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The global existence of a parabolic cross-diffusion system with discontinuous coefficients

Qi-Jian Tan*, Juan Su and Chao-Yi Pan

*Correspondence:
tanqjxxx@163.com
College of Mathematics, Chengdu Normal University, Chengdu, 611130, P.R. China

Abstract

This paper deals with a one-dimensional parabolic cross-diffusion system describing two-species models. The system under consideration is strongly coupled and the coefficients of the equations are allowed to be discontinuous. The existence and uniqueness of nonnegative global solutions are given when one of the cross-diffusion pressures is zero. Our approach to the problem is by an approximation method and various estimates.

MSC: 35K55; 35R05; 35K57

Keywords: parabolic systems; cross-diffusion; discontinuous coefficients; approximation method

1 Introduction

In the previous work [1], we presented a parabolic cross-diffusion system describing two-species models on a bounded domain with different natural conditions, and studied the corresponding steady-state problem, an elliptic cross-diffusion system. In this paper, we deal with this parabolic cross-diffusion system for the one-dimensional case under the homogeneous Neumann boundary conditions.

Assume that $(0,l)$ is partitioned into two intervals $I^{(1)} := (0,l_0)$ and $I^{(2)} := (l_0,l)$ separated by $x = l_0$, and the natural conditions of $I^{(i)}$ are different. For any $T > 0$, set

$$Q_T := (0,l) \times (0,T],$$
$$Q_T^{(i)} := I^{(i)} \times (0,T], \quad i = 1,2,$$
$$S_T := \{(x,t) : x = 0 \text{ or } l, t \in [0,T]\},$$
$$\Gamma_T := \{(x,t) : x = l_0, t \in [0,T]\}.$$

Let $u = u(x,t)$, $v = v(x,t)$ be the population densities of the two species. Assume that the species share the same habitat, the interval $[0,l]$, and that for each species, the density and the flux of the density are all continuous across the inner boundary Γ_T. According to [1], when the cross-diffusion pressure for the second species is zero, u, v are governed by the parabolic cross-diffusion system

$$\begin{cases} u_t = [k_1(x)((1+\beta u+\gamma v)u)_x]_x + uf_1(x,u,v) \quad ((x,t)\in Q_T^{(i)}), i=1,2, \\ v_t = [k_2(x)((1+\delta v)v)_x]_x + vf_2(x,u,v) \quad ((x,t)\in Q_T^{(i)}), i=1,2, \\ u^-(l_0,t) = u^+(l_0,t), \qquad v^-(l_0,t) = v^+(l_0,t) \quad (t\in[0,T]), \\ [k_1(x)((1+\beta u+\gamma v)u)_x]^-(l_0,t) = [k_1(x)((1+\beta u+\gamma v)u)_x]^+(l_0,t) \quad (t\in(0,T]), \\ [k_2(x)((1+\delta v)v)_x]^-(l_0,t) = [k_2(x)((1+\delta v)v)_x]^+(l_0,t) \quad (t\in(0,T]), \\ u_x = v_x = 0 \quad ((x,t)\in S_T), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \quad (x\in(0,l)), \end{cases} \tag{1.1}$$

where $u_t := \partial u/\partial t$, $u_x := \partial u/\partial x$, $u^-(l_0,t)$ and $u^+(l_0,t)$ represent the limits of u from left and right for the space variable, respectively,

$$f_j(x,u,v) := \begin{cases} f_{j,1}(u,v) & (x\in I^{(1)}), \\ f_{j,2}(u,v) & (x\in I^{(2)}), \end{cases} \qquad k_j(x) := \begin{cases} k_{j,1} & (x\in I^{(1)}), \\ k_{j,2} & (x\in I^{(2)}), \end{cases}$$

$k_{j,1}$ and $k_{j,2}$ $(j=1,2)$ are positive constants, and β, γ and δ are nonnegative constants. In (1.1), $k_1(x)$ and $k_2(x)$ are the diffusion rates, $\beta k_1(x)$ and $\delta k_2(x)$ are the self-diffusion rates, and $\gamma k_1(x)$ is cross-diffusion rate (see [1]).

