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Existence and boundary behavior of solutions to p-Laplacian elliptic equations

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Abstract

Under appropriate conditions on $b(x)$ and $g(u)$, we consider the singular Dirichlet problems $-\Delta_p u = b(x)g(u)$, $u > 0$, $x \in \Omega$, $u|_{\partial\Omega} = 0$. These problems are shown to admit weak solutions, and we analyze their exact asymptotic behavior near the boundary. As the main tools, we use Karamata regular variation theory and the method of upper and lower solutions.

Keywords: singular Dirichlet problem; existence of solutions; exact asymptotic behavior; upper and lower solutions

1 Introduction and the main results

The purpose of this paper is to investigate the existence and exact asymptotic behavior of the solution near the boundary to the following problems:

$$-\Delta_p u = b(x)g(u), \quad u > 0, x \in \Omega, u|_{\partial\Omega} = 0, \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p-Laplacian operator with $p > 1$, Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ ($N \geq 2$), b satisfies the condition

(b_1) $b \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$ and is positive in Ω,

and g satisfies the following conditions:

(g_1) $g \in C^1((0,\infty),(0,\infty))$, $\lim_{s\to 0^+} g(s) = \infty$, and $g'(s) \leq 0$ for all $s > 0$;
(g_2) $\int_0^1 \frac{dv}{(g(v))^{q/p}} < \infty$;
(g_3) there exists $C_g > 0$ such that $\lim_{s\to 0} \frac{q}{pg^{1-\frac{q}{p}}(s)} g'(s) \int_0^s g^{-q/p}(v)\,dv = -C_g$,

where q stands for the Hölder conjugate of p.

A solution of (1.1) is meant as a positive function $u \in C^1(\Omega)$ with $u(x) \to 0$ as $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi\,dx = \int_\Omega b(x)g(u)\phi\,dx, \quad \forall \phi \in C_0^\infty(\Omega).$$

The class of problems (1.1) appears in many nonlinear phenomena, for instance, in the theory of quasi-regular and quasi-conformal mappings [1–3], in the generalized reaction-diffusion theory [4], in the turbulent flow of a gas in a porous medium, and in the non-Newtonian fluid theory [5].

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The investigation of problem (1.1) has a long history. Early studies mainly focused on problems involving the classical Laplace operator Δ, that is,

$$-\Delta u = b(x)g(u), \quad u > 0, x \in \Omega, u|_{\partial\Omega} = 0. \tag{1.2}$$

For $b \equiv 1$ and $g(u) = u^{-\gamma}$ with $\gamma > 1$, in 1977, Crandall, Rabinowitz, and Tartar [6] have derived that problem (1.2) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$. This paper is the starting point on semilinear problem with singular nonlinearity. Moreover, the following result was established: there exist positive constants c_1 and c_2 such that

$$c_1\bigl(d(x)\bigr)^{2/(1+\gamma)} \le u(x) \le c_2\bigl(d(x)\bigr)^{2/(1+\gamma)} \quad \text{near } \partial\Omega. \tag{1.3}$$

Lazer and McKenna [7] showed that (1.3) continues to hold on $\bar{\Omega}$, and instead of $b \equiv 1$ on Ω, they assumed that $0 < b_1 \le b(x)(d(x))^{\lambda} \le b_2$ for all $x \in \bar{\Omega}$, where b_1, b_2 are positive constants, and $\lambda \in (0,2)$. Later, a lot of work has been done related to the existence and asymptotic behavior of the solutions to problem (1.2); we refer to [8–16] and the references therein.

It is worth pointing out that Cîrstea and Rădulescu [17–19], Cîrstea and Du [20], and Repovš [21] introduced the Karamata regular variation theory to study the boundary behavior and uniqueness of solutions for boundary blow-up elliptic problems and obtained a series of significant information about the qualitative behavior of large solutions in a general framework.

Recently, by using the Karamata regular variation theory Zhang and Li [22], Zhang [23], Zhang and Cheng [24], and Mi and Liu [25] further studied the boundary behavior of the solutions to problem (1.2).

Now, let us return to problem (1.1). When $b(x) \equiv 1$ and $g(u) = u^m$, the first results concerning (1.1) ($0 < p - 1 < m$) have been obtained by Ni and Serrin [26, 27], who gave a priori estimates near a singularity. In particular, they show that $m = N(p-1)/(N-p)$ is a critical value. They also obtained nonexistence results for positive solutions in an exterior domain for $p - 1 < m < N(p-1)/(N-p)$. Guedda and Veron [28] give Ni and Serrin's estimates under a slightly weaker hypothesis. Later, Bognara and Drabekb [29] deals with the existence and multiplicity results for radial symmetric solutions of problem (1.1) for a more general nonlinearity $g(u)$. In recent years, the existence and uniqueness of positive solutions for the general quasilinear elliptic problem $-\Delta_p u = \lambda h(x, u, \nabla u)$, $u > 0$, $x \in \Omega$, $u|_{\partial\Omega} = 0$, has been studied by many authors. Some sufficient conditions on h and Ω have been proposed to ensure the existence or nonexistence of solutions; see [30–38] and the reference therein. However, to the best of our knowledge, up to now, few papers have addressed the boundary behavior of solutions to problem (1.1) for more general nonlinear terms g.

Inspired by the works mentioned, in this paper, by using Karamata regular variation theory and the method of upper and lower solutions, we show the existence of a solution to problem (1.1) and provide some asymptotic boundary estimates under appropriate conditions on $b(x)$ and $g(u)$.

