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Global existence of solutions for a fluid model of a neutron star

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Abstract

In this paper, we consider an initial-boundary value problem for the equations of a fluid spherical model of neutron star considered by Lattimer *et al.* We establish the global existence and regularity of the spherically symmetric solutions in H^i ($i = 1, 2, 4$) of this fluid model. These results improve and generalize the results of Ducomet and Necasova (Ann. Univ. Ferrara 55(1):153-193, 2009).

MSC: 35Q30; 35D30; 76N10

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1 Introduction

We consider an initial-boundary value problem for a fluid model of neutron star. In the case of a rapid cooling of the core of the star, the model used to describe the evolution of temperature in the star follows Lattimer *et al.* [2]. If a mechanical equilibrium is reached and the specific heat is a linear function of temperature, then the problem reduces to the study of a fast diffusion equation satisfied by the temperature in [3]. In a more general setting, suppose that the temperature is coupled to density and velocity fluctuations through a thermo-mechanical system; the simplest description of such a model is achieved through the compressible Navier-Stokes system in [4].

In this paper, we are interested in the 3D spherical symmetric solutions to the complete system, which has the general formulation as follows (see [5]):

$$\begin{cases} \rho_t + (\rho v) + \dfrac{2\rho v}{r} = 0, & (1.1)\\ \rho(v_t + vv_r) = \left(-p + \mu\left(v_r + \dfrac{2v}{r}\right)\right)_r - 4v_r\dfrac{v}{r} + \rho F(r,t), & (1.2)\\ \rho(e_t + ve_r) = Q_r + \dfrac{2Q}{r} - p\left(v_r + \dfrac{2v}{r}\right) + \mu\left(v_r + \dfrac{2v}{r}\right)^2 - \dfrac{8\nu v v_r}{r} - \dfrac{4\nu v^2}{r^2}, & (1.3) \end{cases}$$

in the domain $\omega \times \mathbb{R}^+$ with $\omega := (R_0, R_1)$, where R_0 is the radius of the internal rigid core of the star and R_1 is the exterior boundary, and $\rho(r,t)$ and $v(r,t)$ denote the density and the velocity, respectively. Let $\eta := \frac{1}{\rho}$ be the specific volume and $\theta(r,t)$ be the temperature, then the pressure $p(\eta,\theta) = \frac{A}{2}\frac{\theta^2}{\eta^{2-\beta}}$ and the internal energy $e(\eta,\theta) = c_v\theta + \frac{A}{2(\beta-1)}\frac{\theta^2}{\eta^{1-\beta}}$, where constants $c_v > 0$, $A > 0$ and $1 < \beta < 2$. The heat flux Q is given by the Fourier law $Q(\eta,\theta) :=$

$\kappa(\eta,\theta)\theta_r$ with the following constraints on the thermal conductivity:

$$\underline{\kappa}\big(1+\theta^q\big) \le \kappa(\eta,\theta) \le \overline{\kappa}\big(1+\theta^q\big), \tag{1.4}$$

$$\big|\kappa_\eta(\eta,\theta)\big| + \big|\kappa_{\eta\eta}(\eta,\theta)\big| \le \overline{K}_1\big(1+\theta^q\big), \tag{1.5}$$

$$\big|\kappa_\theta(\eta,\theta)\big| \le \overline{K}_2\big(1+\theta^{q-1}\big), \tag{1.6}$$

for any $\theta \ge 0$, with positive constants $\underline{\kappa}$, $\overline{\kappa}$, $\overline{K}_1$, $\overline{K}_2$ and $q \ge 4$. $F(r,t)$ is a given external field force (gravitation). Finally, we also assume the bulk viscous coefficient μ is a positive constant and the shear viscous coefficient $\nu = 0$.

As in [1], we transform the system in Eulerian coordinates (r,t) into that in Lagrangian (mass) coordinates (x,t) by

$$r(x,t) := r_0(x) + \int_0^t v(x,s)\,ds, \tag{1.7}$$

where $r_0(x) := (R_0^3 + 3\int_0^x \eta(y,0)\,dy)^{\frac{1}{3}}$ for $x \in (0,M)$, we have

$$\begin{cases} \eta_t = (r^2v)_x, & (1.8)\\ v_t = r^2\left(-p + \mu\dfrac{(r^2v)_x}{\eta}\right)_x + f, & (1.9)\\ e_t = Q_x + \left(-p + \mu\dfrac{(r^2v)_x}{\eta}\right)(r^2v)_x, & (1.10)\\ r_t = v, & (1.11)\end{cases}$$

in the fixed domain $\Omega \times \mathbb{R}^+$ with $\Omega := (0,M)$, where the specific volume η, the velocity v, the temperature θ and the radius r depend on the Lagrangian mass coordinates. Now the heat flux is $Q(\eta,\theta) = \kappa(\eta,\theta)\frac{r^4\theta_x}{\eta}$ and the external field force is given by the Newtonian law $f(x) = -G\frac{M_0}{r^2}$, where G and M_0 are positive constants. Denote the stress σ by

$$\sigma(\eta,\theta) := -p + \mu\frac{(r^2v)_x}{\eta}.$$

System (1.8)-(1.11) is subjected to the following boundary and initial conditions:

$$(\eta,v,r,\theta)|_{t=0} = (\eta_0,v_0,r_0,\theta_0)(x), \quad x \in [0,M], \tag{1.12}$$

$$v|_{x=0,M} = 0, \qquad Q|_{x=0} = 0, \qquad \theta|_{x=M} = \theta_\Gamma, \quad t \ge 0, \tag{1.13}$$

with constant $\theta_\Gamma > 0$.

Now let us recall some known results for the related system. For the full 3D compressible Navier-Stokes system with heat conductivity, we can refer to the basic references on the global existence of a weak solution, such as Lions [6], Feireisl [7], Feireisl and Novotný [8] and Bresch and Desjardins [9] and references therein. For the large-time behavior of the global solutions, we would also like to mention the work of Feireisl and Petzeltová [10, 11] and Feireisl and Novotný [12]. On the subject of the global existence and large-time behavior of smooth/strong solutions for the one-dimensional motions of viscous polytropic ideal gas under various conditions, we refer the reader to Kazhikhov and Shelukhin [13], Kawohl [14], Chen [15], Jiang [16–18], Zheng and Qin [19], Qin [20–22], and so on. For the free boundary problems, we can also refer to the work of Nagasawa [23, 24], Tani [25, 26] and Hsiao and Luo [27]. For the free and pure Neumann boundary value problem, we refer the reader to Umehara and Tani [28, 29], Qin and Huang [30], Qin *et al.* [31], and the references therein.

However, in the major part of astrophysical literature, for example, at least when rotation and magnetic aspects are neglected, a quite reliable approximation is spherical symmetry; see also [4, 32, 33]. In this quasi-monodimensional situation, the global existence and large-time behavior of a classical solution have been established in some spherically symmetric cases, and we refer to [5, 18, 20, 34–38] and the references therein. In addition, for the cylindrically symmetric Navier-Stokes equations with various boundary conditions, the global well-posedness of the solutions has been studied by many researchers, and we can refer to [14, 15, 25, 31, 39–46] and the references therein. For problem (1.8)-(1.13), Ducomet and Nečasová [1] have proved the global well-posedness and large-time asymptotics for the initial data $(\eta_0, v_0, \theta_0) \in H^1 \times H^1 \times H^1$. Ducomet and Nečasová [3] considered a fast diffusion equation satisfied by the temperature and proved well-posedness and large-time asymptotics of global solutions with the initial data $\theta_0 \in L^2$.

In this paper, we shall establish the global existence and regularity of solutions for the spherical symmetric model of neutron star with the initial data $(\eta_0, v_0, \theta_0) \in H^i \times H^i \times H^i$ $(i = 1, 2, 4)$. The main novelty is to establish the H^i $(i = 1, 2, 4)$ regularity of the global solutions to problem (1.8)-(1.13). It is worth pointing out that the boundary condition on the temperature θ is different from general Dirichlet or Neumann boundary condition. Our results improve and generalize the results in [1].

