_{xx}|_{L^2} &\le \big|\eta_t + \big(\eta(u-\Phi_x)\big)_x\big|_{L^2}\\ &\le |\eta_t|_{L^2} + |\eta_x|_{L^2}|u|_{L^\infty} + |\eta_x|_{L^2}|\Phi_x|_{L^\infty} + |\eta|_{L^2}|u_{xx}|_{L^2} + |\eta|_{L^\infty}|\Phi_{xx}|_{L^2}\\ &\le C\Psi^{\frac{8\gamma}{3p-4}}(t). \end{aligned} \tag{2.9}$$

Multiplying (2.1) by ρ, integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho|^2\,\mathrm{d}s + \int_\Omega (\rho u)_x\rho\,\mathrm{d}x = 0.$$

Integrating by parts, using the Sobolev inequality, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|\rho(t)\big|_{L^2}^2 \le \int_\Omega |u_x||\rho|^2\,\mathrm{d}x \le |u_{xx}|_{L^2}|\rho|_{L^2}^2. \tag{2.10}$$

Differentiating $(1.1)_1$ with respect to x, and multiplying it by ρ_x, integrating over Ω, and using the Sobolev inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho_x|^2\mathrm{d}x &= -\int_\Omega\left[\frac{3}{2}u_x(\rho_x)^2 + \rho\rho_x u_{xx}\right](t)\mathrm{d}x\\ &\le C\big[|u_x|_{L^\infty}|\rho_x|_{L^2}^2 + |\rho|_{L^\infty}|\rho_x|_{L^2}|u_{xx}|_{L^2}\big]\\ &\le C|\rho|_{H^1}^2|u_{xx}|_{L^2}. \end{aligned} \tag{2.11}$$

From (2.10)and (2.11), by Gronwall's inequality, it follows that

$$\sup_{0\le t\le T}\big|\rho(t)\big|_{H^1}^2 \le |\rho_0|_{H^1}^2\exp\left\{C\int_0^t |u_{xx}|_{L^2}\,\mathrm{d}s\right\} \le C\exp\left(C\int_0^t \Psi^{\frac{6}{3p-4}\gamma}(s)\,\mathrm{d}s\right), \tag{2.12}$$

we can also get the following estimates. Using $(1.1)_1$ we obtain

$$\big|\rho_t(t)\big|_{L^2} \le \big|\rho_x(t)\big|_{L^2}\big|u(t)\big|_{L^\infty} + \big|\rho(t)\big|_{L^\infty}\big|u_x(t)\big|_{L^2} \le C\Psi^2(t), \tag{2.13}$$

where C is a positive constant, depending only on M_0.

2.2 Estimate for $|\eta_t|_{L^2}$ and $|\eta|_{H^1}$

Multiplying $(1.1)_3$ by η, integrating the resulting equation over Ω_T, using the boundary conditions (1.3) and Young's inequality, we have

$$\begin{aligned}&\int_0^t \big|\eta_x(s)\big|_{L^2}^2\,\mathrm{d}s + \frac{1}{2}\big|\eta(t)\big|_{L^2}^2\\ &\quad\le \iint_{\Omega_T}\big(|\eta u\eta_x| + |\eta\Phi_x\eta_x|\big)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\le \frac{1}{4}\int_0^t \big|\eta_x(s)\big|_{L^2}^2\,\mathrm{d}s + C\int_0^t |u_x|_{L^p}^2|\eta|_{H^1}^2\,\mathrm{d}s + C\int_0^t |\eta|_{H^1}^2 + C\\ &\quad\le \frac{1}{4}\int_0^t \big|\eta_x(s)\big|_{L^2}^2\,\mathrm{d}s + C\left(1 + \int_0^t \Psi^4(t)\,\mathrm{d}s\right). \end{aligned} \tag{2.14}$$

Multiplying $(1.1)_3$ by η_t, integrating (by parts) over Ω_T, using the boundary conditions (1.3) and Young's inequality, we have

$$\begin{aligned}
&\int_0^t \left|\eta_t(s)\right|_{L^2}^2 \mathrm{d}s + \frac{1}{2}\left|\eta_x(t)\right|_{L^2}^2 \\
&\quad \le \iint_{\Omega_T} \left|\eta(u-\Phi_x)\eta_{xt}\right| \mathrm{d}x\,\mathrm{d}s \\
&\quad \le \frac{1}{4}\int_0^t \left|\eta_{xt}(s)\right|_{L^2}^2 \mathrm{d}s + C\int_0^t |\eta|_{H^1}^2 |u_x|_{L^p}^2 \mathrm{d}s + C\int_0^t |\eta|_{H^1}^2 \mathrm{d}s + C \\
&\quad \le \frac{1}{4}\int_0^t \left|\eta_{xt}(s)\right|_{L^2}^2 \mathrm{d}s + C\left(1 + \int_0^t \Psi^4(t)\,\mathrm{d}s\right). \qquad (2.15)
\end{aligned}$$

Differentiating $(1.1)_3$ with respect to t, multiplying the resulting equation by η_t, integrating (by parts) over Ω_T, we get

$$\begin{aligned}
&\int_0^t \left|\eta_{xt}(s)\right|_{L^2}^2 \mathrm{d}s + \frac{1}{2}\left|\eta_t(t)\right|_{L^2}^2 \\
&\quad = \iint_{\Omega_T} \big(\eta(u-\Phi_x)\big)_t \eta_{xt}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad \le C + \iint_{\Omega_T} \left(|\eta_t u\eta_{xt}| + |\eta_t \Phi_x \eta_{xt}| + |\eta_x u_t \eta_t| + |\eta u_{xt}\eta_t|\right) \mathrm{d}x\,\mathrm{d}s \\
&\quad \le C\left(1 + \int_0^t \left(|\eta_t|_{L^2}^2 |u_x|_{L^p}^2 + |\eta_t|_{L^2}^2 + |\eta_x|_{L^2}^2 |\eta_t|_{L^2}^2 + |\eta|_{H^1}^2 |\eta_t|_{L^2}^2\right) \mathrm{d}x\right) \\
&\qquad + \frac{1}{2}\int_0^t |\eta_{xt}|_{L^2}^2 + \frac{1}{2}\int_0^t |u_{xt}|_{L^2}^2 \\
&\quad \le C\left(1 + \int_0^t \Psi^4(s)\,\mathrm{d}s\right). \qquad (2.16)
\end{aligned}$$

Combining (2.14)-(2.16) and (2.28), we get

$$|\eta|_{H^1}^2 + |\eta_t|_{L^2}^2 + \int_0^t \left(|\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2\right)(s)\,\mathrm{d}s \le C\left(1 + \int_0^t \Psi^4(s)\,\mathrm{d}s\right). \qquad (2.17)$$

2.3 Estimate for $|u|_{W_0^{1,p}}$

Using (2.1), we rewrite (2.1) as

$$\rho u_t + \rho u u_x - \left[\left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon}\right)^{\frac{2-p}{2}} u_x\right]_x + (P+\eta)_x = -(\eta + \beta\rho)\Phi_x. \qquad (2.18)$$