In order to present our main results, we introduce the following two kinds of functions.

Let Λ denote the set of positive nondecreasing functions $k \in C^1(0, \nu)$ that satisfy

$$\lim_{t \to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) = C_k, \quad \text{where } K(t) = \int_0^t k(s)\,ds. \tag{1.4}$$

We note that, for each $k \in \Lambda$,

$$\lim_{t \to 0^+} \frac{K(t)}{k(t)} = 0 \quad \text{and} \quad C_k \in [0,1].$$

The set Λ was first introduced by Cîrstea and Rădulescu.

Next, we denote by Θ the set of all Karamata functions $\hat{L}$ that are *normalized* slowly varying at zero (see the definition in Section 2) defined on $(0,\eta]$ for some $\eta > 0$ by

$$\hat{L}(s) = c_0 \exp\left(\int_s^\eta \frac{y(\tau)}{\tau}\,d\tau\right), \quad s \in (0,\eta], \tag{1.5}$$

where $c_0 > 0$, and the function $y \in C([0,\eta])$ with $y(0) = 0$.

The key to our estimates in this paper is the solution to the problem

$$\int_{\phi(t)}^\infty \frac{ds}{(f(s))^{\frac{1}{p-1}}} = t, \quad t > 0. \tag{1.6}$$

Our main results are summarized as follows.

Theorem 1.1 *Let g satisfy* (g_1)-(g_2), *and b satisfy* (b_1)-(b_2). *Suppose that b also satisfies the following condition*:

(b_3) *the linear problem*

$$-\Delta_p u = b(x), \quad u > 0, x \in \Omega, u|_{\partial\Omega} = 0, \tag{1.7}$$

has a unique solution $v_0 \in C^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$ *for some* $\alpha \in (0,1)$.

Then, problem (1.1) *has at least one solution* $u \in C^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$.

Theorem 1.2 *Let g satisfy* (g_1)-(g_2), *and b satisfy* (b_1)-(b_3). *Suppose that b also satisfies the following condition*:

(b_4) *there exist* $k \in \Lambda$ *and a positive constant* $b_0 \in \mathbb{R}$ *such that*

$$\lim_{d(x)\to 0} \frac{b(x)}{k^p(d(x))} = b_0.$$

If

$$C_k + qC_g > q,$$

then any solution u to problem (1.1) *satisfies*

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi(K^q(d(x)))} = A_1^{1-C_g}, \tag{1.8}$$

where ϕ *is uniquely determined by* (1.6), *q stands for the Hölder conjugate of p, and*

$$A_1 = \frac{1}{q}\left(\frac{b_0}{(p-1)(qC_g + C_k - q)}\right)^{\frac{1}{p-1}}.$$

Theorem 1.3 *Let g satisfy* (g_1)-(g_2), *and b satisfy* (b_1)-(b_3). *Suppose that b also satisfies the following condition*:

(b_5) *there exist $L \in \Theta$ and a positive constant $b_1 \in \mathbb{R}$ such that*

$$\lim_{d(x)\to 0} \frac{b(x)}{(d(x))^{-p}L(d(x))} = b_1.$$

Then any solution u to problem (1.1) *satisfies*

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi(h(d(x)))} = A_2^{1-C_g}, \tag{1.9}$$

where ϕ is uniquely determined by (1.6),

$$h(t) = \int_0^t s^{-1}\big(L(s)\big)^{\frac{1}{p-1}}\,ds, \tag{1.10}$$

and

$$A_2 = \left(\frac{b_1}{p-1}\right)^{\frac{1}{p-1}}.$$

The outline of this paper is as follows. In Sections 2-3, we give some preparation that will be used in the next section. The proofs of Theorems 1.1-1.3 will be given in Sections 4-5.

2 Preliminaries

Our approach relies on Karamata regular variation theory established by Karamata in 1930, which is a basic tool in the theory of stochastic processes (see [39–43] and the references therein). In this section, we first give a brief account of the definition and properties of regularly varying functions.

Definition 2.1 A positive measurable function f defined on $[a,\infty)$ for some $a > 0$ is called *regularly varying at infinity* with index ρ, written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, f is called *slowly varying at infinity*.

Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Definition 2.2 A positive measurable function f defined on $[a,\infty)$ for some $a > 0$ is called *rapidly varying at infinity* if for each $\rho > 1$,

$$\lim_{s\to\infty} \frac{f(s)}{s^\rho} = \infty. \tag{2.2}$$

We also see that a positive measurable function g defined on $(0,a)$ for some $a > 0$ is *regularly varying at zero* with index σ (written as $g \in RVZ_\sigma$) if $t \to g(1/t)$ belongs to $RV_{-\sigma}$. Similarly, g is called *rapidly varying at zero* if $t \to g(1/t)$ is rapidly varying at infinity.

Proposition 2.1 (Uniform convergence theorem) *If $f \in RV_\rho$, then* (2.1) *holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then the uniform convergence holds on intervals of the form (a_1, ∞) with $a_1 > 0$; if $\rho > 0$, then the uniform convergence holds on intervals $(0, a_1]$, provided that f is bounded on $(0, a_1]$ for all $a_1 > 0$.*

Proposition 2.2 (Representation theorem) *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(s) = \varphi(s)\exp\left(\int_{a_1}^{s}\frac{y(\tau)}{\tau}\,d\tau\right), \quad s \geq a_1, \tag{2.3}$$

for some $a_1 \geq a$, where the functions φ and y are measurable and $y(s) \to 0$ and $\varphi(s) \to c_0 > 0$ as $s \to \infty$.