In the following, the notations L^p $(1 \le p \le +\infty)$ and $W^{k,p}$ (in particular, $W^{k,2}$ is also denoted by H^k and $H_0^1 = W_0^{1,2}$) stand for the usual Lebesgue spaces and the usual Sobolev spaces on $(0, M)$, respectively. $\|\cdot\|_B$ denotes the norm in the space B, $\|\cdot\| := \|\cdot\|_{L^2}$. $C^{\alpha,\beta} = C^{\alpha,\beta}([0, M] \times [0, T])$ stands for uniformly Hölder continuous space with exponents α in x and β in t. We use C_0 and C_1 to denote a generic positive constant depending only on the parameters of the system and the bounds of the initial data $(\eta_0, v_0, \theta_0) \in (H^1([0, M]))^3$, but being independent of t. Furthermore, $C_i(T)$ $(i = 1, 2, 4)$ is a universal constant only dependent on the given time T, the physical constants and the initial data $(\eta_0, v_0, \theta_0) \in (H^i([0, M]))^3$.

The rest of the paper is arranged as follows. In Section 2, we will state our main theorems about the global existence of the solutions to problem (1.8)-(1.13). Subsequently, by a series of lemmas, we shall prove our main theorems in Section 3.

2 Main results

Let T be an arbitrary positive number. Now we give the definition of $H^i([0, M])$-solution to the initial-boundary problem (1.8)-(1.13).

Definition 2.1 Function $(\eta(x, t), v(x, t), \theta(x, t))$ is called a global $H^i([0, M])$-solution to problem (1.8)-(1.13) if it satisfies the following conditions:

$$\eta(x,t) \in L^2\big([0,T], H^i\big([0,M]\big)\big) \cap L^\infty\big([0,T], H^i\big([0,M]\big)\big), \quad (x,t) \in [0,M] \times [0,T],$$
$$v(x,t) \in L^2\big([0,T], H^{i+1}\big([0,M]\big)\big) \cap L^\infty\big([0,T], H^i\big([0,M]\big)\big), \quad (x,t) \in [0,M] \times [0,T],$$

and

$$\theta(x,t) \in L^2\big([0,T],H^{i+1}\big([0,M]\big)\big) \cap L^\infty\big([0,T],H^i\big([0,M]\big)\big), \quad (x,t) \in [0,M] \times [0,T],$$

where $i = 1, 2, 4$.

For convenience, we first state a proposition from [1].

Proposition 2.1 *The corresponding static problem to problem* (1.8)-(1.13) *has a unique solution* $(\bar\eta, \bar v, \bar\theta)$ *given by*

$$\begin{cases} \bar\eta = [\frac{(\beta-1)GM_0}{(\beta-2)A\theta_\Gamma^2}(\frac{1}{r} - \frac{1}{r_0})]^{-\frac{1}{\beta-1}}, \\ \bar v = 0, \\ \bar\theta = \theta_\Gamma, \end{cases} \tag{2.1}$$

where the constant r_0 *only depends on the initial data.*

We are now in a position to state our main result.

Theorem 2.1 *Let the initial data* $0 < C_0^{-1} < \eta_0(x) < C_0$, $(\eta_0, v_0, \theta_0) \in (H^1[0,M])^3$. *Assume that the heat conductivity* κ *satisfies* (1.4)-(1.6) *and the initial data are compatible with boundary conditions. Then problem* (1.8)-(1.13) *admits a unique global* $H^1([0,M])$*-solution* $(\eta(x,t), v(x,t), \theta(x,t))$ *verifying, for all* $(x,t) \in [0,M] \times [0,T]$,

$$0 < C_1^{-1} \le \eta(x,t) \le C_1, \qquad 0 < C_1^{-1} \le \theta(x,t) \le C_1, \qquad 0 < R_0 \le r(x,t) \le R_1, \tag{2.2}$$

and

$$\begin{aligned} &\big\|\eta(t) - \bar\eta\big\|_{H^1}^2 + \big\|v(t)\big\|_{H^1}^2 + \big\|\theta(t) - \bar\theta\big\|_{H^1}^2 \\ &\quad + \int_0^t \big(\|\eta - \bar\eta\|_{H^1}^2 + \|v\|_{H^2}^2 + \|\theta - \bar\theta\|_{H^2}^2 + \|\eta_t\|^2 + \|v_t\|^2 + \|\theta_t\|^2\big)(s)\,ds \le C_1(T). \end{aligned} \tag{2.3}$$

Theorem 2.2 *Let the initial data* $0 < C_0^{-1} < \eta_0(x) < C_0$, $(\eta_0, v_0, \theta_0) \in (H^2[0,M])^3$. *Assume that the heat conductivity* κ *satisfies* (1.4)-(1.6) *and the initial data are compatible with boundary conditions. Then problem* (1.8)-(1.13) *admits a unique global* $H^2([0,M])$*-solution* $(\eta(x,t), v(x,t), \theta(x,t))$ *verifying, for all* $(x,t) \in [0,M] \times [0,T]$,

$$\begin{aligned} &\big\|\eta(t) - \bar\eta\big\|_{H^2}^2 + \big\|v(t)\big\|_{H^2}^2 + \big\|\theta(t) - \bar\theta\big\|_{H^2}^2 + \big\|v_t(t)\big\|^2 + \big\|\theta_t(t)\big\|^2 \\ &\quad + \int_0^t \big(\|\eta - \bar\eta\|_{H^2}^2 + \|v\|_{H^3}^2 + \|\theta - \bar\theta\|_{H^3}^2 + \|\eta_t\|_{H^1}^2 + \|v_t\|_{H^1}^2 + \|\theta_t\|_{H^1}^2\big)(s)\,ds \\ &\le C_2(T). \end{aligned} \tag{2.4}$$

Theorem 2.3 *Let the initial data* $0 < C_0^{-1} < \eta_0(x) < C_0$, $(\eta_0, v_0, \theta_0) \in (H^4[0,M])^3$. *Assume that the heat conductivity* κ *satisfies* (1.4)-(1.6) *and the initial data are compatible with* *boundary conditions. Then problem* (1.8)-(1.13) *admits a unique global* $H^4([0,M])$*-solution* $(\eta(x,t),v(x,t),\theta(x,t))$ *verifying, for all* $(x,t)\in[0,M]\times[0,T]$,

$$\begin{aligned}
&\|\eta(t)-\bar\eta\|_{H^4}^2+\|v(t)\|_{H^4}^2+\|\theta(t)-\bar\theta\|_{H^4}^2+\|\eta_t(t)\|_{H^2}^2+\|v_t(t)\|_{H^2}^2+\|\theta_t(t)\|_{H^2}^2\\
&\quad+\int_0^t\big(\|\eta-\bar\eta\|_{H^4}^2+\|v\|_{H^5}^2+\|\theta-\bar\theta\|_{H^5}^2+\|\eta_t\|_{H^3}^2+\|v_t\|_{H^3}^2+\|\theta_t\|_{H^3}^2+\|\eta_{tt}\|_{H^1}^2\\
&\quad+\|v_{tt}\|_{H^1}^2+\|\theta_{tt}\|w_{H^1}^2\big)(s)\,ds\le C_4(T).
\end{aligned}\tag{2.5}$$

Corollary 2.1 *Under the assumptions of Theorem* 2.3 *and some suitable compatibility conditions, the global solution* (η,v,θ) *to problem* (1.8)-(1.13) *is the classical solution verifying*

$$\|\eta\|_{C^{3,\frac12}}+\|v\|_{C^{3,\frac12}}+\|\theta\|_{C^{3,\frac12}}\le C_4(T).$$

Remark 2.1 The uniqueness of the global solutions has been obtained in [1].

Remark 2.2 Theorem 2.1 implies that problem (1.8)-(1.13) admits a unique global weak solution. Theorem 2.2 implies that problem (1.8)-(1.13) admits a unique global strong solution.

Remark 2.3 Our results generalize the previous work in [1].

3 Proofs of theorems

In this section, we will give some useful *a priori* estimates of the solutions to complete the proofs of the theorems.

3.1 Global existence of H^1-solution

In this subsection, we shall complete the proof of Theorem 2.1. As in [1], we have the following mass conservation and energy-entropy inequality.