Multiplying (2.18) by u_t, integrating (by parts) over Ω_T, we have

$$\begin{aligned}
&\iint_{\Omega_T} \rho|u_t|^2 \,\mathrm{d}x\,\mathrm{d}s + \iint_{\Omega_T} \left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon}\right)^{\frac{2-p}{2}} u_x u_{xt}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad = -\iint_{\Omega_T} \big(\rho u u_x + P_x + \eta_x + (\eta + \beta\rho)\Phi_x\big) u_t \,\mathrm{d}x\,\mathrm{d}s. \qquad (2.19)
\end{aligned}$$

We deal with each term as follows:

$$\int_\Omega \left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{\frac{2-p}{2}} u_x u_{xt}\,\mathrm{d}x = \frac{1}{2}\int_\Omega \left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{\frac{2-p}{2}} \left(u_x^2\right)_t \mathrm{d}x$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\int_0^{u_x^2}\left(\frac{\varepsilon s+1}{s+\varepsilon}\right)^{\frac{2-p}{2}}\mathrm{d}s\right)\mathrm{d}x,$$

$$\int_0^{u_x^2}\left(\frac{\varepsilon s+1}{s+\varepsilon}\right)^{\frac{2-p}{2}}\mathrm{d}s \ge \int_0^{u_x^2}(s+1)^{\frac{2-p}{2}}\,\mathrm{d}s = \frac{2}{p}\left[\left(u_x^2+1\right)^{\frac{p}{2}}-1\right],$$

$$-\iint_{\Omega_T} P_x u_t\,\mathrm{d}x\,\mathrm{d}s = \iint_{\Omega_T} P u_{xt}\,\mathrm{d}x\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega_T} P u_x\,\mathrm{d}x\,\mathrm{d}s - \iint_{\Omega_T} P_t u_x\,\mathrm{d}x\,\mathrm{d}s.$$

From $(1.1)_1$ we have

$$P_t = -\gamma P u_x - P_x u, \tag{2.20}$$

$$-\iint_{\Omega_T}\eta\Phi_x u_t\,\mathrm{d}x\,\mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega_T}\eta\Phi_x u\,\mathrm{d}x\,\mathrm{d}s + \iint_{\Omega_T}\eta_t\Phi_x u\,\mathrm{d}x\,\mathrm{d}s.$$

Substituting the above into (2.19), using the Sobolev inequality and Young's inequality, we obtain

$$\begin{aligned}
&\int_0^t \left|\sqrt{\rho}u_t(s)\right|_{L^2}^2\mathrm{d}s + \left|u_x(t)\right|_{L^p}^p \\
&\le \iint_{\Omega_T}\left(|\rho u u_x u_t| + \left|\gamma P u_x^2\right| + |P_x u u_x| + |\eta_t u_x| + |\eta_t\Phi_x u| + |\beta\rho\Phi_x u_t|\right)\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_\Omega\left(|Pu_x| + |\eta u_x| + |\eta\Phi_x u|\right)\mathrm{d}x + C \\
&\le C + \int_0^t\Big(|\rho|_{L^2}|u|_{L^\infty}|u_x|_{L^p}^{\frac{p}{2}}|u_x|_{L^\infty}^{1-\frac{p}{2}}|u_t|_{L^\infty} + \gamma|P|_{L^2}|u_x|_{L^p}^{\frac{p}{2}}|u_x|_{L^\infty}^{1-\frac{p}{2}}|u_{xx}|_{L^2} \\
&\quad + a\gamma|\rho|_{L^\infty}^{\gamma-1}|\rho_x|_{L^2}|u_x|_{L^p}|u_{xx}|_{L^2} + |\eta_t|_{L^2}|u_x|_{L^p}^{\frac{p}{2}}|u_x|_{L^\infty}^{1-\frac{p}{2}} + |\eta_t|_{L^2}|\Phi_x|_{L^2}|u|_{L^\infty} \\
&\quad + \beta|\rho|_{L^\infty}^{\frac{1}{2}}|\Phi_x|_{L^2}|\sqrt{\rho}u_t|_{L^2}\Big)\mathrm{d}s + |P|_{L^{\frac{p}{p-1}}}|u_x|_{L^p} + |\eta|_{L^{\frac{p}{p-1}}}|u_x|_{L^p} + |\eta|_{L^{\frac{p}{p-1}}}|\Phi_x|_{L^p}|u|_{L^\infty} \\
&\le C\Bigg(1 + \int_0^t\Big(|\rho|_{L^2}^2|u_x|_{L^p}^{2+p}|u_{xx}|_{L^2}^{2-p} + |P|_{L^\infty}|u_x|_{L^p}^{\frac{p}{2}}|u_{xx}|_{L^2}^{2-\frac{p}{2}} \\
&\quad + |\rho|_{L^\infty}^{\gamma-1}|\rho_x|_{L^2}|u_x|_{L^p}|u_{xx}|_{L^2} + |\eta_t|_{L^2}|u_x|_{L^p}^{\frac{p}{2}}|u_{xx}|_{L^2}^{1-\frac{p}{2}} + |\eta_t|_{L^2}|u_x|_{L^p} + |\rho|_{L^\infty}\Big)\mathrm{d}s\Bigg) \\
&\quad + |P|_{L^{\frac{p}{p-1}}}^{\frac{p}{p-1}} + |\eta|_{L^{\frac{p}{p-1}}}^{\frac{p}{p-1}} + \frac{1}{2}\int_0^t\left|\sqrt{\rho}u_t(s)\right|_{L^2}^2\mathrm{d}s + \frac{1}{2}\left|u_x(t)\right|_{L^p}^p.
\end{aligned} \tag{2.21}$$

To estimate (2.21), combining (2.20) we have the following estimates:

$$\int_\Omega\left|P(t)\right|^{\frac{p}{p-1}}\mathrm{d}x = \int_\Omega\left|P(0)\right|^{\frac{p}{p-1}}\mathrm{d}x + \int_0^t\frac{\partial}{\partial s}\left(\int_\Omega P(s)^{\frac{p}{p-1}}\,\mathrm{d}x\right)\mathrm{d}s$$

$$\le \int_\Omega\left|P(0)\right|^{\frac{p}{p-1}}\mathrm{d}x + \frac{p}{p-1}\int_0^t\int_\Omega a\gamma\rho^{\gamma-1}P(s)^{\frac{p}{p-1}}(-\rho_x u - \rho u_x)\,\mathrm{d}x\,\mathrm{d}s$$

$$\begin{aligned}&\le C + C\int_0^t |\rho|_{L^\infty}^{\gamma-1}|P|_{L^\infty}^{\frac{p}{p-1}}|\rho|_{H^1}|u_x|_{L^p}\,\mathrm{d}s\\&\le C\Big(1+\int_0^t \Psi^{\frac{1}{p-1}+\gamma+1}(s)\,\mathrm{d}s\Big),\end{aligned}\tag{2.22}$$

following the same method, we get

$$\int_\Omega \big|\eta(t)\big|^{\frac{p}{p-1}}\,\mathrm{d}x \le C\Big(1+\int_0^t \Psi^{\frac{1}{p-1}+1}(s)\,\mathrm{d}s\Big).\tag{2.23}$$

Combining (2.21)-(2.23) yields

$$\int_0^t \big|\sqrt{\rho}u_t(s)\big|_{L^2}^2(s)\,\mathrm{d}s + \big|u_x(t)\big|_{L^p}^p \le C\Big(1+\int_0^t \Psi^{\frac{18\gamma}{3p-4}}(s)\,\mathrm{d}s\Big),\tag{2.24}$$

where C is a positive constant, depending only on M_0.

