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On superlinear fractional advection dispersion equation in $\mathbb{R}^N$

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Abstract

We consider the fractional advection dispersion equation in $\mathbb{R}^N$. The nonlinearity is superlinear but does not satisfy the Ambrosetti-Rabinowitz type condition. We obtain the existence of nontrivial solutions of the equations, improving a recent result of Zhang-Sun-Li (Appl. Math. Model. 38:4062-4075, 2014).

MSC: 45B05; 26A33

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1 Introduction and main results

In this paper we are concerned with the following fractional advection dispersion equation:

$$-\int_{|\theta|=1} D_\theta D_\theta^\beta u M(d\theta) + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{P}$$

where $N > 1$, $\beta \in (0,1)$, $\alpha = \frac{\beta+1}{2}$, $M(d\theta)$ is a Borel probability measure on the unit sphere in $\mathbb{R}^N$, D_θ^β denotes directional fractional derivative of order β in the direction of the unit vector θ. We recall that the theory of fractional operators has gained a lot of attention in a large scientific community including pure mathematicians, applied scientists, and numerical analysts: see [1–10] and the references therein. Now, we make the following assumptions on the functions M, V, and f.

(H_M) There is a constant $c_M > 0$ such that $\int_{|\theta|=1} |\xi \cdot \theta|^{2\alpha} M(d\theta) \geq c_M$ for $\xi \in \mathbb{S}^{N-1}$, where $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$.

(H_V) $V \in C(\mathbb{R}^N)$, V is 1-periodic in $x_1, x_2, \ldots, x_N$ and $0 < V_0 \leq V(x) \leq V_1 < +\infty$.

(H_f) (i) $f \in C(\mathbb{R}^N \times \mathbb{R})$ is 1-periodic in $x_1, x_2, \ldots, x_N$, and

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{2_\alpha^*-1}} = 0, \qquad \lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^2} = +\infty \tag{1}$$

uniformly in $x \in \mathbb{R}^N$, where $F(x,t) = \int_0^t f(x,s)\,ds$ and $2_\alpha^* = \frac{2N}{N-2\alpha}$.

(ii) $\lim_{t\to 0} \frac{f(x,t)}{t} = 0$ uniformly with respect to $x \in \mathbb{R}^N$.

(iii) There exists $\vartheta \geq 1$ such that $\vartheta G(x,t) \geq G(x,st)$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $G(x,t) = f(x,t)t - 2F(x,t)$.

Note that if the nonlinearity is subcritical and superlinear, that is, for some positive constants $c_1 > 0$ and $p \in [2, 2_\alpha^* - 1)$,

$$\left|f(x,t)\right| \le c_1\left(1 + |t|^p\right), \qquad \lim_{|t|\to+\infty} \frac{f(x,t)t}{|t|^2} = +\infty, \tag{2}$$

then (1) is satisfied. Our condition (1) is slightly weaker. The condition (1) was first introduced by Liu and Wang [11] and then was used in [12]. The condition (H_f)(iii) is originally due to Jeanjean [13] for semilinear problem in $\mathbb{R}^N$. It is known that (H_f)(iii) is weaker than the condition that

(h_1) for each $x \in \mathbb{R}^N$, $\frac{f(x,t)}{|t|}$ is an increasing function of t in $\mathbb{R}\backslash\{0\}$,

see Proposition 2.3 in [14] for a proof; see also Lemma 2.1 in [15]. Moreover, when dealing with superlinear problem, one usually needs a growth condition together with the following classical condition which was introduced by Ambrosetti and Rabinowitz in [16].