Lemma 3.1 *Under the assumptions in Theorem* 2.1, *the following estimates hold, for any* $t\in[0,T]$,

$$\int_0^M\eta(x,t)\,dx=\int_0^M\eta_0(x)\,dx,\tag{3.1}$$

$$\begin{aligned}
&\int_0^M\left(\frac12v^2+\frac{A}{2(\beta-1)}\eta^{\beta-1}(\theta-\theta_\Gamma)^2\right)dx\\
&\quad+\int_0^t\int_0^M\left(\frac{\kappa(\eta,\theta)r^4}{\eta\theta^2}\theta_x^2+\frac{\mu}{\eta\theta}\big((r^2v)_x\big)^2\right)dx\,ds\le C_1.
\end{aligned}\tag{3.2}$$

Proof See, *e.g.*, Lemma 1 in [1]. □

Lemma 3.2 *Under the assumptions in Theorem* 2.1, *the following estimates hold for all* $(x,t)\in\Omega\times[0,T]$:

$$0<C_1^{-1}\le\eta(x,t)\le C_1,\qquad 0<C_1^{-1}\le\theta(x,t)\le C_1.\tag{3.3}$$

Proof See, *e.g.*, Propositions 2 and 5 in [1]. □

Lemma 3.3 *Under the assumptions in Theorem* 2.1, *the following estimate holds for any* $t \in [0,T]$:

$$\|\eta_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_x(t)\|^2 + \int_0^t \big(\|\eta_x\|^2 + \|v_{xx}\|^2 + \|\theta_t\|^2\big)(s)\,ds \le C_1. \tag{3.4}$$

Proof See, *e.g.*, Propositions 3-5 and Lemma 5 in [1]. □

Lemma 3.4 *Under the assumptions in Theorem* 2.1, *the following estimate holds for any* $t \in [0,T]$:

$$\int_0^t \big(\|\theta_{xx}\|^2 + \|v_t\|^2\big)(s)\,ds \le C_1(T). \tag{3.5}$$

Proof Multiplying (1.9) by v_t over $(0,M) \times (0,T)$, employing an integration by parts and using Lemmas 3.1-3.3 and the Young inequality, we have

$$\begin{aligned}
&\big\|\big(r^2 v\big)_x\big\|^2 + \int_0^t \|v_t(s)\|^2\,ds \\
&\quad\le C_1 + C_1 \int_0^t \int_0^M \big(\big|v_t\big(-r^2 p_x + f\big)\big| + \big|\big(r^2 v\big)_x\big|^3 + |v v_x|\big)\,dx\,ds \\
&\quad\le C_1 + \frac{1}{2}\int_0^t \|v_t(s)\|^2\,ds + C_1 \int_0^t \int_0^M \big(\theta_x^2 + \eta_x^2 + f^2 + v^2 + v_x^2 + \big|\big(r^2 v\big)_x\big|^3\big)\,dx\,ds \\
&\quad\le C_1(T) + \frac{1}{2}\int_0^t \|v_t(s)\|^2\,ds + C_1 \int_0^t \big\|\big(r^2 v\big)_x\big\|_{L^3}^3\,ds \\
&\quad\le C_1(T) + \frac{1}{2}\int_0^t \|v_t(s)\|^2\,ds + C_1 \int_0^t \big\|\big(r^2 v\big)_x\big\|^2 \big\|\big(r^2 v\big)_{xx}\big\|\,ds \\
&\quad\le C_1(T) + \frac{1}{2}\int_0^t \|v_t(s)\|^2\,ds + C_1 \int_0^t \big\|\big(r^2 v\big)_{xx}\big\|^2\,ds \\
&\quad\le C_1(T) + \frac{1}{2}\int_0^t \|v_t(s)\|^2\,ds,
\end{aligned}$$

which implies

$$\big\|\big(r^2 v\big)_x\big\|^2 + \int_0^t \|v_t(s)\|^2\,ds \le C_1(T). \tag{3.6}$$

Equation (1.10) can be rewritten as

$$e_\theta \theta_t = Q_x - \theta p_\theta \big(r^2 v\big)_x + \frac{\mu}{\eta}\big(r^2 v\big)_x^2. \tag{3.7}$$

Multiplying (3.7) by $e_\theta^{-1}\theta_{xx}$, then integrating the result with respect to x over $(0,M)$, using Hölder's inequality, the Sobolev embedding theorem, and Lemmas 3.1-3.3, we have, for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{d}{dt}\left\|\theta_x(t)\right\|^2 + 2\int_0^M \frac{r^4\kappa}{e_\theta\eta}\theta_{xx}^2\,dx \\
&\quad = \int_0^M \left(\left(\frac{r^4\kappa}{\eta}\right)_x \theta_x - \theta p_\theta\left(r^2 v\right)_x + \frac{\mu}{\eta}\left(r^2 v\right)_x^2\right)\frac{\theta_{xx}}{e_\theta}\,dx \\
&\quad \le \varepsilon\|\theta_{xx}\|^2 + C_1(\varepsilon)\left(\|\theta_x\|^2 + \|\eta_x\theta_x\|^2 + \|\theta_x\|_{L^4}^4 + \|v_x\|^2 + \|v\|_{L^4}^4 + \|v_x\|_{L^4}^4\right) \\
&\quad \le \varepsilon\|\theta_{xx}\|^2 + C_1(\varepsilon)\left(\|\theta_x\|^2 + \|\theta_x\|_{L^\infty}^2 + \|\theta_x\|^3\|\theta_{xx}\| + \|v_x\|^2 + \|v\|^3\|v_x\| + \|v_x\|^3\|v_{xx}\|\right) \\
&\quad \le 2\varepsilon\|\theta_{xx}\|^2 + C_1(\varepsilon)\left(\|\theta_x\|^2 + \|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2\right).
\end{aligned} \tag{3.8}$$

Integrating (3.8) with respect to t over $(0,t)$, taking $\varepsilon > 0$ small enough, and using Lemmas 3.1 and 3.3, we can obtain

$$\left\|\theta_x(t)\right\|^2 + \int_0^t \left\|\theta_{xx}(s)\right\|^2 ds \le C_1, \tag{3.9}$$

which, along with (3.6), leads to the estimate (3.5). □

Now combining Lemmas 3.1-3.4 and noting equation (1.8), we complete the proof of Theorem 2.1.

3.2 Global existence of H^2-solution

In this subsection, we shall deal with the H^2-regularity of the global solutions to problem (1.8)-(1.13).

Lemma 3.5 *Under the assumptions in Theorem* 2.2, *the following estimate holds for any* $t \in [0,T]$:

$$\left\|v_{xx}(t)\right\|^2 + \left\|\theta_{xx}(t)\right\|^2 + \left\|v_t(t)\right\|^2 + \left\|\theta_t(t)\right\|^2 + \int_0^t \left(\|v_{xt}\|^2 + \|\theta_{xt}\|^2\right)(s)\,ds \le C_2(T). \tag{3.10}$$

Proof See, *e.g.*, Proposition 6 in [1]. □

Lemma 3.6 *Under the assumptions in Theorem* 2.2, *the following estimate holds for any* $t \in [0,T]$:

$$\left\|\eta_{xx}(t)\right\|^2 + \int_0^t \left\|\eta_{xx}(s)\right\|^2 ds \le C_2(T). \tag{3.11}$$

Proof Differentiating (1.9) with respect to x, we have

$$\begin{aligned}
\mu\frac{d}{dt}\left(\frac{\eta_{xx}}{\eta}\right) - p_\eta\eta_{xx} &= \left(r^{-2}v_t\right)_x + p_\theta\theta_{xx} + p_{\eta\eta}\eta_x^2 + p_{\theta\theta}\theta_x^2 \\
&\quad + 2p_{\eta\theta}\eta_x\theta_x + 2\mu\frac{\eta_x}{\eta}\left(\frac{(r^2 v)_x}{\eta}\right)_x - \left(r^{-2}f\right)_x \\
&=: \mathcal{M},
\end{aligned} \tag{3.12}$$

where

$$\|\mathcal{M}\| \le C_1(T)\big(\|\theta_x\|_{H^1} + \|v_t\| + \|v_{xt}\| + \|\eta_x\|_{L^4}^2 + \|v_x\|_{H^1} + 1\big).$$