2.4 Estimate for $|\sqrt{\rho}u_t|_{L^2}$

Differentiating $(1.1)_2$ with respect to t, we get

$$\begin{aligned}&\rho u_{tt} + \rho u u_{xt} + \rho_t u_t + \rho_t u u_x + \rho u_t u_x + (P+\eta)_{xt}\\&\quad= \left[\left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x\right]_{xt} - (\eta_t+\beta\rho_t)\Phi_x.\end{aligned}$$

Multiplying the result equation by u_t, integrating over Ω, we derive

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|u_t|^2\,\mathrm{d}x + \int_\Omega\left[\left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x\right]_t u_{xt}\,\mathrm{d}x\\&\quad= \int_\Omega\big[(\rho u)_x\big(u_t^2+uu_xu_t\big) - (P+\eta)_t u_{xt} - (\eta_t+\beta\rho_t)\Phi_x u_t\big]\,\mathrm{d}x.\end{aligned}\tag{2.25}$$

Note that

$$\begin{aligned}&\int_\Omega\left[\left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x\right]_t u_{xt}\,\mathrm{d}x\\&\quad= \int_\Omega\left[\left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{-\frac{p}{2}}u_x\right]\frac{(\varepsilon u_x^2+1)(u_x^2+\varepsilon)-(2-p)(1-\varepsilon^2)u_x^2}{(u_x^2+\varepsilon)^2}u_{xt}^2\,\mathrm{d}x\\&\quad\ge (p-1)\int_\Omega\big(u_x^2+1\big)^{\frac{p-2}{2}}|u_{xt}|^2\,\mathrm{d}x.\end{aligned}$$

Let

$$\zeta = \big(u_x^2+1\big)^{\frac{p-2}{4}},$$

from (2.8), it follows that

$$\big|\zeta^{-1}\big|_{L^\infty} = \big|\big(u_x^2+1\big)^{\frac{2-p}{4}}\big|_{L^\infty} \le C\big(|u_{xx}|_{L^2}^{\frac{2-p}{2}}+1\big) \le C\Psi^{\frac{3\gamma}{3p-4}}(t).$$

Combining (2.20), (2.25) can be rewritten as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|u_t|^2\,\mathrm{d}x+\int_\Omega |u_{xt}|^2\,\mathrm{d}x\\
&\quad\le 2\int_\Omega \rho|u||u_t||u_{xt}|\,\mathrm{d}x+\int_\Omega |\rho_x||u|^2|u_x||u_t|\,\mathrm{d}x+\int_\Omega \rho|u||u_x|^2|u_t|\,\mathrm{d}x\\
&\qquad+\int_\Omega \gamma P|u_x||u_{xt}|\,\mathrm{d}x+\int_\Omega |P_x||u||u_{xt}|\,\mathrm{d}x+\int_\Omega |\eta_t||\Phi_x||u_t|\,\mathrm{d}x\\
&\qquad+\int_\Omega |\beta\rho_x||u||\Phi_x||u_t|\,\mathrm{d}x+\int_\Omega |\beta\rho||u_x||\Phi_x||u_t|\,\mathrm{d}x=\sum_{j=1}^{8} I_j. \qquad (2.26)
\end{aligned}$$

Using the Sobolev inequality, Young's inequality, $(1.1)_2$, (2.8), and (2.9), we obtain

$$\begin{aligned}
&I_1\le 2|\rho|_{L^\infty}^{\frac12}|u|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}|\zeta u_{xt}|_{L^2}\big|\zeta^{-1}\big|_{L^\infty}\le C\Psi^{\frac{8\gamma}{3p-4}}(t)+\frac18|u_{xt}|_{L^2}^2,\\
&I_2\le |\rho_x|_{L^2}|u|_{L^\infty}^2|u_x|_{L^p}|u_x|_{L^\infty}^{1-\frac{p}{2}}|u_t|_{L^\infty}\le |\rho_x|_{L^2}|u_x|_{L^p}^3|u_{xx}|_{L^2}^{1-\frac{p}{2}}|\zeta u_{xt}|_{L^2}\big|\zeta^{-1}\big|_{L^\infty}\\
&\quad\le C\Psi^{\frac{12\gamma}{3p-4}}(t)+\frac18|u_{xt}|_{L^2}^2,\\
&I_3\le |\rho|_{L^{\frac{p}{p-1}}}|u|_{L^\infty}|u_x|_{L^p}|u_x|_{L^\infty}|u_t|_{L^\infty}\le |\rho|_{L^\infty}|u_x|_{L^p}^2|u_{xx}|_{L^2}|\zeta u_{xt}|_{L^2}\big|\zeta^{-1}\big|_{L^\infty}\\
&\quad\le C\Psi^{\frac{15\gamma}{3p-4}}(t)+\frac18|u_{xt}|_{L^2}^2,\\
&I_4\le C|P|_{L^2}|u_x|_{L^\infty}|\zeta u_{xt}|_{L^2}\big|\zeta^{-1}\big|_{L^\infty}\le C\Psi^{\frac{7\gamma}{3p-4}}(t)+\frac18|u_{xt}|_{L^2}^2,\\
&I_5\le |P_x|_{L^2}|u|_{L^\infty}|\zeta u_{xt}|_{L^2}\big|\zeta^{-1}\big|_{L^\infty}\le C\Psi^{\frac{7\gamma}{3p-4}}(t)+\frac18|u_{xt}|_{L^2}^2,\\
&I_6\le |\eta_t|_{L^2}|\Phi_x|_{L^2}|u_t|_{L^\infty}\le C|\eta_t|_{L^2}|\zeta u_{xt}|_{L^2}\big|\zeta^{-1}\big|_{L^\infty}\le C\Psi^{\frac{11\gamma}{3p-4}}(t)+\frac18|u_{xt}|_{L^2}^2,\\
&I_7\le \beta|\rho_x|_{L^2}|u|_{L^\infty}|\Phi_x|_{L^2}|u_t|_{L^\infty}\le C|\rho_x|_{L^2}|u_x|_{L^p}|\zeta u_{xt}|_{L^2}\big|\zeta^{-1}\big|_{L^\infty}\\
&\quad\le C\Psi^{\frac{9\gamma}{3p-4}}(t)+\frac18|u_{xt}|_{L^2}^2,\\
&I_8\le \beta|\rho|_{L^2}|u_x|_{L^\infty}|\Phi_x|_{L^2}|u_t|_{L^\infty}\le C|\rho|_{L^2}|u_{xx}|_{L^2}|\zeta u_{xt}|_{L^2}\big|\zeta^{-1}\big|_{L^\infty}\\
&\quad\le C\Psi^{\frac{11\gamma}{3p-4}}(t)+\frac18|u_{xt}|_{L^2}^2.
\end{aligned}$$