In 1979 Shigesada *et al.* [2] proposed a mathematical model with cross-diffusion to investigate the spatial segregation phenomena of two competing species under inter- and intra-species population pressures. It is a strongly coupled quasilinear parabolic system. Over the past 36 years, quasilinear elliptic and parabolic systems with cross-diffusion have been treated extensively in the literature both in theory and in applications, and most of the treatments are for systems with continuous coefficients (see [3–16] and the references therein). In [1], by Schauder's fixed point theorem, we discussed the existence of nonnegative solutions for the elliptic system with cross-diffusion and discontinuous coefficients. In this paper, problem (1.1) is a strongly coupled parabolic system with cross-diffusion. In addition, the diffusion rates, self-diffusion rates, cross-diffusion rates and reaction functions are all allowed to be discontinuous. Based on the result of Lou *et al.* [6] for a parabolic system with continuous coefficients, we shall use approximation method and various estimates to show the existence and uniqueness of global solutions for problem (1.1).

The paper is organized as follows. In the next section we introduce the notations, hypotheses, and main result. In Section 3 we construct an approximation problem of (1.1) and establish the uniform estimates of solutions for the approximation problem. Section 4 is devoted to the existence and uniqueness of solutions for problem (1.1).

2 The notations, hypotheses, and main result

For any set S, $\bar{S}$ denotes the closure of S. The symbol $I' \subset\subset I$ means that I and I' are open intervals and $\bar{I}'$ is a subset of I. Denote

$$\|u\|_{L^{s,r}(Q_T)} := \left[\int_0^T \left(\int_0^l \left|u(x,t)\right|^s \mathrm{d}x\right)^{r/s} \mathrm{d}t\right]^{1/r}, \qquad \|u\|_{L^s(Q_T)} := \|u\|_{L^{s,s}(Q_T)},$$

where $s, r \geq 1$. $W_2^{1,0}(Q_T)$ and $W_2^{1,1}(Q_T)$ are the Hilbert spaces with scalar products

$$(u,v)_{W_2^{1,0}(Q_T)} = \iint_{Q_T} (uv + u_x v_x)\,\mathrm{d}x\,\mathrm{d}t, \qquad (u,v)_{W_2^{1,1}(Q_T)} = \iint_{Q_T} (uv + u_t v_t + u_x v_x)\,\mathrm{d}x\,\mathrm{d}t,$$

respectively. $V_2(Q_T)$ is the Banach space consisting of all elements of $W_2^{1,0}(Q_T)$ having a finite norm

$$\|u\|_{V_2(Q_T)} := \sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^2(0,l)} + \|u_x\|_{L^2(Q_T)}.$$

We let

$$W_r^1(0,l) := \{u = u(x) : u, u_x \in L^r(0,l)\}$$

equipped with the norm

$$\|u\|_{W_r^1(0,l)} := \|u\|_{L^r(0,l)} + \|u_x\|_{L^r(0,l)}.$$

Definition 2.1 A nonnegative vector function (u,v) is called a solution of (1.1) if (u,v) possesses the following properties:

(i) For any $T > 0$, $u, v \in W^{1,1}(Q_T) \cap C^{2,1}(Q_T^{(i)})$, $i = 1, 2$. For any $I' \subset\subset (0,l)$, there exists $\alpha' \in (0,1)$, such that $u, v \in C^{\alpha'}(\bar{I}' \times [0,T])$. For any fixed $i \in \{1,2\}$ and for any $I'' \subset\subset I^{(i)}$, $\tau \in (0,T)$, there exists $\alpha'' \in (0,1)$, such that $u, v \in C^{2+\alpha'',1+\alpha''/2}(\bar{I}'' \times [\tau,T])$.