By Theorem 2.1 and Lemma 3.5, using Young's inequality, we get, for any $\varepsilon > 0$,

$$\int_0^t \|\mathcal{M}\|^2\,ds \le C_2(T) + \varepsilon \int_0^t \big\|\eta_{xx}(s)\big\|^2\,ds. \tag{3.13}$$

Multiplying (3.12) by $\frac{\eta_{xx}}{\eta}$, then integrating the result over $[0,M]\times[0,t]$ and using Young's inequality and (3.13), taking $\varepsilon > 0$ sufficiently small, we can obtain (3.11). Thus we complete the proof. □

Lemma 3.7 *Under the assumptions in Theorem* 2.2, *the following estimate holds for any* $t \in [0,T]$:

$$\int_0^t \big(\|v_{xxx}\|^2 + \|\theta_{xxx}\|^2\big)(s)\,ds \le C_2(T). \tag{3.14}$$

Proof Differentiating (1.9) and (1.10) with respect to x, respectively, and using the Cauchy inequality, we easily obtain

$$\big\|v_{xxx}(t)\big\| \le C_1(T)\big(\big\|v_{xt}(t)\big\| + \big\|v_x(t)\big\|_{H^1} + \big\|\eta_x(t)\big\|_{H^1} + \big\|\theta_x(t)\big\|_{H^1}\big) \tag{3.15}$$

and

$$\big\|\theta_{xxx}(t)\big\| \le C_1(T)\big(\big\|\theta_{xt}(t)\big\| + \big\|\theta_x(t)\big\|_{H^1} + \big\|\eta_x(t)\big\|_{H^1} + \big\|v_x(t)\big\|_{H^1}\big). \tag{3.16}$$

By virtue of Theorem 2.1 and Lemmas 3.5-3.6, we complete the proof. □

Now combining Lemmas 3.5-3.7, we have completed the proof of Theorem 2.2.

3.3 Global existence of H^4-solution

In this subsection, we shall complete the proof of Theorem 2.3, which can be divided into the following lemmas.

Lemma 3.8 *Under the assumptions of Theorem* 2.3, *we see that for any* $t \in [0,T]$ *and for* $\varepsilon > 0$ *small enough*,

$$\begin{aligned}&\big\|v_{xt}(x,0)\big\| + \big\|\theta_{xt}(x,0)\big\| + \big\|v_{tt}(x,0)\big\| + \big\|\theta_{tt}(x,0)\big\| \\ &\quad + \big\|v_{txx}(x,0)\big\| + \big\|\theta_{txx}(x,0)\big\| \le C_4(T),\end{aligned} \tag{3.17}$$

$$\big\|v_{tt}(t)\big\|^2 + \int_0^t \big\|v_{ttx}(s)\big\|^2\,ds \le C_4(T) + C_2(T)\int_0^t \big(\|\theta_{txx}\|^2 + \|v_{txx}\|^2\big)(s)\,ds, \tag{3.18}$$

$$\begin{aligned}\big\|\theta_{tt}(t)\big\|^2 + \int_0^t \big\|\theta_{ttx}(s)\big\|^2\,ds &\le C_4(T) + C_2(T)\varepsilon^{-1}\int_0^t \big\|\theta_{txx}(s)\big\|^2\,ds \\ &\quad + C_1\varepsilon\int_0^t \big(\|v_{ttx}\|^2 + \|v_{txx}\|^2\big)(s)\,ds.\end{aligned} \tag{3.19}$$

Proof Differentiating (1.9) and (1.10) with respect to x, respectively, using Theorems 2.1 and 2.2, we can get

$$\|v_{xt}(t)\| \le C_2(T)\big(\|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^1} + \|\eta_x(t)\|_{H^1} + 1\big), \tag{3.20}$$

$$\|\theta_{xt}(t)\| \le C_2(T)\big(\|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^1} + \|\eta_x(t)\|_{H^1} + \|\theta_x(t)\|\big). \tag{3.21}$$

Similarly, differentiating (1.9) and (1.10) with respect to x twice, respectively, we can infer from Theorems 2.1 and 2.2 that

$$\begin{aligned}\|v_{xxt}(t)\| &\le C_2(T)\big(\|v_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{L^\infty}\|\eta_{xxx}(t)\| \\ &\qquad + \|\eta_x(t)\|_{L^\infty}\|v_{xxx}(t)\| + \|v_{xx}(t)\|_{L^\infty}\|\eta_{xx}(t)\| + \|\eta_x(t)\|\big) \\ &\le C_2(T)\big(\|v_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2}\big), \end{aligned} \tag{3.22}$$

$$\|\theta_{xxt}(t)\| \le C_2(T)\big(\|\theta_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2}\big), \tag{3.23}$$

or

$$\|v_{xxxx}(t)\| \le C_2(T)\big(\|v_x(t)\|_{H^2} + \|\eta_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_{txx}(t)\|\big), \tag{3.24}$$

$$\|\theta_{xxxx}(t)\| \le C_2(T)\big(\|\theta_x(t)\|_{H^2} + \|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_{txx}(t)\|\big). \tag{3.25}$$

It follows from (1.8) and (1.10) that

$$\|\eta_t(t)\| \le C_1\big(\|v(t)\| + \|v_x(t)\|\big), \tag{3.26}$$

$$\|\theta_t(t)\| \le C_1\big(\|\theta_{xx}(t)\| + \|\eta_x(t)\| + \|v_x(t)\| + \|v_{xx}(t)\|\big). \tag{3.27}$$

Differentiating (1.9) and (1.10) with respect to t, respectively, using Theorems 2.1-2.2 and (3.20)-(3.27), we have

$$\begin{aligned}\|v_{tt}(t)\| &\le C_2(T)\big(\|v_x(t)\|_{H^1} + \|\eta_x(t)\| + \|\theta_t(t)\| \\ &\qquad + \|\theta_{xt}(t)\| + \|v_{tx}(t)\| + \|v_{txx}(t)\| + \|\eta_t(t)\|\big) \qquad (3.28)\\ &\le C_2(T)\big(\|v_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + 1\big), \qquad (3.29)\\ \|\theta_{tt}(t)\| &\le C_2(T)\big(\|v_x(t)\|_{H^1} + \|\eta_x(t)\| + \|\theta_t(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_{tx}(t)\|\big) \qquad (3.30)\\ &\le C_2(T)\big(\|v_x(t)\|_{H^2} + \|\eta_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^3} + 1\big). \qquad (3.31)\end{aligned}$$

Thus the estimate (3.17) follows from (3.20)-(3.23), (3.29), and (3.31).

Differentiating (1.9) with respect to t twice, multiplying the resultant by v_{tt} and performing an integration by parts in $L^2(0,M)$, and using Theorem 2.2, the embedding theorem, and the Young inequality, we can derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_{tt}\|^2 &= -\int_0^M (r^2 v_{tt})_x\left(\mu\frac{(r^2v)_x}{\eta} - p\right)_{tt}dx - 2\int_0^M \big((r^2)_t v_{tt}\big)_x\left(\mu\frac{(r^2v)_x}{\eta} - p\right)_t dx \\ &\quad - \int_0^M \big((r^2)_{tt} v_{tt}\big)_x\left(\mu\frac{(r^2v)_x}{\eta} - p\right)dx\end{aligned}$$

$$\begin{aligned}
&\leq -\int_0^M \mu \frac{r_x^4}{\eta} v_{ttx}^2\,dx + C_2(T)\big(\|v_{tt}\| + \|v_{xt}v_x\| + \big\|v_x^3\big\| + \|\theta_t v_x\| \\
&\quad + \|v_{xt}\| + \|\theta_{tt}\| + \big\|v_x^2\big\|\big)\|v_{ttx}\| \\
&\leq -C_1^{-1}\|v_{ttx}\|^2 + C_2(T)\big(\|v_x\|_{H^1}^2 + \|\theta_t\|^2 + \|v_{xt}\|^2 + \|\theta_{tt}\|^2 + \|v_{tt}\|^2\big). \qquad (3.32)
\end{aligned}$$

Thus, by Theorem 2.2,

$$\big\|v_{tt}(t)\big\|^2 + \int_0^t \big\|v_{ttx}(s)\big\|^2\,ds \leq C_4(T) + C_2(T)\int_0^t \big(\|v_{tt}\|^2 + \|\theta_{tt}\|^2\big)(s)\,ds,$$

which, together with (3.28) and (3.30), gives estimate (3.18).