Substituting I_j ($j=1,2,\dots,8$) into (2.26), and integrating over $(\tau,t)\subset(0,T)$ over the time variable, we have

$$\big|\sqrt{\rho}u_t(t)\big|_{L^2}^2+\int_0^t |u_{xt}|_{L^2}^2(s)\,\mathrm{d}s\le \big|\sqrt{\rho}u_t(\tau)\big|_{L^2}^2+\int_0^t \Psi^{\frac{15\gamma}{3p-4}}(s)\,\mathrm{d}s. \qquad (2.27)$$

To obtain the estimate of $|\sqrt{\rho}u_t(t)|_{L^2}^2$, we need to estimate $\lim_{\tau\to0}|\sqrt{\rho}u_t(\tau)|_{L^2}^2$. Multiplying (2.18) by u_t and integrating over Ω, we get

$$\begin{aligned}
&\int_\Omega \rho|u_t|^2\,\mathrm{d}x\\
&\quad\le 2\int_\Omega\left(\rho|u|^2|u_x|^2+\beta^2\rho|\Phi_x|^2+\rho^{-1}\left|\left[\left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x\right]_x+(P+\eta)_x+\eta\Phi_x\right|^2\right)\mathrm{d}x.
\end{aligned}$$

According to the smoothness of (ρ, u, η), we obtain

$$\begin{aligned}
&\lim_{\tau\to 0}\int_\Omega \left(\rho|u|^2|u_x|^2+\beta^2\rho|\Phi_x|^2+\rho^{-1}\left|\left[\left(\frac{\varepsilon u_x^2+1}{u_x^2+\varepsilon}\right)^{\frac{2-p}{2}}u_x\right]_x+(P+\eta)_x+\eta\Phi_x\right|^2\right)\mathrm{d}x\\
&\quad=\int_\Omega\left(\rho|u_0|^2|u_{0x}|^2+\beta^2\rho_0|\Phi_x|^2\right.\\
&\qquad\left.+\rho_0^{-1}\left|\left[\left(\frac{\varepsilon u_{0x}^2+1}{u_{0x}^2+\varepsilon}\right)^{\frac{2-p}{2}}u_{0x}\right]_x+(P_0+\eta_0)_x+\eta_0\Phi_x\right|^2\right)\mathrm{d}x\\
&\quad\le|\rho_0|_{L^\infty}|u_0|_{L^\infty}^2|u_{0x}|_{L^2}^2+\beta^2|\rho_0|_{L^\infty}|\Phi_x|^2+|g|_{L^2}^2+\beta|\Phi_x|_{L^2}^2\le C.
\end{aligned}$$

Therefore, taking the limit on τ in (2.27), as $\tau\to 0$, we conclude that

$$\left|\sqrt{\rho}u_t(t)\right|_{L^2}^2+\int_0^t|u_{xt}|_{L^2}^2(s)\,\mathrm{d}s\le C\left(1+\int_0^t\Psi^{\frac{15\gamma}{3p-4}}(s)\,\mathrm{d}s\right),\tag{2.28}$$

where C is a positive constant, depending only on M_0.

Combining the estimates of (2.8), (2.9), (2.12), (2.13), (2.17), (2.24), (2.28), and the definition of $\Psi(t)$, we conclude that

$$\Psi(t)\le C\exp\left(\tilde{C}\int_0^t\Psi^{\frac{18\gamma}{3p-4}}(s)\,\mathrm{d}s\right),\tag{2.29}$$

where C, $\tilde{C}$ are positive constant, depending only on M_0. This means that there exist a time $T_1>0$ and a constant C, such that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\le t\le T_1}\left(|\rho|_{H^1}+|u|_{W_0^{1,p}\cap H^2}+|\eta|_{H^2}+|\eta_t|_{L^2}+|\sqrt{\rho}u_t|_{L^2}+|\rho_t|_{L^2}\right)\\
&\quad+\int_0^{T_1}\left(|\sqrt{\rho}u_t|_{L^2}^2+|u_{xt}|_{L^2}^2+|\eta_x|_{L^2}^2+|\eta_t|_{L^2}^2+|\eta_{xt}|_{L^2}^2\right)\mathrm{d}s\le C,
\end{aligned}\tag{2.30}$$

where C is a positive constant, depending only on M_0.

3 Proof of the main theorem

In this section, the existence of strong solutions can be established by a standard argument, we construct the approximate solutions by using the iterative scheme, derive uniform bounds and thus obtain solutions of the original problem by passing to the limit. Our proof will be based on the usual iteration argument and some ideas developed in [25, 26]. Precisely, we first define $u^0=0$ and assuming that u^{k-1} was defined for $k\ge 1$, let ρ^k, u^k, η^k be the unique smooth solution to the following problems:

$$\rho_t^k+\rho_x^k u^{k-1}+\rho^k u_x^{k-1}=0,\tag{3.1}$$

$$\rho^k u_t^k+\rho^k u^{k-1}u_x^k+L_p u^k+P_x^k+\eta_x^k=-\left(\eta^k+\beta\rho^k\right)\Phi_x,\tag{3.2}$$

$$\eta_t^k+\left(\eta^k\left(u^{k-1}-\Phi_x\right)\right)_x=\eta_{xx}^k,\tag{3.3}$$

with the initial and boundary conditions

$$\left(\rho^k,u^k,\eta^k\right)|_{t=0}=(\rho_0,u_0,\eta_0),\tag{3.4}$$

$$u^k|_{\partial\Omega} = \left(\eta_x^k + \eta^k \Phi_x\right)|_{\partial\Omega} = 0, \tag{3.5}$$

where

$$L_p u^k = -\left[\left(\frac{\varepsilon(u_x^k)^2 + 1}{(u_x^k)^2 + \varepsilon}\right)^{\frac{2-p}{2}} u_x^k\right]_x.$$

With the process, the nonlinear coupled system has been reduced to a sequence of decoupled problems and each problem admits a smooth solution. The following estimates hold:

$$\begin{aligned}&\operatorname*{ess\,sup}_{0\le t\le T_1}\left(\left|\rho^k\right|_{H^1} + \left|u^k\right|_{W_0^{1,p}\cap H^2} + \left|\eta^k\right|_{H^2} + \left|\eta_t^k\right|_{L^2} + \left|\sqrt{\rho^k}u_t^k\right|_{L^2} + \left|\rho_t^k\right|_{L^2}\right)\\ &\quad + \int_0^{T_1}\left(\left|\sqrt{\rho^k}u_t^k\right|_{L^2}^2 + \left|u_{xt}^k\right|_{L^2}^2 + \left|\eta_x^k\right|_{L^2}^2 + \left|\eta_t^k\right|_{L^2}^2 + \left|\eta_{xt}^k\right|_{L^2}^2\right)\mathrm{d}s \le C,\end{aligned} \tag{3.6}$$

where C is a generic constant depending only on M_0, but independent of k.