(AR) There exist $\mu > 2$ such that

$$0 < \mu F(x,t) \le tf(x,t), \quad \forall x \in \mathbb{R}^N, t \neq 0.$$

It is well known that the condition (AR) is crucial in verifying the boundedness of the family of the corresponding functionals $(PS)_c$, $c \in \mathbb{R}$. This is very crucial in applying the critical point theory. However, there are many functions which are superlinear at infinity, but do not satisfy the condition (AR) for any $\mu > 2$. In fact, the condition (AR) implies that $F(x,t) \ge C|t|^\mu$ for some $C > 0$. Thus, for example the superlinear function $f(x,t) = t\log(1 + |t|)$ does not satisfy the condition (AR). However, it satisfies our condition (H_f). From the above we can see the (AR) condition is stronger than (H_f).

We look for solution of (P) in the space

$$E = \left\{ u \in J_M^\alpha\left(\mathbb{R}^N\right) : \int_{|\theta|=1} \int_{\mathbb{R}^N} \left|D_\theta^\alpha u\right|^2 dx M(d\theta) + \int_{\mathbb{R}^N} V(x)|u|^2 dx < +\infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_{|\theta|=1} \int_{\mathbb{R}^N} \left|D_\theta^\alpha u\right|^2 dx M(d\theta) + \int_{\mathbb{R}^N} V(x)|u|^2 dx \right)^{\frac{1}{2}}.$$

The fractional Sobolev space $J_M^\alpha(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_M = \left(\int_{|\theta|=1} \int_{\mathbb{R}^N} \left|D_\theta^\alpha u\right|^2 dx M(d\theta) + \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}}.$$

From Lemma 2.6 in [1], we have $E \subset J_M^\alpha(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $2 \le p \le 2_\alpha^*$. By conditions (H_f)(i) and (H_f)(ii), there exists a constant $C > 0$ such that

$$F(x,t) \le C\left(|t|^2 + |t|^{2_\alpha^*}\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence the energy functional $\varphi : E \to \mathbb{R}$ given by

$$\varphi(u) = -\frac{1}{2}\int_{|\theta|=1}\bigl(D_\theta^\alpha u, D_{-\theta}^\alpha u\bigr)M(d\theta) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx \tag{3}$$

is well defined and of class C^1. The derivative of φ is given by

$$\varphi'(u)v = -\int_{|\theta|=1}\bigl(D_\theta^\alpha u, D_{-\theta}^\alpha v\bigr)M(d\theta) + \int_{\mathbb{R}^N} V(x)uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx \tag{4}$$

for $v \in C_0^\infty(\mathbb{R}^N)$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^N)$. Therefore, the critical points of φ are weak solutions of (P). We are now in the position to state our main results.

Theorem 1 *Suppose the measure M is symmetric and satisfies* (H_M). *If* (H_V) *and* (H_f) *hold, then problem* (P) *has at least one nontrivial solution.*

Corollary 1 *Suppose the measure M is symmetric and satisfies* (H_M). *If $V(x) = V_0, f(x,u) = f(u)$ and* (H_f) *are satisfied, then problem* (P) *has at least one nontrivial solution.*

This corollary improves a recent result of Zhang-Sun-Li [1], Theorem 3.1. In the some paper [1], in order to get the existence of a nontrivial solution, by substituting the condition (H_M), Zhang-Sun-Li assumed in addition to (H_f)(ii), (AR), and the well-known subcritical growth condition, that is:

(h_2) there are positive constants c and $p \in [2, \frac{N+2\alpha}{N-2\alpha})$ such that

$$\bigl|f(x,u)\bigr| \le c\bigl(1+|u|^p\bigr), \quad x \in \mathbb{R}^N, u \in \mathbb{R}.$$

In Corollary 1, the condition (AR) is completely removed, and our assumption (H_f)(ii)-(iii) is weaker than (AR) condition.

In this paper, we consider problem (P) in the case when the nonlinear term $f(x,t)$ is superlinear at infinity but does not satisfy the (AR) type condition. Thus, the variational functional φ may possess unbounded (PS) sequences. To overcome this difficulty, the Cerami sequences are employed (see Lemma 5).

The rest of this paper is organized as follows. In Section 2, we state some preliminary results that will be used later. We will finish the proof of our main result (Theorem 1) in Section 3.