We say that

$$\hat{L}(s) = c_0\exp\left(\int_{a_1}^{s}\frac{y(\tau)}{\tau}\,d\tau\right), \quad s \geq a_1, \tag{2.4}$$

is *normalized* slowly varying at infinity and

$$f(s) = c_0 s^\rho \hat{L}(s), \quad s \geq a_1, \tag{2.5}$$

is *normalized* regularly varying at infinity with index ρ (and written as $f \in NRV_\rho$).

Similarly, g is called *normalized* regularly varying at zero with index σ, written as $g \in NRVZ_\sigma$ if $t \to g(1/t)$ belongs to $NRV_{-\sigma}$.

A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1, \infty) \quad \text{for some } a_1 > 0 \quad \text{and} \quad \lim_{s\to\infty}\frac{sf'(s)}{f(s)} = \rho. \tag{2.6}$$

Proposition 2.3 *If functions L, L_1 are slowly varying at infinity, then*

(i) *L^σ for every $\sigma \in \mathbb{R}$, $c_1L + c_2L_1$ ($c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \circ L_1$ (if $L_1(t) \to +\infty$ as $t \to +\infty$) are also slowly varying at infinity.*

(ii) *For every $\theta > 0$, $t^\theta L(t) \to +\infty$ and $t^{-\theta}L(t) \to 0$ as $t \to +\infty$,*

(iii) *For $\rho \in \mathbb{R}$, $\frac{\ln(L(t))}{\ln t} \to 0$ and $\frac{\ln(t^\rho L(t))}{\ln t} \to \rho$ as $t \to +\infty$.*

Proposition 2.4

(i) *If $f_1 \in RV_{\rho_1}$ and $f_2 \in RV_{\rho_2}$ with $\lim_{t\to\infty} f_2(t) = \infty$, then $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$.*

(ii) *If $f \in RV_\rho$, then $f^\alpha \in RV_{\rho\alpha}$ for every $\alpha \in \mathbb{R}$.*

Proposition 2.5 *If a function L defined on $(0, \eta]$ is slowly varying at zero, then*

$$\lim_{t\to 0^+}\frac{L(t)}{\int_t^\eta \frac{L(s)}{s}\,ds} = 0. \tag{2.7}$$

If, moreover, $\int_0^\eta \frac{L(s)}{s}\,ds$ converges, then

$$\lim_{t\to 0^+}\frac{L(t)}{\int_0^t \frac{L(s)}{s}\,ds} = 0. \tag{2.8}$$

Proposition 2.6 (Asymptotic behavior) *If a function L is slowly varying at zero, then, for $a > 0$ and $t \to 0^+$,*

(i) $\int_0^t s^\rho L(s)\,ds \cong (\rho+1)^{-1} t^{1+\rho} L(t)$ *for* $\rho > -1$;

(ii) $\int_t^a s^\rho L(s)\,ds \cong (-\rho-1)^{-1} t^{1+\rho} L(t)$ *for* $\rho < -1$.

Proposition 2.7 (Proposition 2.6 in [44]) *Let $Z \in C^1(0,\eta]$ be positive and $\lim_{t\to 0^+} \frac{sZ'(s)}{Z(s)} = +\infty$. Then Z is rapidly varying to zero at zero.*

Proposition 2.8 (Proposition 2.7 in [44]) *Let $Z \in C^1(0,\eta)$ be positive and $\lim_{t\to 0^+} \frac{sZ'(s)}{Z(s)} = -\infty$. Then Z is rapidly varying to infinity at zero.*

3 Some auxiliary results

In this section, we collect some useful results.

Lemma 3.1 *Let $k \in \Lambda$. Then*

(i) $\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0$, $\lim_{t\to 0^+} \frac{tk(t)}{K(t)} = C_k^{-1}$, *i.e.*, $K \in NRVZ_{C_k^{-1}}$;

(ii) $\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \frac{1-C_k}{C_k}$, *i.e.*, $k \in NRVZ_{(1-C_k)/C_k}$; $\lim_{t\to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - C_k$.

Proof The proof is similar to that of Lemma 2.1 in [23]; so we omit it. □

Lemma 3.2 *Let*

$$a(t) = t^{-p} L(t)$$

and

$$h(t) = \int_0^t s^{-1} \bigl(L(s)\bigr)^{\frac{1}{p-1}}\,ds,$$

where $t \in (0,\delta_0)$, $\int_0^\eta s^{-1}(L(s))^{\frac{1}{p-1}}\,ds < \infty$ for some $\eta > 0$, and $L(s) \in \Theta$. Then

(i) $\lim_{t\to 0^+} \frac{(h'(t))^p}{h(t)a(t)} = 0$ *and* $\lim_{t\to 0^+} \frac{th'(t)}{h(t)} = 0$;

(ii) $\lim_{t\to 0^+} \frac{th''(t)}{h'(t)} = -1$;

(iii) $\lim_{t\to 0^+} \frac{(h'(t))^{p-2}h''(t)}{a(t)} = -1$.