In addition, we first find ρ^k from the initial problem

$$\rho_t^k + u^{k-1}\rho_x^k + u_x^{k-1}\rho^k = 0 \quad \text{and} \quad \rho^k|_{t=0} = \rho_0,$$

with smooth function u^{k-1}, obviously, there is a unique solution ρ^k on the above problem and also by a standard argument, we obtain

$$\rho^k(x,t) \ge \delta \exp\left[-\int_0^{T_1}\left|u_x^{k-1}(\cdot,s)\right|_{L^\infty}\mathrm{d}s\right] > 0, \quad \text{for all } t \in (0,T_1).$$

Next, we will prove that the approximate solution (ρ^k, u^k, η^k) converges to a limit $(\rho^\varepsilon, u^\varepsilon, \eta^\varepsilon)$ in a strong sense. To this end, let us define

$$\bar\rho^{k+1} = \rho^{k+1} - \rho^k, \qquad \bar u^{k+1} = u^{k+1} - u^k, \qquad \bar\eta^{k+1} = \eta^{k+1} - \eta^k,$$

then we easily verify that the functions $\bar\rho^{k+1}$, $\bar u^{k+1}$, $\bar\eta^{k+1}$ satisfy the system of equations

$$\bar\rho_t^{k+1} + \left(\bar\rho^{k+1}u^k\right)_x + \left(\rho^k\bar u^k\right)_x = 0, \tag{3.7}$$

$$\begin{aligned}&\rho^{k+1}\bar u_t^{k+1} + \rho^{k+1}u^k\bar u_x^{k+1} + \left(L_p u^{k+1} - L_p u^k\right)\\ &\quad = -\bar\rho^{k+1}u_t^k - \rho^k\bar u^k u_x^k - \bar\rho^{k+1}u^k u_x^k - \left(P_x^{k+1} - P_x^k\right) - \bar\eta_x^{k+1} - \left(\bar\eta^{k+1} + \beta\bar\eta^{k+1}\right)\Phi_x,\end{aligned} \tag{3.8}$$

$$\bar\eta_t^{k+1} + \left(\eta^k\bar u^k\right)_x + \left(\bar\eta^{k+1}\left(u^k - \Phi_x\right)\right)_x = \bar\eta_{xx}^{k+1}. \tag{3.9}$$

Multiplying (3.7) by $\bar\rho^{k+1}$, integrating over Ω and using Young's inequality, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|\bar\rho^{k+1}\right|_{L^2}^2 &\le C\left|\bar\rho^{k+1}\right|_{L^2}^2\left|u_x^k\right|_{L^\infty} + \left|\rho^k\right|_{H^1}\left|\bar u_x^k\right|_{L^2}\left|\bar\rho^{k+1}\right|_{L^2}\\ &\le C\left|u_{xx}^k\right|_{L^2}\left|\bar\rho^{k+1}\right|_{L^2}^2 + C_\xi\left|\rho^k\right|_{H^1}^2\left|\bar\rho^{k+1}\right|_{L^2}^2 + \xi\left|\bar u_x^k\right|_{L^2}^2\\ &\le C_\xi\left|\bar\rho^{k+1}\right|_{L^2}^2 + \xi\left|\bar u_x^k\right|_{L^2}^2,\end{aligned} \tag{3.10}$$

where C_ζ is a positive constant, depending on M_0 and ζ for all $t < T_1$ and $k \ge 1$.

Multiplying (3.8) by $\bar{u}^{k+1}$, integrating over Ω, and using Young's inequality, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}\left|\bar{u}^{k+1}\right|^2\mathrm{d}x+\int_\Omega\left(L_pu^{k+1}-L_pu^k\right)\bar{u}^{k+1}\,\mathrm{d}x\\
&\quad\le C\int_\Omega\left(\left|\bar{\rho}^{k+1}\right|\left(\left|u_t^k\right|+\left|u^ku_x^k\right|\right)\left|\bar{u}^{k+1}\right|+\rho^k\left|\bar{u}^k\right|\left|u_x^k\right|\left|\bar{u}^{k+1}\right|+\left|P^{k+1}-P^k\right|\left|\bar{u}_x^{k+1}\right|\right.\\
&\qquad\left.+\left|\bar{\eta}^{k+1}\right|\left|\bar{u}_x^{k+1}\right|+\left|\bar{\eta}^{k+1}+\beta\bar{\rho}^{k+1}\right|\left|\Phi_x\bar{u}^{k+1}\right|\right)\mathrm{d}x\\
&\quad\le C\left(\left|\bar{\rho}^{k+1}\right|_{L^2}\left|u_{xt}^k\right|_{L^2}\left|\bar{u}_x^{k+1}\right|_{L^2}+\left|\bar{\rho}^{k+1}\right|_{L^2}\left|u_x^k\right|_{L^p}\left|u_{xx}^k\right|_{L^2}\left|\bar{u}_x^{k+1}\right|_{L^2}\right.\\
&\qquad+\left|\rho^k\right|_{L^2}^{\frac{1}{2}}\left|\sqrt{\rho^k}\bar{u}^k\right|_{L^2}\left|u_{xx}^k\right|_{L^2}\left|\bar{u}_x^{k+1}\right|_{L^2}+\left|P^{k+1}-P^k\right|_{L^2}\left|\bar{u}_x^{k+1}\right|_{L^2}\\
&\qquad\left.+\left|\bar{\eta}^{k+1}\right|_{L^2}\left|\bar{u}_x^{k+1}\right|_{L^2}+\left|\bar{\rho}^{k+1}\right|_{L^2}\left|\bar{u}_x^{k+1}\right|_{L^2}+\left|\bar{\eta}^{k+1}\right|_{L^2}\left|\bar{u}_x^{k+1}\right|_{L^2}\right).
\end{aligned}\tag{3.11}$$

Let

$$\sigma(s)=\left(\frac{\varepsilon s^2+1}{s^2+\varepsilon}\right)^{\frac{2-p}{2}}s,$$

then

$$\sigma'(s)=\left(\frac{\varepsilon s^2+1}{s^2+\varepsilon}\right)^{-\frac{p}{2}}\frac{(\varepsilon s^2+1)(s^2+\varepsilon)-(2-p)(1-\varepsilon^2)s^2}{(s^2+\varepsilon)^2}\ge\frac{p-1}{(s^2+\varepsilon)^{\frac{2-p}{2}}}.$$

We estimate the second term of (3.11) as follows:

$$\begin{aligned}
\int_\Omega\left(L_pu^{k+1}-L_pu^k\right)\bar{u}^{k+1}\,\mathrm{d}x&=\int_\Omega\int_0^1\sigma'\left(\theta u_x^{k+1}+(1-\theta)u_x^k\right)\mathrm{d}\theta\left|\bar{u}_x^{k+1}\right|^2\mathrm{d}x\\
&\ge\int_\Omega\left[\int_0^1\frac{\mathrm{d}\theta}{|\theta u_x^{k+1}+(1-\theta)u_x^k|_{L^\infty}^{2-p}+1}\right]\left(\bar{u}_x^{k+1}\right)^2\\
&\ge C^{-1}\int_\Omega\left|\bar{u}_x^{k+1}\right|^2\mathrm{d}x.
\end{aligned}\tag{3.12}$$