Similarly, differentiating (1.10) with respect to t twice, multiplying the result by θ_{tt} and performing an integration by parts over $L^2(0,M)$, and using the embedding theorem and the Young inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^M e_\theta \theta_{tt}^2\,dx \\
&\quad = -\int_0^M \left(\frac{r^4\kappa\theta_x}{\eta}\right)_{tt}\theta_{ttx}\,dx - \int_0^M \big(e_{\theta tt}\theta_t + e_{\eta tt}\big(r^2 v\big)_x\big)\theta_{tt}\,dx - \frac{3}{2}\int_0^M e_{\theta t}\theta_{tt}^2\,dx \\
&\quad\quad - \int_0^M \left(e_\eta + p - \mu\frac{(r^2v)_x}{\eta}\right)\big(r^2v\big)_{xtt}\theta_{tt}\,dx + \int_0^M \left(\mu\frac{(r^2v)_x}{v} - p\right)_{tt}\big(r^2v\big)_x\theta_{tt}\,dx \\
&\quad\quad - 2\int_0^M \left(e_{\eta t} + \left(p - \mu\frac{(r^2v)_x}{\eta}\right)_t\right)\big(r^2v\big)_{xt}\theta_{tt}\,dx \\
&\quad =: \sum_{i=1}^6 P_i. \qquad (3.33)
\end{aligned}$$

By virtue of Theorems 2.1-2.2 and the embedding theorem, we deduce that, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
P_1 &\leq -C_1\|\theta_{ttx}\|^2 + C_2\big(\|\theta_x\|_{L^\infty}\|v_{xt}\| + \|v_x\|_{L^\infty}\|\theta_{xt}\| + \|v_x\|_{L^\infty}^2\|\theta_x\| \\
&\quad + \|\theta_x\|_{L^\infty}\|\theta_t\| + \|\theta_x\|_{L^\infty}\|\theta_{tt}\|\big)\|\theta_{ttx}\| \\
&\leq -(2C_1)^{-1}\|\theta_{ttx}\|^2 + C_2(T)\big(\|\theta_{xt}\|^2 + \|v_{xt}\|^2 + \|v_x\|_{H^1}^2 + \|\theta_{tt}\|^2\big), \qquad (3.34) \\
P_2 &\leq C_1\int_0^M \big(\big(|v_x| + |\theta_t|\big)^2 + |v_{xt}| + |\theta_{tt}|\big)\big(|v_x| + |\theta_t|\big)|\theta_{tt}|\,dx \\
&\leq C_1\|\theta_{tt}\|_{L^\infty}\big(\|v_x\| + \|\theta_t\|\big)\big(\big(\|v_x\|_{L^\infty} + \|\theta_t\|_{L^\infty}\big)\big(\|v_x\| + \|\theta_t\|\big) + \|v_{xt}\| + \|\theta_{tt}\|\big) \\
&\leq C_2(T)\big(\|\theta_{tt}\| + \|\theta_{ttx}\|\big)\big(\|v_x\|_{H^1} + \|\theta_t\| + \|\theta_{xt}\| + \|v_{xt}\| + \|\theta_{tt}\|\big) \\
&\leq \varepsilon\|\theta_{ttx}\|^2 + C_2(T)\varepsilon^{-1}\big(\|v_x\|_{H^1}^2 + \|\theta_t\|^2 + \|\theta_{xt}\|^2 + \|v_{xt}\|^2 + \|\theta_{tt}\|^2\big), \qquad (3.35) \\
P_3 &\leq C_1\int_0^M \big(|v_x| + |\theta_t|\big)\theta_{tt}^2\,dx \leq C_1\|\theta_{tt}\|_{L^\infty}\big(\|v_x\| + \|\theta_t\|\big)\|\theta_{tt}\| \\
&\leq C_1\big(\|\theta_{tt}\| + \|\theta_{ttx}\|\big)\big(\|v_x\| + \|\theta_t\|\big)\|\theta_{tt}\| \leq \varepsilon\|\theta_{ttx}\|^2 + C_2(T)\varepsilon^{-1}\|\theta_{tt}\|^2, \qquad (3.36) \\
P_4 &\leq \varepsilon\|v_{ttx}\|^2 + C_2(T)\varepsilon^{-1}\|\theta_{tt}\|^2, \qquad (3.37) \\
P_5 &\leq C_2(T)\|v_x\|_{L^\infty}\|\theta_{tt}\|\big(\big(\|v_x\|_{L^\infty} + \|\theta_t\|_{L^\infty}\big)\big(\|v_x\| + \|\theta_t\|\big) + \|v_{xt}\|
\end{aligned}$$

$$\begin{aligned}
&+ \|\theta_{tt}\| + \|v_{xtt}\| + \|v_{tt}\| + \|v_x\|\big) \\
&\le C_2(T)\|\theta_{tt}\|\big(\|v_x\|_{H^1} + \|\theta_t\| + \|\theta_{xt}\| + \|v_{xt}\| + \|\theta_{tt}\| + \|v_{xtt}\| + \|v_{tt}\|\big) \\
&\le \varepsilon\|v_{ttx}\|^2 + C_2(T)\varepsilon^{-1}\big(\|\theta_{tt}\|^2 + \|v_x\|_{H^1}^2 + \|\theta_t\|^2 + \|\theta_{xt}\|^2 + \|v_{xt}\|^2\big), \qquad (3.38)
\end{aligned}$$

$$\begin{aligned}
P_6 &\le C_1\int_0^M \big(|v_x| + |\theta_t| + |v_{xt}| + |v_x|^2 + |v_t|\big)\big(|v_{xt}| + |v_t|\big)|\theta_{tt}|\,dx \\
&\le C_2(T)\|v_{tx}\|^{\frac{1}{2}}\|v_{txx}\|^{\frac{1}{2}}\big(\|v_x\| + \|\theta_t\| + \|v_{xt}\|\big)\|\theta_{tt}\|, \qquad (3.39)
\end{aligned}$$

which, by Hölder's inequality, implies

$$\begin{aligned}
\int_0^t P_6\,ds &\le C_2(T)\sup_{0\le s\le t}\big\|\theta_{tt}(s)\big\|\left(\int_0^t \big\|v_{txx}(s)\big\|^2\,ds\right)^{\frac{1}{4}}\left(\int_0^t \big\|v_{tx}(s)\big\|^2\,ds\right)^{\frac{1}{4}} \\
&\quad\times\left(\int_0^t \big(\|v_x\|^2 + \|\theta_t\|^2 + \|v_{tx}\|^2\big)(s)\,ds\right)^{\frac{1}{2}} \\
&\le \varepsilon\left(\sup_{0\le s\le t}\big\|\theta_{tt}(s)\big\|^2 + \int_0^t \big\|vtxx(s)\big\|^2\,ds\right) + C_2(T)\varepsilon^{-3}. \qquad (3.40)
\end{aligned}$$

Thus it follows from (3.33)-(3.40) that, for any $\varepsilon \in (0,1)$ small enough,

$$\begin{aligned}
&\big\|\theta_{tt}(t)\big\|^2 + \int_0^t \big\|\theta_{ttx}(s)\big\|^2\,ds \\
&\quad\le C_4(T)\varepsilon^{-3} + C_2(T)\varepsilon^{-1}\int_0^t \big\|\theta_{tt}(s)\big\|^2\,ds \\
&\qquad + C_1\varepsilon\left(\sup_{0\le s\le t}\big\|\theta_{tt}(s)\big\|^2 + \int_0^t \big(\|v_{txx}\|^2 + \|v_{ttx}\|^2\big)(s)\,ds\right). \qquad (3.41)
\end{aligned}$$

Therefore taking the supremum in t on the left-hand side of (3.41) and choosing $\varepsilon \in (0,1)$ small enough, we can derive estimate (3.19) from (3.30). The proof is complete. □