Proof (i) Since $h'(t) = t^{-1}(L(t))^{\frac{1}{p-1}}$, we have

$$\frac{(h'(t))^p}{h(t)a(t)} = \frac{t^{-p} L^{\frac{p}{p-1}}(t)}{t^{-p}L(t)\int_0^t s^{-1} L^{\frac{1}{p-1}}(s)\,ds} = \frac{L^{\frac{1}{p-1}}(t)}{\int_0^t s^{-1} L^{\frac{1}{p-1}}(s)\,ds}$$

and

$$\frac{th'(t)}{h(t)} = \frac{L^{\frac{1}{p-1}}(t)}{\int_0^t s^{-1} L^{\frac{1}{p-1}}(s)\,ds}$$

Hence, by Proposition 2.5 we get $\lim_{t\to 0^+} \frac{(h'(t))^p}{h(t)a(t)} = \lim_{t\to 0^+} \frac{th'(t)}{h(t)} = 0$.

(ii) By a direct computation we get

$$h''(t) = -t^{-2}\big(L(t)\big)^{\frac{1}{p-1}} + \frac{1}{p-1}t^{-1}\big(L(t)\big)^{\frac{1}{p-1}-1}L'(t)$$

and

$$\frac{th''(t)}{h'(t)} = \frac{1}{p-1}\frac{tL'(t)}{L(t)} - 1.$$

Since $L \in \Theta$, it follows that $\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$. Hence,

$$\lim_{t\to 0^+} \frac{th''(t)}{h'(t)} = -1.$$

(iii) Since

$$\frac{(h'(t))^{p-2}h''(t)}{a(t)} = \frac{th''(t)}{h'(t)}\frac{(h'(t))^{p-1}}{ta(t)} = \frac{th''(t)}{h'(t)},$$

by (ii) we get

$$\lim_{t\to 0^+} \frac{(h'(t))^{p-2}h''(t)}{a(t)} = -1. \qquad \square$$

Lemma 3.3 *Let g satisfy* (g_1)-(g_2).

(i) *If g satisfies* (g_3), *then* $C_g \le 1$;

(ii) (g_3) *holds for* $C_g \in (0,1)$ *if and only if* $g \in NRV_{-pC_g/(q(1-C_g))}$;

(iii) (g_3) *holds for* $C_g = 0$ *if and only if g is normalized slowly varying at zero*;

(iv) *if* (g_3) *holds with* $C_g = 1$, *then g is rapidly varying to infinity at zero.*

Proof Since g satisfies (g_1) and is strictly decreasing on $(0,S_0)$, we see that

$$0 < \int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu < \frac{s}{g^{q/p}(s)}, \quad \forall s \in (0,S_0),$$

that is,

$$0 < g^{q/p}(s)\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu < s, \quad \forall s \in (0,S_0), \tag{3.1}$$

and

$$\lim_{s\to 0} g^{q/p}(s)\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu = 0. \tag{3.2}$$

(i) Let

$$I(s) = -\frac{q}{pg^{1-\frac{q}{p}}(s)}g'(s)\int_0^s g^{-q/p}(\nu)\,d\nu, \quad \forall s \in (0,s_0).$$

Integrating $I(t)$ from 0 to s and integrating by parts, we obtain by (3.2) that

$$\int_0^s I(t)\,dt = -g^{q/p}(s)\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu + s, \quad \forall s \in (0, s_0),$$

that is,

$$0 < \frac{g^{q/p}(s)}{s}\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu = 1 - \frac{\int_0^s I(t)\,dt}{s}, \quad \forall s \in (0, s_0).$$

It follows by l'Hospital's rule that

$$0 \le \lim_{s\to 0^+}\frac{g^{q/p}(s)}{s}\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu = 1 - \lim_{s\to 0^+} I(s) = 1 - C_g. \tag{3.3}$$

So (i) holds.

(ii) When (g_3) holds with $C_g \in (0,1)$, it follows by (3.3) that

$$\lim_{s\to 0^+}\frac{g(s)}{sg'(s)} = \lim_{s\to 0^+}\frac{\frac{q}{p}g^{q/p}(s)\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu}{\frac{q}{p}sg'(s)\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu g^{\frac{q}{p}-1}(s)} = -\frac{q(1-C_g)}{pC_g}, \tag{3.4}$$

that is, $g \in NRV_{-pC_g/(q(1-C_g))}$.

Conversely, when $g \in NRV_{-\gamma}$ with $\gamma > 0$, that is, $\lim_{s\to 0^+}\frac{sg'(s)}{g(s)} = -\gamma$ and there exist a positive constant η and $\hat{L} \in \Theta$ such that $g(s) = c_0 s^{-\gamma}\hat{L}(s)$, $s \in (0,\eta]$, it follows by (2.6) and Proposition 2.6(i) that

$$\begin{aligned}
-\lim_{s\to 0^+}\frac{q}{pg^{1-\frac{q}{p}}(s)}g'(s)\int_0^s g^{-q/p}(\nu)\,d\nu &= -\frac{q}{p}\lim_{s\to 0^+}\frac{sg'(s)}{g(s)}\lim_{s\to 0^+}\frac{g^{q/p}(s)}{s}\int_0^s g^{-q/p}(\nu)\,d\nu \\
&= \frac{q\gamma}{p}\lim_{s\to 0^+} s^{-\frac{q\gamma}{p}-1}\big(\hat{L}(s)\big)^{\frac{q}{p}}\int_0^s \nu^{\frac{q\gamma}{p}}\big(\hat{L}(\nu)\big)^{-\frac{q}{p}}\,d\nu \\
&= \frac{q\gamma}{p+q\gamma}.
\end{aligned}$$