Using (3.6), (3.12), and Young's inequality, (3.11) can be rewritten as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho^{k+1}\left|\bar{u}^{k+1}\right|^2\mathrm{d}x+C^{-1}\int_\Omega\left|\bar{u}_x^{k+1}\right|^2\mathrm{d}x\\
&\quad\le B_\xi(t)\left|\bar{\rho}^{k+1}\right|_{L^2}^2+C\left(\left|\sqrt{\rho^k}\bar{u}^k\right|_{L^2}^2+\left|\bar{\eta}^{k+1}\right|_{L^2}^2\right)+\xi\left|\bar{u}_x^{k+1}\right|_{L^2}^2,
\end{aligned}\tag{3.13}$$

where $B_\xi(t)=C(1+|u_{xt}^k(t)|_{L^2}^2)$, for all $t\le T_1$ and $k\ge1$. Using (3.6) we derive

$$\int_0^t B_\xi(s)\,\mathrm{d}s\le C+Ct.$$

Multiplying (3.9) by $\bar{\eta}^{k+1}$, integrating over Ω, and using (3.6) and Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\bar{\eta}^{k+1}\right|^2\mathrm{d}x+\int_\Omega\left|\bar{\eta}_x^{k+1}\right|^2\mathrm{d}x\\
&\quad\le\int_\Omega\left|\bar{\eta}^{k+1}\right|\left|u^k-\Phi_x\right|\left|\bar{\eta}_x^{k+1}\right|\mathrm{d}x+\int_\Omega\left(\left|\eta^k\right|\left|\bar{u}^k\right|\right)_x\left|\bar{\eta}^{k+1}\right|\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}&\le \left|\bar{\eta}^{k+1}\right|_{L^2}\left|u^k-\Phi_x\right|_{L^\infty}\left|\bar{\eta}_x^{k+1}\right|_{L^2}+\left|\eta_x^k\right|_{L^2}\left|\bar{u}^k\right|_{L^\infty}\left|\bar{\eta}^{k+1}\right|_{L^2}+\left|\eta^k\right|_{L^\infty}\left|\bar{u}_x^k\right|_{L^2}\left|\bar{\eta}^{k+1}\right|_{L^2}\\&\le C_\xi\left|\bar{\eta}^{k+1}\right|_{L^2}^2+\xi\left|\bar{\eta}_x^{k+1}\right|_{L^2}^2+\xi\left|\bar{u}_x^k\right|_{L^2}^2.\end{aligned}\tag{3.14}$$

Collecting (3.10), (3.13), and (3.14), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\bar{\rho}^{k+1}(t)\right|_{L^2}^2+\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)\right|_{L^2}^2+\left|\bar{\eta}^{k+1}(t)\right|_{L^2}^2\right)+\left|\bar{u}_x^{k+1}(t)\right|_{L^2}^2+\left|\bar{\eta}_x^{k+1}\right|_{L^2}^2\\&\quad\le E_\xi(t)\left|\bar{\rho}^{k+1}(t)\right|_{L^2}^2+C\left|\sqrt{\rho^k}\bar{u}^k\right|_{L^2}^2+C_\xi\left|\bar{\eta}^{k+1}\right|_{L^2}^2+\xi\left|\bar{u}_x^k\right|_{L^2}^2,\end{aligned}\tag{3.15}$$

with $E_\zeta(t)$ depending only on $B_\zeta(t)$ and C_ξ, for all $t\le T_1$ and $k\ge 1$. Using (3.6), we have

$$\int_0^t E_\xi(s)\,\mathrm{d}s\le C+C_\xi t.$$

Integrating (3.15) over $(0,t)\subset(0,T_1)$ with respect to t, using Gronwall's inequality, we have

$$\begin{aligned}&\left|\bar{\rho}^{k+1}(t)\right|_{L^2}^2+\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)\right|_{L^2}^2+\left|\bar{\eta}^{k+1}(t)\right|_{L^2}^2+\int_0^t\left|\bar{u}_x^{k+1}(t)\right|_{L^2}^2\,\mathrm{d}s+\int_0^t\left|\bar{\eta}_x^{k+1}\right|_{L^2}^2\,\mathrm{d}s\\&\quad\le C\exp(C_\xi t)\int_0^t\left(\left|\sqrt{\rho^k}\bar{u}^k(s)\right|_{L^2}^2+\left|\bar{u}_x^k(s)\right|_{L^2}^2\right)\mathrm{d}s.\end{aligned}\tag{3.16}$$

From the above recursive relation, choose $\xi>0$ and $0<T_*<T_1$ such that $C\exp(C_\xi T_*)<\frac{1}{2}$, using Gronwall's inequality, we deduce that

$$\begin{aligned}&\sum_{k=1}^K\Bigg[\sup_{0\le t\le T_*}\left(\left|\bar{\rho}^{k+1}(t)\right|_{L^2}^2+\left|\sqrt{\rho^{k+1}}\bar{u}^{k+1}(t)\right|_{L^2}^2+\left|\bar{\eta}^{k+1}(t)\right|_{L^2}^2\right)\\&\quad+\int_0^{T_*}\left|\bar{u}_x^{k+1}(t)\right|_{L^2}^2\,\mathrm{d}t+\int_0^{T_*}\left|\bar{\eta}_x^{k+1}(t)\right|_{L^2}^2\,\mathrm{d}t\Bigg]<C,\end{aligned}\tag{3.17}$$

where C is a positive constant, depending only on M_0.

Therefore, as $k\to+\infty$, the sequence (ρ^k,u^k,η^k) converges to a limit $(\rho^\varepsilon,u^\varepsilon,\eta^\varepsilon)$ in the following strong sense:

$$\rho^k\to\rho^\varepsilon\quad\text{in } L^\infty\left(0,T_*;L^2(\Omega)\right),\tag{3.18}$$

$$u^k\to u^\varepsilon\quad\text{in } L^\infty\left(0,T_*;L^2(\Omega)\right)\cap L^2\left(0,T_*;H_0^1(\Omega)\right),\tag{3.19}$$

$$\eta^k\to\eta^\varepsilon\quad\text{in } L^\infty\left(0,T_*;L^2(\Omega)\right)\cap L^2\left(0,T_*;H^1(\Omega)\right).\tag{3.20}$$

By virtue of the lower semi-continuity of various norms, we deduce from the uniform estimate (3.6) that $(\rho^\varepsilon,u^\varepsilon,\eta^\varepsilon)$ satisfies the following uniform estimate:

$$\begin{aligned}&\operatorname*{ess\,sup}_{0\le t\le T_1}\left(\left|\rho^\varepsilon\right|_{H^1}+\left|u^\varepsilon\right|_{W_0^{1,p}\cap H^2}+\left|\eta^\varepsilon\right|_{H^2}+\left|\eta_t^\varepsilon\right|_{L^2}+\left|\sqrt{\rho^\varepsilon}u_t^\varepsilon\right|_{L^2}+\left|\rho_t^\varepsilon\right|_{L^2}\right)\\&\quad+\int_0^{T_*}\left(\left|\sqrt{\rho}^\varepsilon u_t^\varepsilon\right|_{L^2}^2+\left|u_{xt}^\varepsilon\right|_{L^2}^2+\left|\eta_x^\varepsilon\right|_{L^2}^2+\left|\eta_t^\varepsilon\right|_{L^2}^2+\left|\eta_{xt}^\varepsilon\right|_{L^2}^2\right)\mathrm{d}s\le C.\end{aligned}\tag{3.21}$$