(ii) For any $I' \subset\subset (0,l)$ and $\tau \in (0,T)$, there exists $\bar{\alpha}' \in (0,1)$, such that $u_t, v_t \in C^{\bar{\alpha}'}(\bar{I}' \times [\tau,T])$ and $u_x, v_x \in C^{\bar{\alpha}'}((\bar{I}' \cap \bar{I}^{(i)}) \times [\tau,T])$, $i = 1, 2$.

(iii) (u,v) satisfies pointwise the equations in (1.1) on $Q_T^{(i)}$ ($i = 1,2$), the initial conditions on $\{(x,t) : x \in (0,l), t = 0\}$ and the inner boundary conditions on Γ_T, and satisfies the homogeneous Neumann boundary conditions on S_T for almost all t.

We see from property (ii) in Definition 2.1 that if (u,v) is a solution of (1.1), then u_x, v_x are Hölder continuous up to the inner boundary Γ_T, and u_t, v_t are Hölder continuous across Γ_T.

Throughout the paper we make the following hypotheses on the various functions in (1.1):

(H) For each $j, i = 1, 2$, $f_{j,i}(u,v) \in C^1(\mathbb{R}_+^2)$, where $\mathbb{R}_+ := [0,+\infty)$. There exist positive constants $\underline{d}_1$, $\bar{d}_1$, $\underline{e}_1$, $\underline{d}_2$, $\underline{e}_2$, $\bar{e}_2$, such that, for any $u, v \geq 0$,

$$\begin{cases} -\underline{d}_1 \leq \frac{\partial f_{1,i}(u,v)}{\partial u} \leq -\bar{d}_1, & -\underline{e}_1 \leq \frac{\partial f_{1,i}}{\partial v} \leq 0, \\ -\underline{d}_2 \leq \frac{\partial f_{2,i}(u,v)}{\partial u} \leq 0, & -\underline{e}_2 \leq \frac{\partial f_{2,i}}{\partial v} \leq -\bar{e}_2. \end{cases} \tag{2.1}$$

Assume that $u_0 = u_0(x)$, $v_0 = v_0(x)$ possess the properties

$$\begin{cases} u_0(x), v_0(x) \in W_{r_0}^1(0,l) \cap C^1((0,l_0]) \cap C^1([l_0,l)) \quad \text{for some } r_0 > 2, \\ u_0(x), v_0(x) \geq 0 \quad \text{for } x \in [0,l], \end{cases} \tag{2.2}$$

and satisfy the compatibility conditions

$$\begin{cases} [k_1(x)((1+\beta u_0+\gamma v_0)u_0)_x]^-(l_0) = [k_1(x)((1+\beta u_0+\gamma v_0)u_0)_x]^+(l_0), \\ [k_2(x)((1+\delta v_0)v_0)_x]^-(l_0) = [k_2(x)((1+\delta v_0)v_0)_x]^+(l_0). \end{cases} \tag{2.3}$$

From (2.2) it follows that $u_0(x), v_0(x) \in C^{1/2}([0,l])$.

The main result of this paper is the following theorem.

Theorem 2.1 *Let hypothesis* (H) *hold. Then problem* (1.1) *has a unique solution* (u,v).

Since T is an arbitrary positive number, the solution (u,v) given by Theorem 2.1 is global.

3 An approximation problem of (1.1)

In order to show the existence and uniqueness of solutions for problem (1.1), in this section we shall construct an approximation problem and establish the uniform estimates of its solutions.

3.1 The global solution for the approximation problem

To construct an approximation problem of (1.1), we first construct some approximation functions. For an arbitrary $\varepsilon > 0$, let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ a smooth function with values between 0 and 1, such that $\zeta_\varepsilon = 1$ for $x \le l_0$, $\zeta_\varepsilon = 0$ for $x \ge l_0 + \varepsilon$ and $|\zeta_{\varepsilon x}| \le C/\varepsilon$ for all $x \in \mathbb{R}$. For each $j = 1, 2$, define

$$\begin{cases} k_{j\varepsilon}(x) := \zeta_\varepsilon(x)k_{j,1} + (1-\zeta_\varepsilon(x))k_{j,2} \quad (x \in [0,l]), \\ f_{j\varepsilon}(x,u,v) := \zeta_\varepsilon(x)f_{j,1}(u,v) + (1-\zeta_\varepsilon(x))f_{j,2}(u,v) \quad ((x,u,v) \in [0,l] \times \mathbb{R}_+^2). \end{cases} \tag{3.1}$$