Lemma 3.9 *Under the assumptions of Theorem* 2.3, *the following estimates hold for any* $t \in [0,T]$ *and for* $\varepsilon > 0$ *small enough*:

$$\big\|v_{xt}(t)\big\|^2 + \int_0^t \big\|v_{xxt}(s)\big\|^2\,ds \le C_4(T) + C_2(T)\varepsilon^2\int_0^t \big(\|v_{xtt}\|^2 + \|\theta_{xxt}\|^2\big)(s)\,ds, \qquad (3.42)$$

$$\big\|\theta_{xt}(t)\big\|^2 + \int_0^t \big\|\theta_{xxt}(s)\big\|^2\,ds \le C_4(T) + C_2(T)\varepsilon^2\int_0^t \big(\|v_{xxt}\|^2 + \|\theta_{xtt}\|^2\big)(s)\,ds. \qquad (3.43)$$

Proof Differentiating (1.9) with respect to x and t, multiplying the result by v_{xt} and integrating by parts in $L^2(0,M)$, we have

$$\frac{1}{2}\frac{d}{dt}\|v_{xt}\|^2 = N_0(t) + N_1(t) \qquad (3.44)$$

with

$$N_0(t) = \left(r^2\left(\mu\frac{(r^2v)_x}{v} - p\right)_x\right)_t v_{xt}\Big|_{x=0}^{x=L}, \qquad N_1(t) = -\int_0^M \left(r^2\left(\mu\frac{(r^2v)_x}{v} - P\right)_x\right)_t v_{xxt}\,dx.$$

Using Theorem 2.2 and Lemma 3.8, the interpolation inequality, and Poincaré's inequality, we can get

$$\begin{aligned} N_0(t) \le{}& C_1\big(\big(\|v_x\|_{L^\infty} + \|\theta_x\|_{L^\infty}\big)\big(\|v_x\|_{L^\infty} + \|\theta_x\|_{L^\infty} + \|\eta_x\|_{L^\infty}\big) + \|v_{xx}\|_{L^\infty} + \|\theta_{xt}\|_{L^\infty} \\ & + \|v_{xxt}\|_{L^\infty} + \|\eta_x\|_{L^\infty}\|v_{xt}\|_{L^\infty} + \|v_x\|_{L^\infty}\|v_{xx}\|_{L^\infty} + \big\|v_x^2\big\|_{L^\infty} + \|\eta_{xt}\|_{L^\infty} \\ & + \|\eta_x\|_{L^\infty}\|\theta_t\|_{L^\infty} + \|v_x\|_{L^\infty}\|\theta_x\|_{L^\infty} \\ & + \|\theta_x\|_{L^\infty}\|\theta_t\|_{L^\infty} + \|v_x\|_{L^\infty}\|\eta_x\|_{L^\infty}\big)\|v_{xt}\|_{L^\infty} \\ \le{}& C_2(T)(N_{01} + N_{02})\|v_{xt}\|^{\frac{1}{2}}\|v_{xxt}\|^{\frac{1}{2}}, \end{aligned} \tag{3.45}$$

where

$$N_{01} = \|v_x\|_{H^2} + \|\theta_t\| + \|\theta_{xt}\|$$

and

$$N_{02} = \|\theta_{xt}\|^{\frac{1}{2}}\|\theta_{xxt}\|^{\frac{1}{2}} + \|v_{xxt}\|^{\frac{1}{2}}\|v_{xxxt}\|^{\frac{1}{2}} + \|v_{xxt}\| + \|v_{xt}\|^{\frac{1}{2}}\|v_{xxt}\|^{\frac{1}{2}}.$$

Applying Young's inequality several times, we have, for any $\varepsilon \in (0,1)$,

$$\begin{aligned} C_2(T)N_{01}\|v_{xt}\|^{\frac{1}{2}}\|v_{xxt}\|^{\frac{1}{2}} \le{}& \frac{\varepsilon^2}{2}\|v_{xxt}\|w^2 \\ & + C_2(T)\varepsilon^{-1}\big(\|v_x\|_{H^2}^2 + \|\theta_t\|_{H^1}^2 + \|v_{xt}\|^2\big) \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} C_2(T)N_{02}\|v_{xt}\|^{\frac{1}{2}}\|v_{xxt}\|^{\frac{1}{2}} \le{}& \frac{\varepsilon^2}{2}\|v_{xxt}\|^2 + \varepsilon^2\big(\|\theta_{txx}\|^2 + \|v_{xxxt}\|^2\big) \\ & + C_2(T)\varepsilon^{-6}\big(\|\theta_{tx}\|^2 + \|v_{xt}\|^2\big). \end{aligned} \tag{3.47}$$

Thus it follows from (3.45)-(3.47) and Theorem 2.1 and Lemma 3.8 that

$$\begin{aligned} N_0(t) \le{}& \varepsilon^2\big(\|v_{xxt}\|^2 + \|\theta_{txx}\|^2 + \|v_{xxxt}\|^2\big) \\ & + C_2(T)\varepsilon^{-6}\big(\|\theta_x\|^2 + \|v_x\|_{H^2}^2 + \|\theta_{tx}\|^2 + \|v_{xt}\|^2\big), \end{aligned} \tag{3.48}$$

which, along with Theorem 2.2, further yields

$$\int_0^t N_0(s)\,ds \le \varepsilon^2\int_0^t\big(\|v_{xxt}\|^2 + \|\theta_{txx}\|^2 + \|v_{xxxt}\|^2\big)(s)\,ds + C_2(T)\varepsilon^{-6}. \tag{3.49}$$

Analogously, from Lemma 3.8, Theorem 2.1, and the embedding theorem, we can also derive that, for any $\varepsilon \in (0,1)$,

$$\begin{aligned} N_1(t) \le{}& -\int_0^M \mu\frac{r^4}{\eta}v_{txx}^2\,dx \\ & + C_1\big(\big(\|v_x\| + \|\theta_t\| + \|\eta_x\|\big)\big(\|\theta_x\|_{L^\infty} + \|\eta_x\|_{L^\infty}\big) + \|v_{xx}\| + \|\theta_{xt}\| \end{aligned}$$

$$+ \|\eta_x\|_{L^\infty}\|v_{xt}\| + \|v_x\|_{L^\infty}\|v_{xx}\| + \|v_x\|_{L^\infty}^2\|\eta_x\| + \|v_x\| + \|\theta_x\| + \|v_x\|^2\big)\|v_{xxt}\|$$
$$\le -(2C_1)^{-1}\|v_{xxt}\|^2 + C_2(T)\big(\|v_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|v_{xt}\|^2 + \|\eta_x\|^2\big), \tag{3.50}$$

which, combined with (3.44), (3.49), and Theorem 2.2, shows that, for any $\varepsilon \in (0,1)$ small enough,

$$\begin{aligned} &\big\|v_{xt}(t)\big\|^2 + \int_0^t \big\|v_{xxt}(s)\big\|^2\,ds \\ &\quad \le C_2(T)\varepsilon^{-6} + C_1\varepsilon^2\int_0^t \big(\|\theta_{txx}\|^2 + \|v_{xxxt}\|^2\big)(s)\,ds. \end{aligned} \tag{3.51}$$

On the other hand, differentiating (1.9) with respect to x and t, we can derive from Theorem 2.2 and Lemma 3.8 that

$$\begin{aligned} \big\|v_{xxxt}(t)\big\| \le{}& C_1\big\|v_{xtt}(t)\big\| + C_2(T)\big(\big\|v_x(t)\big\|_{H^2} + \big\|\theta_x(t)\big\|_{H^1} \\ &+ \big\|\eta_x(t)\big\|_{H^1} + \big\|\theta_t(t)\big\|_{H^2}\big). \end{aligned} \tag{3.52}$$

Thus inserting (3.52) into (3.51) leads to (3.42).