Since all of the constants do not depend on ε, there exists a subsequence $(\rho^{\varepsilon_j}, u^{\varepsilon_j}, \eta^{\varepsilon_j})$ of $(\rho^{\varepsilon}, u^{\varepsilon}, \eta^{\varepsilon})$, that, without loss of generality, we denote $(\rho^{\varepsilon}, u^{\varepsilon}, \eta^{\varepsilon})$. Let $\varepsilon \to 0$, then we obtain the following convergence:

$$\rho^{\varepsilon} \to \rho^{\delta} \quad \text{in } L^{\infty}\big(0, T_*; L^2(\Omega)\big), \tag{3.22}$$

$$u^{\varepsilon} \to u^{\delta} \quad \text{in } L^{\infty}\big(0, T_*; L^2(\Omega)\big) \cap L^2\big(0, T_*; H_0^1(\Omega)\big), \tag{3.23}$$

$$\eta^{\varepsilon} \to \eta^{\delta} \quad \text{in } L^{\infty}\big(0, T_*; L^2(\Omega)\big) \cap L^2\big(0, T_*; H^1(\Omega)\big), \tag{3.24}$$

and also

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\le t\le T_1} \big(\big|\rho^{\delta}\big|_{H^1} + \big|u^{\delta}\big|_{W_0^{1,p}\cap H^2} + \big|\eta^{\delta}\big|_{H^2} + \big|\eta_t^{\delta}\big|_{L^2} + \big|\sqrt{\rho^{\delta}}u_t^{\delta}\big|_{L^2} + \big|\rho_t^{\delta}\big|_{L^2}\big) \\ &\quad + \int_0^{T_*} \big(\big|\sqrt{\rho}^{\delta}u_t^{\delta}\big|_{L^2}^2 + \big|u_{xt}^{\delta}\big|_{L^2}^2 + \big|\eta_x^{\delta}\big|_{L^2}^2 + \big|\eta_t^{\delta}\big|_{L^2}^2 + \big|\eta_{xt}^{\delta}\big|_{L^2}^2\big)\,\mathrm{d}s \le C. \end{aligned} \tag{3.25}$$

For each small $\delta > 0$, let $\rho_0^{\delta} = J_{\delta} * \rho_0 + \delta$, J_{δ} is a mollifier on Ω, and $u_0^{\delta} \in H_0^1(\Omega) \cap H^2(\Omega)$ is a smooth solution of the boundary value problem

$$\begin{cases} L_p u_0^{\delta} = (P(\rho_0^{\delta}) + \eta_0^{\delta})_x + \eta_0^{\delta}\Phi_x + \rho_0^{\delta}(g^{\delta} + \beta\Phi_x), \\ u_0^{\delta}(0) = u_0^{\delta}(1) = 0, \end{cases} \tag{3.26}$$

where $g^{\delta} \in C_0^{\infty}$ and satisfies $|g^{\delta}|_{L^2} \le |g|_{L^2}$, $\lim_{\delta\to 0^+} |g^{\delta} - g|_{L^2} = 0$.

We deduce that $(\rho^{\delta}, u^{\delta}, \eta^{\delta})$ is a solution of the following initial boundary value problem:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x - \lambda(|u_x|^{p-2}u_x)_x + (P + \eta)_x = -(\eta + \beta\rho)\Phi_x, \\ \eta_t + (\eta(u - \Phi_x))_x = \eta_{xx}, \\ (\rho, u, \eta)|_{t=0} = (\rho_0^{\delta}, u_0^{\delta}, \eta_0^{\delta}), \\ u|_{\partial\Omega} = (\eta_x + \eta\Phi_x)|_{\partial\Omega} = 0, \end{cases}$$

where $\rho_0^{\delta} \ge \delta$, $\frac{4}{3} < p < 2$.

By the proof of Lemma 2.3 in [16], there exists a subsequence $\{u_0^{\delta_j}\}$ of $\{u_0^{\delta}\}$, as $\delta_j \to 0^+$, $u_0^{\delta} \to u_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$, $-(|u_{0x}^{\delta_j}|^{p-2}u_{0x}^{\delta_j})_x \to -(|u_{0x}|^{p-2}u_{0x})_x$ in $L^2(\Omega)$, Hence, u_0 satisfies the compatibility condition (1.5) of Theorem 1.1. By virtue of the lower semi-continuity of various norms, we deduce that (ρ, u, η) satisfies the following uniform estimate:

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\le t\le T_1} \big(|\rho|_{H^1} + |u|_{W_0^{1,p}\cap H^2} + |\eta|_{H^2} + |\eta_t|_{L^2} + |\sqrt{\rho}u_t|_{L^2} + |\rho_t|_{L^2}\big) \\ &\quad + \int_0^{T_*} \big(|\sqrt{\rho}u_t|_{L^2}^2 + |u_{xt}|_{L^2}^2 + |\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2\big)\,\mathrm{d}s \le C, \end{aligned} \tag{3.27}$$

where C is a positive constant, depending only on M_0. The uniqueness of the solution can also be obtained by the same method as the above proof of convergence, we omit the details here. This completes the proof.

Appendix

In this section, for the sake of completeness, we give a slightly more particular result for the previous case.

Lemma A.1 *Let* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\rho_0 \in H^1(\Omega)$, $\eta_0 \in H^2(\Omega)$, $\Phi \in C^2(\Omega)$, $g \in L^2(\Omega)$, u_0^ε *is a solution of the boundary value problem*

$$\begin{cases} [(\frac{\varepsilon(u_{0x}^\varepsilon)^2+1}{(u_{0x}^\varepsilon)^2+\varepsilon})^{\frac{2-p}{2}} u_{0x}^\varepsilon]_x - (P(\rho_0)+\eta_0)_x - \eta_0\Phi_x = \rho_0(g+\beta\Phi_x), \\ u_0^\varepsilon(0) = u_0^\varepsilon(1) = 0. \end{cases} \tag{A.1}$$

Then there are a subsequence $\{u_0^{\varepsilon_j}\}$, $j = 1,2,3,\dots$, *of* $\{u_0^\varepsilon\}$ *and* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *such that, as* $\varepsilon_j \to 0$,

$$u_0^{\varepsilon_j} \to u_0 \quad \text{in } H_0^1(\Omega) \cap H^2(\Omega),$$

$$\left[\left(\frac{\varepsilon_j(u_{0x}^{\varepsilon_j})^2+1}{(u_{0x}^{\varepsilon_j})^2+\varepsilon_j}\right)^{\frac{2-p}{2}} u_{0x}^{\varepsilon_j}\right]_x \to \left(|u_{0x}|^{p-2}u_{0x}\right)_x \quad \text{in } L^2(\Omega).$$