(iii) By $C_g = 0$ and the proof of (ii) we can see that

$$\begin{aligned}
\lim_{s\to 0^+}\frac{sg'(s)}{g(s)} &= \lim_{s\to 0^+}\frac{\frac{q}{p}sg'(s)\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu g^{\frac{q}{p}-1}(s)}{\frac{q}{p}g^{q/p}(s)\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu} \\
&= \frac{p}{q}\left(\lim_{s\to 0^+}\frac{g^{q/p}(s)}{s}\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu\right)^{-1}\lim_{s\to 0^+}\frac{q}{pg^{1-\frac{q}{p}}(s)}g'(s)\int_0^s g^{-q/p}(\nu)\,d\nu \\
&= 0,
\end{aligned}$$

that is, g is normalized slowly varying at zero.

Conversely, when g is normalized slowly varying at zero, that is, $\lim_{s\to 0^+}\frac{sg'(s)}{g(s)} = 0$, it follows by (3.3) that

$$\lim_{s\to 0^+}\frac{q}{pg^{1-\frac{q}{p}}(s)}g'(s)\int_0^s g^{-q/p}(\nu)\,d\nu = \lim_{s\to 0^+}\frac{q}{p}\frac{sg'(s)}{g(s)}\frac{g^{q/p}(s)}{s}\int_0^s \frac{1}{g^{q/p}(\nu)}\,d\nu = 0.$$

(iv) By $C_g = 1$ and the proof of (ii) we see that $\lim_{s\to 0^+} \frac{g(s)}{sg'(s)} = 0$, that is, $\lim_{s\to 0^+} \frac{sg'(s)}{g(s)} = -\infty$, and by Proposition 2.8 we get that g is rapidly varying to infinity at zero. $\square$

Lemma 3.4 *Let g satisfy* (g_1)-(g_3), *and ϕ be the solution to the problem*

$$\int_0^{\phi(t)} \frac{ds}{(g(s))^{\frac{1}{p-1}}} = t, \quad \forall t > 0.$$

Then

(i) $\phi'(t) = (g(\phi(t)))^{\frac{1}{p-1}}$, $\phi(t) > 0$, $t > 0$, $\phi(0) = 0$, *and* $\phi''(t) = \frac{q}{p}(g(\phi(t)))^{\frac{2q-p}{p}} g'(\phi(t))$, $t > 0$;
(ii) $\phi \in NRVZ_{1-C_g}$ *and* $\phi' \in NRVZ_{-C_g}$;
(iii) *when* $C_k + qC_g > q$ *and* $k \in \Lambda$, $\lim_{t\to 0^+} \frac{t}{\phi(\xi K^q(t))} = 0$ *uniformly for* $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$, *where q stands for the Hölder conjugate of p*;
(iv) $\lim_{t\to 0^+} \frac{t}{\phi(\xi h(t))} = 0$ *uniformly for* $\xi \in [c_1, c_2]$ *with* $0 < c_1 < c_2$, *where h is given as in* (1.10).

Proof By the definition of ϕ and a direct calculation we show that (i) holds.

(ii) It follows from (i), (3.4), and (g_3) that

$$\begin{aligned}\lim_{t\to 0^+} \frac{t\phi'(t)}{\phi(t)} &= \lim_{t\to 0^+} \frac{t(g(\phi(t)))^{\frac{1}{p-1}}}{\phi(t)} \\ &= \lim_{s\to 0} \frac{(g(s))^{\frac{1}{p-1}} \int_0^s \frac{dv}{(g(v))^{\frac{1}{p-1}}}}{s} = 1 - C_g,\end{aligned}$$

that is, $\phi \in NRVZ_{1-C_g}$, and

$$\begin{aligned}\lim_{t\to 0^+} \frac{t\phi''(t)}{\phi'(t)} &= \frac{q}{p} \lim_{t\to 0^+} \frac{g'(\phi(t))(g(\phi(t)))^{\frac{q}{p}} \int_0^{\phi(t)} (g(v))^{-\frac{1}{p-1}}\, dv}{g(\phi(t))} \\ &= \frac{q}{p} \lim_{s\to 0^+} \frac{g'(s)(g(s))^{\frac{q}{p}} \int_0^s (g(v))^{-\frac{1}{p-1}}\, dv}{g(s)} \\ &= -C_g.\end{aligned}$$

(iii) By Lemma 3.1(i) we see that $K \in NRVZ_{C_k^{-1}}$. It follows by Proposition 2.4 that $\phi \circ K^q \in NRVZ_{\frac{q(1-C_g)}{C_k}}$. Since $C_k + qC_g > q$, the result follows by Proposition 2.3(ii).