Similarly, by (1.10), we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L e_\theta\theta_{tx}^2\,dx =: \sum_{i=0}^3 L_i(t), \tag{3.53}$$

where

$$\begin{aligned} &L_0(t) = \left(\frac{r^4\kappa\theta_x}{\eta}\right)_{xt}\theta_{xt}\bigg|_{x=0}^{x=M}, \qquad L_1(t) = -\int_0^M \left(\frac{r^4\kappa\theta_x}{\eta}\right)_{tx}\theta_{txx}\,dx, \\ &L_2(t) = -\int_0^M \left(\left(e_\eta + p - \mu\frac{(r^2v)_x}{\eta}\right)\big(r^2v\big)_x\right)_{xt}\theta_{tx}\,dx, \\ &L_3(t) = -\int_0^M \left(e_{\theta tx}\theta_t + \frac{1}{2}e_{\theta t}\theta_{tx} + e_{\theta x}\theta_{tt}\right)\theta_{tx}\,dx. \end{aligned}$$

By virtue of the embedding theorem and the Young inequality, we derive from Lemmas 3.1, 3.8, and (3.42) that, for any $\varepsilon \in (0,1)$,

$$\begin{aligned} L_0(t) \le{}& C_2(T)\big(\|v_x\|_{H^2} + \|\theta_x\|_{H^2} + \|\theta_t\|_{H^2} + \|\theta_{xt}\|^{\frac{1}{2}}\|\theta_{xxt}\|^{\frac{1}{2}} \\ &+ \|\theta_{xxt}\|^{\frac{1}{2}}\|\theta_{xxxt}\|^{\frac{1}{2}}\big)\|\theta_{xt}\|^{\frac{1}{2}}\|\theta_{xxt}\|^{\frac{1}{2}} \\ \le{}& \varepsilon^2\big(\|\theta_{txx}\|^2 + \|\theta_{txxx}\|^2\big) + C_2(T)\varepsilon^{-6}\big(\|v_x\|_{H^2}^2 + \|\theta_x\|_{H^2}^2 + \|\theta_{xt}\|^2\big), \end{aligned} \tag{3.54}$$

$$L_1(t) \le -(2C_1)^{-1}\|\theta_{txx}\|^2 + C_2(T)\big(\|v_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2\big), \tag{3.55}$$

$$L_2(t) \le \varepsilon^2\|v_{txx}\|^2 + C_2(T)\varepsilon^{-2}\big(\|v_x\|_{H^2}^2 + \|\theta_t\|_{H^1}^2 + \|v_{xt}\|^2 + \|\eta_x\|_{H^1}^2\big), \tag{3.56}$$

$$L_3(t) \le \varepsilon^2\|\theta_{txx}\|^2 + C_2(T)\varepsilon^{-2}\big(\|v_x\|w_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|\theta_x\|_{H^2}^2 + \|v_{xt}\|^2 + \|\eta_x\|^2\big). \tag{3.57}$$

Differentiating (1.10) with respect to x and t, we can derive from Theorems 2.1-2.2 and Lemma 3.8 that

$$\begin{aligned}\|\theta_{txxx}(t)\| &\le C_1\big(\|\theta_{ttx}(t)\| + \|v_{xxt}(t)\|\big)\\ &\quad + C_2(T)\big(\|v_x(t)\|_{H^2} + \|\eta_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_{xt}(t)\|\big).\end{aligned} \tag{3.58}$$

Inserting (3.54)-(3.58) into (3.53) yields (3.43). □

Lemma 3.10 *Under the assumptions of Theorem* 2.3, *we have, for any* $t \in [0, T]$,

$$\begin{aligned}&\|v_{tt}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2\\ &\quad + \int_0^t \big(\|v_{ttx}\|^2 + \|v_{xxt}\|^2 + \|\theta_{ttx}\|^2 + \|\theta_{xxt}\|^2\big)(s)\,ds \le C_4(T),\end{aligned} \tag{3.59}$$

$$\begin{aligned}&\|\eta_{xxx}(t)\|_{H^1}^2 + \|v_{xxx}(t)\|_{H^1}^2 + \|\theta_{xxx}(t)\|_{H^1}^2 + \|v_{txx}(t)\|^2 + \|\theta_{txx}(t)\|^2\\ &\quad + \int_0^t \big(\|v_{tt}\|^2 + \|v_{xxt}\|_{H^1}^2 + \|\theta_{tt}\|^2 + \|\theta_{xxt}\|_{H^1}^2\big)(s)\,ds \le C_4(T),\end{aligned} \tag{3.60}$$

$$\int_0^t \big(\|\eta_{xxx}\|_{H^1}^2 + \|v_{xxxx}\|_{H^1}^2 + \|\theta_{xxxx}\|_{H^1}^2\big)(s)\,ds \le C_4(T). \tag{3.61}$$

Proof Adding (3.42)-(3.43) and choosing $\varepsilon > 0$ small enough, we get

$$\begin{aligned}&\|v_{xt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \int_0^t \big(\|v_{xxt}\|^2 + \|\theta_{xxt}\|^2\big)(s)\,ds\\ &\quad \le C_4(T) + C_2(T)\varepsilon^2 \int_0^t \big(\|v_{xtt}\|^2 + \|\theta_{xtt}\|^2\big)(s)\,ds.\end{aligned} \tag{3.62}$$

Now multiplying (3.18) and (3.19) by ε and $\varepsilon^{\frac{3}{2}}$, respectively, then adding the results to (3.62) and taking ε sufficiently small, we obtain (3.59).

Differentiating (1.9) with respect to x and noting that $\eta_{xxt} = (r^2 v)_{xxx}$, we get

$$\mu \frac{\partial}{\partial t}\left(\frac{\eta_{xx}}{\eta}\right) - p_\eta \eta_{xx} = r^{-2} v_{tx} + K(x,t) - \big(r^{-2} f\big)_x - 2r^{-5}\eta v_t, \tag{3.63}$$

where

$$\begin{aligned}K(x,t) &= p_{\eta\eta}\eta_x^2 + 2p_{\eta\theta}\theta_x\eta_x + p_{\theta\theta}\theta_x^2 + p_\theta\theta_{xx} - 2\mu\frac{\eta_x^2}{\eta^3}\big(r^2 v\big)_x + 2\mu\frac{\eta_x}{\eta^2}\big(r^2 v\big)_{xx}\\ &= \frac{A(\beta-2)(\beta-3)}{2}\theta^2\eta^{\beta-4}\eta_x^2 + 2A(\beta-2)\theta\eta^{\beta-3}\theta_x\eta_x + A\eta^{\beta-2}\theta_x^2\\ &\quad + A\theta\eta^{\beta-2}\theta_{xx} + 2\mu\left(\frac{\eta_x}{\eta^2}\big(r^2 v\big)_{xx} - \frac{\eta_x^2}{\eta^3}\big(r^2 v\big)_x\right).\end{aligned}$$

Differentiating (3.63) with respect to x, we have

$$\mu \frac{\partial}{\partial t}\left(\frac{\eta_{xxx}}{\eta}\right) - p_\eta \eta_{xxx} = K_1(x,t), \tag{3.64}$$

where

$$K_1(x,t) = K_x(x,t) + p_{\eta x}\eta_{xx} + \mu\left(\frac{\eta_{xx}\eta_x}{\eta^2}\right)_t + r^{-2}v_{txx} - 4r^{-5}\eta v_{tx}$$
$$+ 10r^{-8}\eta^2 v_t - 2r^{-5}\eta_x v_t - \left(r^{-2}f\right)_{xx}.$$

Obviously, it follows from Theorem 2.1 and Lemmas 3.8-3.9 that

$$\left\|K_1(t)\right\| \le C_2(T)\left(\left\|\eta_x(t)\right\|_{H^1} + \left\|v_x(t)\right\|_{H^2} + \left\|\theta_x(t)\right\|_{H^2} + \left\|v_{txx}(t)\right\|\right) \tag{3.65}$$

and

$$\int_0^t \left\|K_1(s)\right\|^2 ds \le C_4(T). \tag{3.66}$$

Multiplying (3.64) by $\frac{\eta_{xxx}}{\eta}$ over $L^2(0,M)$, we can obtain

$$\frac{d}{dt}\left\|\frac{\eta_{xxx}}{\eta}\right\|^2 + C_1^{-1}\left\|\frac{\eta_{xxx}}{\eta}\right\|^2 \le C_1\left\|K_1(t)\right\|^2, \tag{3.67}$$

which, along with (3.66), gives

$$\left\|\eta_{xxx}(t)\right\|^2 + \int_0^t \left\|\eta_{xxx}(s)\right\|^2 ds \le C_4(T). \tag{3.68}$$