2 Preliminaries

In this section, we introduce some basic definitions and properties of the fractional calculus which are used further in this paper. For the proofs, which are omitted, we refer the reader to [1] or other texts on basic fractional calculus.

Definition 1 Let $u : \mathbb{R}^N \to \mathbb{R}$, $\alpha > 0$, θ be a unit vector in $\mathbb{R}^N$. The αth order fractional integral in the direction of θ of u is given by

$$D_\theta^{-\alpha} u(x) = \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} \xi^{\alpha-1} u(x - \xi\theta)\,d\xi$$

and the αth order directional derivative in the direction of θ is defined by

$$D_\theta^\alpha u(x) = (\nabla \cdot \theta)^n D_\theta^{\alpha-n} u(x),$$

where n denotes the smallest integer greater than or equal to α.

Definition 2 Let $u : \mathbb{R}^N \to \mathbb{R}$, $\alpha > 0$ (or $\alpha < 0$) be given. Then the αth order fractional integral (derivative) with respect to the measure M is defined as

$$D_M^\alpha u(x) = \int_{|\theta|=1} D_\theta^\alpha u(x) M(d\theta),$$

where $M(d\theta)$ is a Borel probability measure on the unit sphere in $\mathbb{R}^N$.

Definition 3 Let $\alpha = \frac{1+\beta}{2}$, a function $u \in J_M^\alpha(\mathbb{R}^N)$ is called of (P) if

$$-\int_{|\theta|=1} \left(D_\theta^\alpha u, D_{-\theta}^\alpha v\right) M(d\theta) + \int_{\mathbb{R}^N} V(x) uv \, dx = \int_{\mathbb{R}^N} f(x,u) v \, dx$$

for all $v \in C_0^\infty(\mathbb{R}^N)$.

Lemma 1 *If $\alpha \in (0,1)$, then for $u \in C_0^\infty(\mathbb{R}^N)$, we have the following Fourier transform property*:

$$\mathcal{F}\left(D_\theta^{-\alpha} u\right)(\xi) = (i\xi \cdot \theta)^{-\alpha} \mathcal{F}(u)(\xi) \quad \textit{and} \quad \mathcal{F}\left(D_\theta^\alpha u\right)(\xi) = (i\xi \cdot \theta)^\alpha \mathcal{F}(u)(\xi),$$

where $\mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} u(x) \, dx$.

Lemma 2 *Assume that the measure M is symmetric. Let $\alpha > 0$, $u, v \in J_M^\alpha(\mathbb{R}^N)$. Then, for M-a.e. $\theta \in S^{N-1}$, we have*

$$\left(D_\theta^\alpha u, D_{-\theta}^\alpha v\right) + \left(D_\theta^\alpha v, D_{-\theta}^\alpha u\right) = 2\cos(\pi\alpha)\left(D_\theta^\alpha u, D_\theta^\alpha v\right),$$

especially

$$\left(D_\theta^\alpha u, D_{-\theta}^\alpha u\right) = \cos(\pi\alpha)\left(D_\theta^\alpha u, D_\theta^\alpha u\right).$$

Throughout this paper, we will use the norm $\|\cdot\|$ in E. As usual, for $1 \le \nu < \infty$, we let

$$|u|_\nu = \left(\int_{\mathbb{R}^N} \left|u(x)\right|^\nu dx\right)^{\frac{1}{\nu}}, \quad u \in L^\mu\left(\mathbb{R}^N\right).$$

For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces.

Lemma 3 ([1], Lemma 2.6) *E continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [2, 2_\alpha^*]$, and compactly embedded into $L_{\mathrm{loc}}^p(\mathbb{R}^N)$ for $p \in [2, 2_\alpha^*)$.*

In order to prove Theorem 1, we need the following lemma whose proof is analogous to that of Lemma 1.21 in [17] (see also [18]).