(iv) As in the proof of (iii), by Lemma 3.2(i) we see $h \in NRVZ_0$. It follows by Proposition 2.4 that $\phi \circ h \in NRVZ_0$. Then the result follows by Proposition 2.3(ii). $\square$

4 Existence of solutions to problem (1.1)

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1 Let

$$H(u) = \int_0^u \frac{1}{(g(s))^{\frac{1}{p-1}}}\, ds \quad \text{for } u > 0.$$

It follows that $H : [0,\infty) \to [0,\infty)$ is strictly increasing and

$$H'(u) = \frac{1}{(g(u))^{\frac{1}{p-1}}} \quad \text{for } u > 0.$$

Let $\bar{u}(x) := H^{-1}(v_0(x))$, $x \in \Omega$, where H^{-1} denotes the inverse function of H, and v_0 is the unique classical solution of problem (1.7). We see that $u|_{\partial\Omega} = 0$ and

$$-\Delta_p \bar{u} + \frac{g'(\bar{u})|\nabla \bar{u}|^p}{g(\bar{u})} = b(x)g(\bar{u}), \quad x \in \Omega.$$

It follows by (g_1) that

$$-\Delta_p \bar{u} \geq b(x)g(\bar{u}), \quad x \in \Omega,$$

that is, $\bar{u} = H^{-1}(v_0)$ is a supersolution of problem (1.1).

On the other hand, hypothesis (g_1) implies that $\lim_{s\to 0^+} g(s) \in (0,\infty]$, so that

$$\lim_{s\to 0^+} \frac{g(s)}{s} = +\infty \quad \text{and} \quad \lim_{s\to 0^+} \frac{(g(s))^{\frac{1}{p-1}}}{s} = +\infty.$$

There then exists $c_0 \in (0,1)$ such that

$$\frac{g(c_0|v_0|_\infty)}{c_0} \geq 1 \quad \text{and} \quad \frac{(g(c_0|v_0|_\infty))^{\frac{1}{p-1}}}{c_0} \geq 1.$$

Let $\underline{u} = c_0 v_0$. It follows that

$$-\Delta_p \underline{u} = c_0 b(x) \leq b(x)g\big(c_0|v_0|_\infty\big) \leq b(x)g(\underline{u}), \quad x \in \Omega,$$

that is, $\underline{u} = c_0 v_0$ is a subsolution of problem (1.1). Moreover, we see that

$$H\big(c_0 v_0(x)\big) = \int_0^{c_0 v_0(x)} \frac{1}{(g(s))^{\frac{1}{p-1}}}\, ds \leq \frac{c_0 v_0(x)}{(g(c_0|v_0|_\infty))^{\frac{1}{p-1}}} \leq v_0(x), \quad x \in \Omega,$$

that is, $\underline{u} \leq \bar{u}$ on Ω. Therefore, by the lower and upper theorem the claim follows. □

5 Boundary behaviors of solutions to problem (1.1)

In this section, we prove Theorems 1.2-1.3.

First, we need the following comparison principle for weak solutions to quasilinear equations (see [45] for a proof).

Lemma 5.1 (Weak comparison principle) *Let $D \subset \mathbb{R}^N$ be a bounded domain, $G : D \times \mathbb{R} \to \mathbb{R}$ be nonincreasing in the second variable and continuous. Let $u, w \in W^{1,p}(D)$ satisfy the respective inequalities*

$$\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \leq \int_D G(x,u)\phi \quad \textit{and}$$

$$\int_D |\nabla w|^{p-2}\nabla w\cdot\nabla\phi \geq \int_D G(x,w)\phi$$

for all nonnegative $\phi \in W_0^{1,p}(D)$. *Then the inequality* $u \leq w$ *on* ∂D *implies* $u \leq w$ *in* D.

Fix $\varepsilon > 0$. For any $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}$. Since Ω is C^2-smooth, choose $\delta_1 \in (0,\delta_0)$ such that $d \in C^2(\Omega_{\delta_1})$ and

$$\big|\nabla d(x)\big| = 1, \qquad \Delta d(x) = -(N-1)H(\bar{x}) + o(1), \quad \forall x \in \Omega_{\delta_1}, \tag{5.1}$$

where, for $x \in \Omega_{\delta_1}$, $\bar{x}$ denotes the unique point of the boundary such that $d(x) = |x-\bar{x}|$, and $H(\bar{x})$ denotes the mean curvature of the boundary at that point.

5.1 Proof of Theorem 1.2

Define $r = d(x)$ and

$$\begin{aligned} I_{1\pm}(r) &= (A_1 \pm \varepsilon)^{p-1}(p-1)q^{p-1}\biggl(p\frac{(A_1 \pm \varepsilon)K^q(r)\phi''((A_1 \pm \varepsilon)K^q(r))}{\phi'((A_1 \pm \varepsilon)K^q(r))} + 1 + \frac{p}{q}\frac{K(r)k'(r)}{k^2(r)}\biggr); \\ I_2(x) &= (A_1 \pm \varepsilon)^{p-1}q^{p-1}\frac{K(r)}{k(r)}\Delta d(x) + \frac{b(x)}{k^p(r)}\frac{g(\phi((A_1 \pm \varepsilon)K^q(r)))}{(\phi'((A_1 \pm \varepsilon)K^q(r)))^{p-1}}. \end{aligned}$$

By Lemmas 3.1 and 3.4, combined with the choices of A_1 in Theorem 1.2, we get the following lemma.

Lemma 5.2 *Suppose that g satisfies* (g_1)-(g_3) *and b satisfies* (b_1)-(b_4). *Then*

(i) $\lim_{r\to 0} I_{1\pm}(r) = (A_1 \pm \varepsilon)^{p-1}(p-1)q^{p-1}(q - qC_g - C_k)$;
(ii) $\lim_{d(x)\to 0} I_2(x) = b_0 = -A_1^{p-1}(p-1)q^{p-1}(q - qC_g - C_k)$;
(iii) $\lim_{d(x)\to 0}(I_{1\pm}(r) + I_2(x)) = (p-1)q^{p-1}(q - qC_g - C_k)((A_1 \pm \varepsilon)^{p-1} - A_1^{p-1})$.