It follows from (1.8)-(1.10) that

$$\left\|v_{xxx}(t)\right\| \le C_2(T)\left(\left\|v(t)\right\|_{H^2} + \left\|\eta_x(t)\right\|_{H^1} + \left\|\theta_x(t)\right\|_{H^1} + \left\|v_{xt}(t)\right\|\right), \tag{3.69}$$
$$\left\|\theta_{xxx}(t)\right\| \le C_2(T)\left(\left\|\theta(t)\right\|_{H^2} + \left\|\eta_x(t)\right\|_{H^1} + \left\|v_x(t)\right\|_{H^1} + \left\|\theta_{xt}(t)\right\|\right). \tag{3.70}$$

Using the embedding theorem, Theorems 2.1-2.2 and Lemmas 3.8-3.9, we can derive from (3.24)-(3.25), (3.59), and (3.68)-(3.70) that, for any $t \in [0,T]$,

$$\begin{aligned}&\left\|v_{xxx}(t)\right\|^2 + \left\|\theta_{xxx}(t)\right\|^2 + \left\|v_{xx}(t)\right\|_{L^\infty}^2 + \left\|\theta_{xx}(t)\right\|_{L^\infty}^2 \\ &\quad + \int_0^t \left(\|v_{xxx}\|_{H^1}^2 + \|\theta_{xxx}\|_{H^1}^2 + \|v_{xx}\|_{W^{1,\infty}}^2 + \|\theta_{xx}\|_{W^{1,\infty}}^2\right)(s)\,ds \le C_4(T).\end{aligned} \tag{3.71}$$

Differentiating (1.9)-(1.10) with respect to t and using Theorems 2.1-2.2 and Lemmas 3.8-3.9, we can deduce from (3.59), (3.68)-(3.71) that

$$\begin{aligned}\left\|v_{txx}(t)\right\| &\le C_1\left\|v_{tt}(t)\right\| + C_2(T)\left(\left\|v_x(t)\right\|_{H^1} + \left\|\eta_x(t)\right\| + \left\|\theta_x(t)\right\|\right. \\ &\quad \left. + \left\|\theta_t(t)\right\| + \left\|\theta_{xt}(t)\right\| + \left\|v_{xt}(t)\right\|\right) \le C_4(T),\end{aligned} \tag{3.72}$$
$$\begin{aligned}\left\|\theta_{txx}(t)\right\| &\le C_1\left\|\theta_{tt}(t)\right\| + C_2(T)\left(\left\|v_x(t)\right\|_{H^1} + \left\|\eta_x(t)\right\| + \left\|\theta_x(t)\right\|_{H^2}\right. \\ &\quad \left. + \left\|\theta_t(t)\right\|_{H^1} + \left\|v_{xt}(t)\right\|\right) \le C_4(T),\end{aligned} \tag{3.73}$$

which, combined with (3.24)-(3.25) and (3.72), implies

$$\begin{aligned} &\|v_{xxxx}(t)\|^2 + \|\theta_{xxxx}(t)\|^2 \\ &\quad + \int_0^t \left(\|v_{txx}\|^2 + \|\theta_{txx}\|^2 + \|v_{xxxxx}\|^2 + \|\theta_{xxxxx}\|^2\right)(s)\,ds \le C_4(T). \end{aligned} \tag{3.74}$$

Therefore it follows from (3.71), (3.74), and the embedding theorem that

$$\|v_{xxx}(t)\|_{L^\infty}^2 + \|\theta_{xxx}(t)\|_{L^\infty}^2 + \int_0^t \left(\|v_{xxxx}\|_{L^\infty}^2 + \|\theta_{xxxx}\|_{L^\infty}^2\right)(s)\,ds \le C_4(T). \tag{3.75}$$

Now differentiating (3.64) with respect to x, we find

$$\epsilon \frac{\partial}{\partial t}\left(\frac{\eta_{xxxx}}{\eta}\right) - p_\eta \eta_{xxxx} = K_2(x,t), \tag{3.76}$$

where

$$K_2(x,t) = K_{1x}(x,t) + p_{\eta x}\eta_{xxx} + \mu\left(\frac{\eta_{xxx}\eta_x}{\eta^2}\right)_t.$$

From the embedding theorem and Lemmas 3.8-3.9 and (3.68)-(3.75), we can derive

$$\begin{aligned} \|K_{xx}(t)\| &\le C_4(T)\left(\|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2}\right), \\ \|K_{1x}(t)\| &\le C_1\Bigg(\|K_{xx}(t)\| + \|v_{xxxt}\| + \|v_{xxt}\| + \|(p_{\eta x}\eta_{xx})_x\| + \|\eta_x v_{xt}\| \\ &\qquad + \|\eta_{xx}\| + \|\eta_x v_t\| + \|\eta_{xx} v_t\| + \left\|\left(\frac{\eta_x\eta_{xx}}{\eta^2}\right)_{xt}\right\|\Bigg) \\ &\le C_1\|v_{xxxt}(t)\| + C_4(T)\left(\|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2}\right), \end{aligned}$$

whence

$$\|K_2(t)\| \le C_1\|v_{xxxt}(t)\| + C_4(T)\left(\|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2}\right). \tag{3.77}$$

It follows from (3.28)-(3.31) that

$$\int_0^t \left(\|v_{tt}\|^2 + \|\theta_{tt}\|^2\right)(s)\,ds \le C_4(T), \tag{3.78}$$

which, along with (3.52) and (3.59), gives

$$\int_0^t \|v_{xxxt}(s)\|^2\,ds \le C_4(T). \tag{3.79}$$

Thus from (3.68), (3.74), (3.77), and (3.79), it follows that

$$\int_0^t \|K_2(s)\|^2\,ds \le C_4(T). \tag{3.80}$$

Multiplying (3.76) by $\frac{\eta_{xxxx}}{\eta}$ in $L^2(0,M)$, we can get

$$\frac{d}{dt}\left\|\frac{\eta_{xxxx}}{\eta}\right\|^2 + C_1^{-1}\left\|\frac{\eta_{xxxx}}{\eta}\right\|^2 \le C_1\left\|K_2(t)\right\|^2, \tag{3.81}$$

whence, by (3.80),

$$\left\|\eta_{xxxx}(t)\right\|^2 + \int_0^t \left\|\eta_{xxxx}(s)\right\|^2 ds \le C_4(T). \tag{3.82}$$

Differentiating (1.10) with respect to x and t, we can derive from Theorems 2.1-2.2 and Lemmas 3.8-3.9 that

$$\left\|\theta_{txxx}(t)\right\| \le C_1\left\|\theta_{ttx}(t)\right\| + C_2(T)\left(\left\|v_x(t)\right\|_{H^3} + \left\|\eta_x(t)\right\|_{H^2} + \left\|\theta_x(t)\right\|_{H^3} + \left\|\theta_{xt}(t)\right\|\right). \tag{3.83}$$

Thus,

$$\int_0^t \left\|\theta_{txxx}(s)\right\|^2 ds \le C_4(T). \tag{3.84}$$

Differentiating (1.9) with respect to x three times, applying Lemmas 3.8-3.9, Theorems 2.1-2.2, and Poincaré's inequality, we have

$$\left\|v_{xxxxx}(t)\right\| \le C_1\left\|v_{txxx}(t)\right\| + C_2(T)\left(\left\|v_x(t)\right\|_{H^3} + \left\|\eta_x(t)\right\|_{H^3} + \left\|\theta_x(t)\right\|_{H^3}\right). \tag{3.85}$$

Thus it follows from (3.74), (3.79), and (3.82) that

$$\int_0^t \left\|v_{xxxxx}(s)\right\|^2 ds \le C_4(T). \tag{3.86}$$

Similarly, we can differentiate (1.10) with respect to x three times and use Lemmas 3.8-3.9, Theorems 2.1-2.2, Poincaré's inequality, (3.74), (3.82), and (3.84) to find

$$\int_0^t \left\|\theta_{xxxxx}(s)\right\|^2 ds \le C_4(T). \tag{3.87}$$

Hence, (3.60)-(3.61) follow from (3.74), (3.82), (3.86), and (3.87). □

Finally, combining Lemmas 3.8-3.10, we complete the proof of Theorem 2.3.

Competing interests
The author declares that they have no competing interests.

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