Lemma 4 *Let $r > 0$ and $2 \le p < 2_\alpha^*$. If $\{u_n\}$ is bounded in $J_M^\alpha(\mathbb{R}^N)$ and if*

$$\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^p \, dx \to 0, \quad \textit{as } n \to +\infty,$$

where $B_r(y) = \{x \in \mathbb{R}^N : |x-y| < r\}$, then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in (2, 2_\alpha^*)$.

In order to ensure the existence of solutions for the problem (P), our main tool will be the mountain pass theorem [19], p.140, Theorem 6, which will be used in our proof.

Lemma 5 *Let X be a Banach space and $I : X \to \mathbb{R}$ a continuous, Gâteaux-differentiable function, such that $I' : X \to X$ is continuous from the norm topology of X to the weak* topology of X^*. Take two points (z_0, z_1) in X and consider the set Γ of all continuous paths from z_0 to z_l:*

$$\Gamma = \bigl\{\gamma \in C\bigl([0,1], X\bigr) : \gamma(0) = z_0, \gamma(1) = z_1\bigr\}.$$

Define a number c by $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$. Assume there is a closed subset Θ of X such that $\Theta \cap I^c$ separates z_0 and z_1 with $I^c = \{x \in X : I(x) \ge c\}$. Then there is a sequence $\{x_n\}$ in X such that

$$\delta(x_n, \Theta) \to 0, \qquad I(x_n) \to c \quad \textit{and} \quad \bigl(1 + \|x_n\|_X\bigr)\bigl\|I'(x_n)\bigr\|_{X^*} \to 0.$$

3 Proof of Theorem 1

In this section, for the notation in Lemma 5, the space $X = E$, and related functional on E is $I = \varphi$. Recall that a sequence $\{u_n\} \subset E$ is called a Palais-Smale sequence of φ the level c, a $(PS)_c$ sequence for short, if $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$. A sequence $\{u_n\} \subset E$ is called a Cerami sequence ϕ at the level c, a $(C)_c$ sequence for short, if

$$\varphi(u_n) \to c \quad \text{and} \quad \bigl(1 + \|u_n\|\bigr)\varphi'(u_n) \to 0.$$

We deduce from (H_f)(iii) that

$$f(x,t)t - 2F(x,t) \ge 0, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Let $t > 0$. For $x \in \mathbb{R}^N$, by direct computations we have

$$\frac{\partial}{\partial t}\left(\frac{F(x,t)}{t^2}\right) = \frac{t^2 f(x,t) - 2tF(x,t)}{t^4} \ge 0. \tag{5}$$

Taking into account hypothesis (H_f)(ii) we deduce

$$\lim_{t\to 0^+} \frac{F(x,t)}{t^2} = 0. \tag{6}$$

So it follows from (5) and (6) that $F(x,t) \geq 0$ for all $x \in \mathbb{R}^N$ and $t \geq 0$. Arguing similarly for the case $t \leq 0$, eventually we obtain

$$F(x,t) \geq 0, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{7}$$

Lemma 6 *There exists $r > 0$ and $\eta_0 \in E$ such that $\|\eta_0\| > r$ and*

$$\inf_{\|u\|=r} \varphi(u) > \varphi(0) = 0 \geq \varphi(\eta_0).$$

Proof First of all, from (H_f)(i) and (H_f)(ii) it follows that, for all given $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon |t|^2 + c_\varepsilon |t|^{2_\alpha^*}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Consequently, using Lemma 2, we have

$$\begin{aligned}
\varphi(u) &= -\frac{1}{2} \int_{|\theta|=1} \bigl(D_\theta^\alpha u, D_{-\theta}^\alpha u\bigr) M(d\theta) + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 \,dx - \int_{\mathbb{R}^N} F(x,u)\,dx \\
&= -\frac{\cos(\pi\alpha)}{2} \int_{|\theta|=1} \bigl|D_\theta^\alpha u\bigr|_2^2 M(d\theta) + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 \,dx - \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\geq -\frac{\cos(\pi\alpha)}{2} \biggl[\int_{|\theta|=1} \bigl|D_\theta^\alpha u\bigr|_2^2 M(d\theta) + \int_{\mathbb{R}^N} V(x) u^2 \,dx \biggr] - \varepsilon \int_{\mathbb{R}^N} |u|^2 \,dx \\
&\quad - c_\varepsilon \int_{\mathbb{R}^N} |u|^{2_\alpha^*} \,dx.
\end{aligned}$$