Proof of Theorem 1.2 Let $v \in C^{1+\alpha}(\Omega) \cap C^1(\bar{\Omega})$ be the unique solution of the problem

$$-\Delta_p v = 1, \quad v > 0, x \in \Omega, v|_{\partial\Omega} = 0. \tag{5.2}$$

Then, we see that

$$\nabla v(x) \neq 0, \quad \forall x \in \partial\Omega \quad \text{and} \quad c_3 d(x) \leq v(x) \leq c_4 d(x), \quad \forall x \in \Omega, \tag{5.3}$$

where c_3, c_4 are positive constants.

By Lemma 5.2, since $K \in C[0,\delta_0)$ with $K(0) = 0$, we see that there exist $\delta_{1\varepsilon}, \delta_{2\varepsilon} \in (0, \min\{1,\delta_0\})$ (which corresponds to ε) sufficiently small such that

(i) $0 \leq K^q(r) \leq \delta_{1\varepsilon}$, $r \in (0,\delta_{2\varepsilon})$;
(ii) $I_{1+}(r) + I_2(x) \leq 0$, $\forall (x,r) \in \Omega_{\delta_{1\varepsilon}} \times (0,\delta_{2\varepsilon})$;
(iii) $I_{1-}(r) + I_2(x) \geq 0$, $\forall (x,r) \in \Omega_{\delta_{1\varepsilon}} \times (0,\delta_{2\varepsilon})$.

Now we define

$$\bar{u}_\varepsilon = \phi\bigl((A_1 + \varepsilon)K^q\bigl(d(x)\bigr)\bigr), \quad x \in \Omega_{\delta_{1\varepsilon}}.$$

Before we prove the theorem, let us note the following. Suppose that z is a C^2 function on a domain Ω in $\mathbb{R}^N$ and $v = \phi(z)$, where ϕ is uniquely determined by (1.6). A direct computation shows that

$$\Delta_p v = (p-1)\left|\phi'(z)\right|^{p-2}\phi''(z)|\nabla z|^p + \left|\phi'(z)\right|^{p-2}\phi'(z)\Delta_p z. \tag{5.4}$$

Hence, by (5.4), Lemma 5.2, and a direct calculation we see that, for $x \in \Omega_{\delta_{1\varepsilon}}$

$$\begin{aligned}&\Delta_p \bar{u}_\varepsilon(x) + b(x)g\big(\bar{u}_\varepsilon(x)\big)\\ &\quad = \big(\phi'\big(K^q\big(d(x)\big)\big)\big)^{p-1}k^p\big(d(x)\big)\big(I_{1+}(r) + I_2(x)\big) \le 0,\end{aligned}$$

where $r = d(x)$, that is, $\bar{u}_\varepsilon$ is a supersolution of problem (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

In a similar way, we show that

$$\underline{u}_\varepsilon = \phi\big((A_1-\varepsilon)K^q\big(d(x)\big)\big), \quad x \in \Omega_{\delta_{1\varepsilon}},$$

is a subsolution of problem (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

Let $u \in C(\bar{\Omega}) \cap C^{1,\alpha}(\Omega)$ be the unique solution to problem (1.1). We assert that there exists M large enough such that

$$u(x) \le Mv(x) + \bar{u}_\varepsilon(x), \qquad \underline{u}_\varepsilon(x) \le u(x) + Mv(x), \quad x \in \Omega_{\delta_{1\varepsilon}}, \tag{5.5}$$

where v is the solution of problem (5.2).

In fact, we can choose M large enough such that

$$u(x) \le \bar{u}_\varepsilon(x) + Mv(x) \quad \text{and} \quad \underline{u}_\varepsilon(x) \le u(x) + Mv(x) \quad \text{on } \big\{x \in \Omega : d(x) = \delta_{1\varepsilon}\big\}.$$

We see by (g_1) that $\bar{u}_\varepsilon(x) + Mv(x)$ and $u(x) + Mv(x)$ are also supersolutions of problem (1.1) in $\Omega_{\delta_{1\varepsilon}}$. Since $u = \bar{u}_\varepsilon + Mv = u + Mv = \underline{u}_\varepsilon = 0$ on $\partial\Omega$, (5.5) follows by (g_1) and the weak comparison principle (Lemma 5.1). Hence, for $x \in \Omega_{\delta_{1\varepsilon}}$

$$\frac{u(x)}{\phi((A_1+\varepsilon)K^q(d(x)))} \le \frac{Mv(x)}{\phi((A_1+\varepsilon)K^q(d(x)))} + 1$$

and

$$1 - \frac{Mv(x)}{\phi((A_1-\varepsilon)K^q(d(x)))} \le \frac{u(x)}{\phi((A_1-\varepsilon)K^q(d(x)))}.$$

Consequently, by (5.3) and Lemma 3.4(iii),

$$1 \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi((A_1-\varepsilon)K^q(d(x)))} \le \limsup_{d(x)\to 0} \frac{u(x)}{\phi((A_1+\varepsilon)K^q(d(x)))} \le 1$$

and

$$\lim_{d(x)\to 0} \frac{\phi((A_1-\varepsilon)K^q(d(x)))}{\phi(K^q(d(x)))} = (A_1-\varepsilon)^{1-C_g}.$$

Thus, letting $\varepsilon \to 0$, we obtain (1.8). $\square$

5.2 Proof of Theorem 1.3

As before, fix $\varepsilon > 0$. For any $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}$. Since Ω is C^2-smooth, choose $\delta_1 \in (0, \delta_0)$ such that $d \in C^2(\Omega_{\delta_1})$ and (5.1) holds.