Proof According to (A.1), we have

$$u_{0xx}^\varepsilon = \left(\frac{\varepsilon(u_{0x}^\varepsilon)^2+1}{(u_{0x}^\varepsilon)^2+\varepsilon}\right)^{\frac{p}{2}} \frac{((u_{0x}^\varepsilon)^2+\varepsilon)^2((P(\rho_0)+\eta_0)_x + \eta_0\Phi_x + \rho_0(g+\beta\Phi_x))}{(\varepsilon(u_{0x}^\varepsilon)^2+1)((u_{0x}^\varepsilon)^2+\varepsilon) - (2-p)(1-\varepsilon^2)(u_{0x}^\varepsilon)^2}.$$

Then

$$\begin{aligned} |u_{0xx}^\varepsilon|_{L^2} &\le \left|\left(\frac{\varepsilon(u_{0x}^\varepsilon)^2+1}{(u_{0x}^\varepsilon)^2+\varepsilon}\right)^{1-\frac{p}{2}}\right|_{L^\infty} |(P(\rho_0)+\eta_0)_x + \eta_0\Phi_x + \rho_0(g+\beta\Phi_x)|_{L^2} \\ &\le \left(|u_{0x}^\varepsilon|_{L^\infty}^2 + 1\right)^{1-\frac{p}{2}} |(P(\rho_0)+\eta_0)_x + \eta_0\Phi_x + \rho_0(g+\beta\Phi_x)|_{L^2}, \end{aligned}$$

then we get

$$|u_{0xx}^\varepsilon|_{L^2} \le C\left(1 + |(P(\rho_0)+\eta_0)_x + \eta_0\Phi_x + \rho_0(g+\beta\Phi_x)|_{L^2}\right)^{\frac{1}{p-1}} \le C.$$

Therefore, by the above inequality, as $\varepsilon_j \to 0$,

$$u_0^{\varepsilon_j} \to u_0 \quad \text{in } C^{\frac{3}{2}}(\Omega),$$

$$u_{0xx}^{\varepsilon_j} \to u_{0xx} \quad \text{in } L^2(\Omega) \text{ weakly.}$$

Thus, we see that $\{u_{0x}^{\varepsilon_j}\}$ is a Cauchy subsequence of $C^{\frac{3}{2}}(\Omega)$, for all $\alpha_1 > 0$, we find N, as $i,j > N$, we have

$$|u_{0x}^{\varepsilon_i} - u_{0x}^{\varepsilon_j}|_{L^\infty(\Omega)} < \alpha_1.$$

Now, we prove that $\{u_{0xx}^\varepsilon\}$ has a Cauchy sequence in L_2 norm.

Let

$$\phi_i = \phi\left(\left(u_{0x}^{\varepsilon_i}\right)^2\right) = \left(\frac{\varepsilon_i(u_{0x}^{\varepsilon})^2+1}{(u_{0x}^{\varepsilon_i})^2+\varepsilon_i}\right)^{\frac{p}{2}} \frac{((u_{0x}^{\varepsilon_i})^2+\varepsilon_i)^2}{(\varepsilon_i(u_{0x}^{\varepsilon_i})^2+1)((u_{0x}^{\varepsilon_i})^2+\varepsilon_i)-(2-p)(1-\varepsilon_i{}^2)(u_{0x}^{\varepsilon_i})^2}.$$

For all $\alpha > 0$, there exists N, as $i, j > N$, we can deduce that

$$\left|u_{0xx}^{\varepsilon_i} - u_{0xx}^{\varepsilon_j}\right|_{L^2(\Omega)} \le |\phi_i - \phi_j|_{L^\infty(\Omega)} \left|\left(P(\rho_0)+\eta_0\right)_x + \eta_0\Phi_x + \rho_0(g+\beta\Phi_x)\right|_{L^2(\Omega)}.$$

With the assumption, we obtain

$$\left|\left(P(\rho_0)+\eta_0\right)_x + \eta_0\Phi_x + \rho_0(g+\beta\Phi_x)\right|_{L^2(\Omega)} \le C,$$

where C is a positive constant, depending only on $|\rho_0|_{H^1(\Omega)}$, $|g|_{L^2(\Omega)}$, and $|\eta_0|_{H^2(\Omega)}$. Use the following inequality:

$$|\phi_i - \phi_j|_{L^\infty(\Omega)} \le \left|\int_0^1 \phi'\left(\theta\left(u_{0x}^{\varepsilon_i}\right)^2 + (1-\theta)\left(u_{0x}^{\varepsilon_j}\right)^2\right)d\theta\left(\left(u_{0x}^{\varepsilon_i}\right)^2 - \left(u_{0x}^{\varepsilon_j}\right)^2\right)\right|_{L^\infty(\Omega)}, \tag{A.2}$$

where $0 < \theta < 1$.

By a simple calculation, we can get $\phi'(s) \le \frac{2}{p-1}(1+s^{-\frac{p}{2}})$, where C depends only on p, then

$$\begin{aligned}
|\phi_i - \phi_j|_{L^\infty(\Omega)} &\le \frac{2}{p-1}\left|\left(1+\int_0^1 \left(\theta\left(u_{0x}^{\varepsilon_i}\right)^2 + (1-\theta)\left(u_{0x}^{\varepsilon_j}\right)^2\right)d\theta\right)\left(\left(u_{0x}^{\varepsilon_i}\right)^2 - \left(u_{0x}^{\varepsilon_j}\right)^2\right)\right|_{L^\infty(\Omega)} \\
&\le \frac{2}{p-1}\left|u_{0x}^{\varepsilon_i} - u_{0x}^{\varepsilon_j}\right|_{L^\infty(\Omega)}\left|u_{0x}^{\varepsilon_i} + u_{0x}^{\varepsilon_j}\right|_{L^\infty(\Omega)} \\
&\quad + \frac{4}{(2-p)(p-1)}\left|u_{0x}^{\varepsilon_i} - u_{0x}^{\varepsilon_j}\right|_{L^\infty(\Omega)}^{\frac{2-p}{2}}\left|u_{0x}^{\varepsilon_i} + u_{0x}^{\varepsilon_j}\right|_{L^\infty(\Omega)}^{\frac{2-p}{2}} \le \alpha.
\end{aligned}$$

Substituting this into (A.1), we have

$$\left|u_{0xx}^{\varepsilon_i} - u_{0xx}^{\varepsilon_j}\right|_{L^\infty(\Omega)} < \alpha,$$

then there is a subsequence $\{u_{0xx}^{\varepsilon_j}\}$ and $\{u_{0xx}^{\varepsilon}\}$, such that

$$\left\{u_{0xx}^{\varepsilon_j}\right\} \to \chi \quad \text{in } L^2(\Omega).$$

By the uniqueness of the weak convergence, we have

$$\chi = \left\{u_{0xx}^{\varepsilon}\right\}.$$

Since $(P(\rho_0)+\eta)_x + \eta_0\Phi_x + \rho_0(g+\beta\Phi_x)$ are independent of ε, in the same way we obtain, as $\varepsilon_j \to 0$,

$$\left[\left(\frac{\varepsilon_j(u_{0x}^{\varepsilon_j})^2+1}{(u_{0x}^{\varepsilon_j})^2+\varepsilon_j}\right)^{\frac{2-p}{2}} u_{0x}^{\varepsilon_j}\right]_x \to \left(|u_{0x}|^{p-2}u_{0x}\right)_x \quad \text{in } L^2(\Omega).$$

This completes the proof of Lemma A.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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