Since $E \hookrightarrow L^p(\mathbb{R}^N)$, $2 \leq p \leq 2_\alpha^*$, we know there exists a constant $C > 0$ such that $|u|_p \leq C\|u\|$. So

$$\varphi(u) \geq -\frac{\cos(\pi\alpha)}{2} \|u\|^2 - \varepsilon C^2 \|u\|^2 - c_\varepsilon C^{2_\alpha^*} \|u\|^{2_\alpha^*}.$$

By choosing $\varepsilon > 0$ such that $-\frac{\cos(\pi\alpha)}{2} - \varepsilon C^2 \geq -\frac{\cos(\pi\alpha)}{4}$, we obtain

$$\varphi(u) \geq -\frac{\cos(\pi\alpha)}{4} \|u\|^2 - c_\varepsilon C^{2_\alpha^*} \|u\|^{2_\alpha^*}, \quad \forall u \in E.$$

That is, there exist $r > 0$ and $\rho > 0$, such that

$$\varphi(u) \geq \rho > 0, \qquad \|u\| = r.$$

Using (1), it is easy to see that for any $u \neq 0$, we have $\varphi(tu) \to -\infty$ as $t \to +\infty$. Hence, there is a point $\eta_0 \in E \backslash \overline{B}_r$ such that $\varphi(\eta_0) \leq 0$. $\square$

By Lemma 6 we see that φ has a mountain pass geometry: that is, setting

$$\Gamma = \bigl\{ \gamma \in C\bigl([0,1], E\bigr) : \gamma(0) = 0, \varphi\bigl(\gamma(1)\bigr) < 0 \bigr\},$$

we have $\Gamma \neq \emptyset$. Moreover, it is easy to see that

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi\bigl(\gamma(t)\bigr) > 0. \tag{8}$$

Take $F = E$ in Lemma 5. Equation (8) implies that φ^c separates $\gamma(0) = 0$ and $\gamma(1) = \eta_0$, and there exists a $(C)_c$ sequence $\{u_n\}$ for φ.

Claim *The sequence $\{u_n\}$ is bounded.*

If $\{u_n\}$ is unbounded, up to a subsequence we may assume that

$$\varphi(u_n) \to c, \qquad \|u_n\| \to +\infty, \qquad \bigl\|\varphi'(u_n)\bigr\| \|u_n\| \to 0.$$

In particular,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx = \lim_{n\to\infty} \left(\varphi(u_n) - \frac{1}{2}\varphi'(u_n)u_n\right) = c. \tag{9}$$

Let $w_n = \frac{u_n}{\|u_n\|}$, then $\{w_n\}$ is bounded in E. We claim that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |w_n|^2\, dx = 0. \tag{10}$$

Otherwise, for some $\delta > 0$, up to a subsequence we have

$$\sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |w_n|^2\, dx \geq \delta > 0.$$

So we can choose $\{y_n\} \subset \mathbb{R}^N$ such that

$$\sup_{y\in\mathbb{R}^N} \int_{B_2(y_n)} |w_n|^2\, dx \geq \frac{\delta}{2}.$$

It is easy to see that the number of points in $Z^N \cap B_2(y_n)$ is less than 4^N; there exists $z_n \in B_2(y_n)$ such that

$$\int_{B_2(z_n)} |w_n|^2\, dx \geq \frac{\delta}{2^{2N+1}}. \tag{11}$$

Let $\overline{w}_n(x) = w_n(x + z_n)$. Then $\{\overline{w}_n\}$ is also bounded in E. Passing to a subsequence we have

$$\overline{w}_n \to \overline{w} \quad \text{in } L^2_{\mathrm{loc}}\bigl(\mathbb{R}^N\bigr) \quad \text{and} \quad \overline{w}_n(x) \to \overline{w}(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