Define $r = d(x)$ and

$$\begin{aligned} I_{1\pm}(r) &= (A_2 \pm \varepsilon)^{p-1}(p-1)\biggl(\frac{(A_2 \pm \varepsilon)h(r)\phi''((A_2 \pm \varepsilon)h(r))}{\phi'((A_1 \pm \varepsilon)h(r))}\frac{(h'(r))^p}{h(r)r^{-p}L(r)} \\ &\quad + \frac{(h'(r))^{p-2}h''(r)}{r^{-p}L(r)}\biggr), \\ I_2(x) &= (A_2 \pm \varepsilon)^{p-1}\frac{(h'(r))^{p-1}}{r^{-p}L(r)}\Delta d(x) + \frac{b(x)}{k^p(r)}\frac{g(\phi((A_2 \pm \varepsilon)h(r)))}{(\phi'((A_2 \pm \varepsilon)h(r)))^{p-1}}. \end{aligned}$$

By Lemmas 3.2 and 3.4, combined with the choices of A_2 in Theorem 1.3, we get the following lemma.

Lemma 5.3 *Suppose that g satisfies* (g_1)-(g_3), *b satisfies* (b_1)-(b_3), *and* (b_5) *holds. Then*

(i) $\lim_{r\to 0} I_{1\pm}(r) = -(p-1)(A_2 \pm \varepsilon)^{p-1}$;
(ii) $\lim_{d(x)\to 0} I_2(x) = b_1 = (p-1)A_2^{p-1}$;
(iii) $\lim_{d(x)\to 0}(I_{1\pm}(r) + I_2(x)) = -(p-1)((A_2 \pm \varepsilon)^{p-1} - A_2^{p-1})$.

Proof of Theorem 1.3 By Lemma 5.3, since $h \in C[0, \delta_0)$ with $h(0) = 0$, we see that there exist $\delta_{1\varepsilon}, \delta_{2\varepsilon} \in (0, \min\{1, \delta_0\})$ (which corresponds to ε) sufficiently small such that

(i) $0 \le h(r) \le \delta_{1\varepsilon}$, $r \in (0, \delta_{2\varepsilon})$;
(ii) $I_{1+}(r) + I_2(x) \le 0$, $\forall (x, r) \in \Omega_{\delta_{1\varepsilon}} \times (0, \delta_{2\varepsilon})$;
(iii) $I_{1-}(r) + I_2(x) \ge 0$, $\forall (x, r) \in \Omega_{\delta_{1\varepsilon}} \times (0, \delta_{2\varepsilon})$.

As in the proof of Theorem 1.2, we define

$$\bar{u}_\varepsilon = \phi\bigl((A_1 + \varepsilon)h\bigl(d(x)\bigr)\bigr), \quad x \in \Omega_{\delta_{1\varepsilon}},$$

where

$$h(t) = \int_0^t s^{-1}\bigl(L(s)\bigr)^{\frac{1}{p-1}}\,ds.$$

By (5.4), Lemma 5.3, and a direct calculation we see that, for $x \in \Omega_{\delta_{1\varepsilon}}$

$$\begin{aligned} &\Delta_p \bar{u}_\varepsilon(x) + b(x)g\bigl(\bar{u}_\varepsilon(x)\bigr) \\ &\quad = \bigl(\phi'\bigl(h(r)\bigr)\bigr)^{p-1}r^{-p}L(r)\bigl(I_{1+}(r) + I_2(x)\bigr) \le 0, \end{aligned}$$

where $r = d(x)$, that is, $\bar{u}_\varepsilon$ is a supersolution of problem (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

In a similar way, we show that

$$\underline{u}_\varepsilon = \phi\bigl((A_2 - \varepsilon)h\bigl(d(x)\bigr)\bigr), \quad x \in \Omega_{\delta_{1\varepsilon}},$$

is a subsolution of problem (1.1) in $\Omega_{\delta_{1\varepsilon}}$.

As in the proof of Theorem 1.2, we obtain, for $x \in \Omega_{\delta_{1\varepsilon}}$

$$\frac{u(x)}{\phi((A_2 + \varepsilon)h(d(x)))} \le \frac{Mv(x)}{\phi((A_2 + \varepsilon)h(d(x)))} + 1$$

and

$$1 - \frac{Mv(x)}{\phi((A_2 - \varepsilon)h(d(x)))} \le \frac{u(x)}{\phi((A_2 - \varepsilon)h(d(x)))}.$$

Consequently, by (5.3) and Lemma 3.4(iv),

$$1 \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi((A_2 - \varepsilon)h(d(x)))} \le \limsup_{d(x)\to 0} \frac{u(x)}{\phi((A_2 + \varepsilon)h(d(x)))} \le 1$$

and

$$\lim_{d(x)\to 0} \frac{\phi((A_2 - \varepsilon)h(d(x)))}{\phi(h(d(x)))} = (A_2 - \varepsilon)^{1-C_g}.$$

Thus, letting $\varepsilon \to 0$, we obtain (1.9). □

Competing interests
The author declares to have no competing interests.

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