Remark that

$$\int_{B_2(0)} |\overline{w}_n|^2\, dx = \int_{B_2(z_n)} |w_n|^2\, dx \tag{12}$$

and so $\overline{w} \neq 0$. Let $\overline{u}_n = \|u_n\|\overline{w}_n$. It is easy to see that $\overline{u}_n(x) \to +\infty$ as $n \to +\infty$. Using (1) we have

$$\lim_{n\to+\infty} \frac{F(x,\overline{u}_n(x))}{|\overline{u}_n(x)|^2} \left|\overline{w}_n(x)\right|^2 = +\infty. \tag{13}$$

Recall that $f(x,u)$ is 1-periodic with respect to x. Using this fact, we obtain

$$\int_{\mathbb{R}^N} F(x,u_n)\,dx = \int_{\mathbb{R}^N} F(x,\overline{u}_n)\,dx.$$

Since the set $\Omega_0 = \{x \in \mathbb{R}^N : \overline{w}(x) \neq 0\}$ has positive Lebesgue measure, using (13) we have

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{2\|u_n\|^2} \int_{\mathbb{R}^N} \int_{|\theta|=1} \int_{\mathbb{R}^N} \left|D_\theta^\alpha u_n\right|^2 dx M(d\theta) + \int_{\mathbb{R}^N} V(x)|u_n|^2\,dx \\
&= \frac{1}{\|u_n\|^2} \left(\varphi(u_n) + \int_{\mathbb{R}^N} F(x,u_n)\,dx\right) \\
&\geq \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} F(x,u_n)\,dx - 1 \\
&= \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} F(x,\overline{u}_n)\,dx - 1 \\
&\geq \int_{\overline{w}\neq 0} \frac{F(x,\overline{u}_n(x))}{|\overline{u}_n(x)|^2} \left|\overline{w}_n(x)\right|^2 dx - 1 \to +\infty.
\end{aligned}$$

This is impossible. Therefore we have proved (10). Hence, using Lemma 4 and (10), we obtain

$$w_n \to 0 \quad \text{in } L^s\big(\mathbb{R}^N\big), s \in \big(2, 2_\alpha^*\big). \tag{14}$$

Next, we shall derive a contradiction as follows. Given a real number $R > 0$, by (H_f)(i) and (H_f)(ii), for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$F(x,Rt) \leq \varepsilon\big(|t|^2 + |t|^{2_\alpha^*}\big) + c_\varepsilon |t|^s. \tag{15}$$

Note that $\|w_m\| = 1$. So from (14), (15), and the continuity of the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ (since $p \in [2, 2_\alpha^*]$), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(x,Rw_n)\,dx &\leq \limsup_{n\to\infty} \big[\varepsilon C^2\|w_n\|^2 + \varepsilon C^{2_\alpha^*}\|w_n\|^{2_\alpha^*} + c_\varepsilon |w_n|_s^s\big] \\
&\leq \varepsilon\big(C^2 + C^{2_\alpha^*}\big).
\end{aligned}$$

Now let $\varepsilon \to 0$, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x,Rw_n)\,dx = 0. \tag{16}$$

Let $t_n \in [0,1]$ such that $\varphi(t_n u_n) = \max_{t\in[0,1]} \varphi(tu_n)$. Given $m > 0$. Since for n large enough we have $(-\frac{4m}{\cos(\pi\alpha)})^{\frac{1}{2}} \|u_n\|^{-1} \in (0,1)$, using (16) with $R = (-\frac{4m}{\cos(\pi\alpha)})^{\frac{1}{2}}$, we have

$$\begin{aligned}
\varphi(t_n u_n) &\geq \varphi\left(\left(-\frac{4m}{\cos(\pi\alpha)}\right)^{\frac{1}{2}} \|u_n\|^{-1} u_n\right) = \varphi\left(\left(-\frac{4m}{\cos(\pi\alpha)}\right)^{\frac{1}{2}} w_n\right) \\
&= 2m \int_{|\theta|=1} \left|D_\theta^\alpha w_n\right|_2^2 M(d\theta) - \frac{2m}{\cos(\pi\alpha)} \int_{\mathbb{R}^N} V(x) w_n^2 \, dx \\
&\quad - \int_{\mathbb{R}^N} F\left(x, \left(-\frac{4m}{\cos(\pi\alpha)}\right)^{\frac{1}{2}} w_n\right) dx \\
&\geq 2m \int_{|\theta|=1} \left|D_\theta^\alpha w_n\right|_2^2 M(d\theta) + 2m \int_{\mathbb{R}^N} V(x) w_n^2 \, dx \\
&\quad - \int_{\mathbb{R}^N} F\left(x, \left(-\frac{4m}{\cos(\pi\alpha)}\right)^{\frac{1}{2}} w_n\right) dx \\
&\geq m.
\end{aligned}$$

That is, $\varphi(t_n u_n) \to +\infty$. But $\varphi(0) = 0$, $\varphi(u_n) \to c$, we see that $t_n \in (0,1)$, and

$$\varphi'(t_n u_n) t_n u_n = t_n \frac{d}{dt}\bigg|_{t=t_n} \varphi(tu_n) = 0.$$

Because of hypothesis (H_f)(iii),

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right) dx &\geq \frac{1}{\vartheta} \int_{\mathbb{R}^N} \left(\frac{1}{2} f(x,t_n u_n) t_n u_n - F(x,t_n u_n)\right) dx \\
&= \frac{1}{\vartheta}\left[\varphi(t_n u_n) - \frac{1}{2}\varphi'(t_n u_n) t_n u_n\right] \\
&= \frac{1}{\vartheta} \varphi(t_n u_n) \to +\infty.
\end{aligned}$$

This contradicts (9) and consequently we have proved that $\{u_n\}$ is bounded.

Let

$$\tau := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |u_n|^2 \, dx.$$

If $\tau = 0$, using Lemma 4, similarly to (16) we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x,u_n)\, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_n) u_n \, dx = 0. \tag{17}$$

Hence using (9) we have $c = 0$, a contradiction. Therefore $\tau > 0$. Similarly to (12), we can choose a sequence $\{z_n\} \subset Z^N$ such that setting $\overline{u}_n(x) = u_n(x + z_n)$, we have

$$\int_{B_2(0)} |\overline{u}_n|^2 \, dx = \int_{B_2(z_n)} |u_n|^2 \, dx \geq \frac{\tau}{2^{2N+1}}. \tag{18}$$

Note that $\|\overline{u}_n\| = \|u_n\|$, we see that $\{\overline{u}_n\}$ is bounded. Going if necessary to a subsequence, we obtain

$$\overline{u}_n \rightharpoonup \overline{u} \quad \text{in } E \quad \text{and} \quad \overline{u}_n \to \overline{u} \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^N).$$

Returning to (18), we see that $\overline{u} \neq 0$. Moreover, by the Z^N invariance of the problem, $\{\overline{u}_n\}$ is also a $(C)_c$ sequence of φ. Thus for every $v \in C_0^\infty(\mathbb{R}^N)$, we have

$$\varphi'(\overline{u})v = \lim_{n\to\infty} \varphi'(\overline{u}_n)v = 0.$$

So $\varphi'(\overline{u}) = 0$ and $\overline{u}$ is a nontrivial solution of (P).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

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