Then (2.1) in hypothesis (H) implies that, for all $(x,u,v) \in [0,l] \times \mathbb{R}_+^2$,

$$\mu_0 := \min_{j,m=1,2} k_{j,m} \le k_{j\varepsilon}(x) \le \mu_1 := \max_{j,m=1,2} k_{j,m}, \tag{3.2}$$

$$\begin{cases} -\underline{d}_1 \le \frac{\partial f_{1\varepsilon}(x,u,v)}{\partial u} \le -\bar{d}_1, \qquad -\underline{e}_1 \le \frac{\partial f_{1\varepsilon}}{\partial v} \le 0, \\ -\underline{d}_2 \le \frac{\partial f_{2\varepsilon}(x,u,v)}{\partial u} \le 0, \qquad -\underline{e}_2 \le \frac{\partial f_{2\varepsilon}}{\partial v} \le -\bar{e}_2, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \min_{i=1,2} f_{1,i}(0,0) - \underline{d}_1 u - \underline{e}_1 v \le f_{1\varepsilon}(x,u,v) \le \max_{i=1,2} f_{1,i}(0,0) - \bar{d}_1 u, \\ \min_{i=1,2} f_{2,i}(0,0) - \underline{d}_2 u - \underline{e}_2 v \le f_{2\varepsilon}(x,u,v) \le \max_{i=1,2} f_{2,i}(0,0) - \bar{e}_2 v. \end{cases} \tag{3.4}$$

By employing $k_{j\varepsilon}(x), f_{j\varepsilon}(x,u,v)$, we consider the following approximation problem of (1.1):

$$\begin{cases} u_t = [k_{1\varepsilon}(x)((1+\beta u+\gamma v)u)_x]_x + uf_{1\varepsilon}(x,u,v) \quad ((x,t) \in Q_T), \\ v_t = [k_{2\varepsilon}(x)((1+\delta v)v)_x]_x + vf_{2\varepsilon}(x,u,v) \quad ((x,t) \in Q_T), \\ u_x = v_x = 0 \quad ((x,t) \in S_T), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \quad (x \in (0,l)). \end{cases} \tag{3.5}$$

Proposition 3.1 *Let hypothesis* (H) *be satisfied. Then problem* (3.5) *has a unique nonnegative global solution* $(u_\varepsilon, v_\varepsilon) = (u_\varepsilon(x,t), u_\varepsilon(x,t))$ *satisfying* $u_\varepsilon(\cdot,t), v_\varepsilon(\cdot,t) \in C([0,\infty), W^1_{r_0}(0,l)) \cap C^\infty((0,\infty), C^\infty(0,l))$.

Proof By (2.2) and (3.3), and by a minor modification, we can use the arguments in [6] to prove this proposition. The detailed proofs are omitted here. □

3.2 Uniform estimates of $\max_{Q_T} u_\varepsilon$, $\max_{Q_T} v_\varepsilon$

In order to investigate the limit of vector sequence $\{(u_\varepsilon, v_\varepsilon)\}$ governed by (3.5), in the rest of this sections we derive some uniform estimates of u_ε, v_ε. We shall use the following relations several times:

$$\big((1+\beta_\varepsilon u_\varepsilon+\gamma v_\varepsilon)u_\varepsilon\big)_x=(1+2\beta u_\varepsilon+\gamma v_\varepsilon)u_{\varepsilon x}+\gamma u_\varepsilon v_{\varepsilon x},$$
$$\big((1+\delta v_\varepsilon)v_\varepsilon\big)_x=(1+2\delta v_\varepsilon)v_{\varepsilon x}.$$

For notational simplicity, we shall use α', α'', $\bar{\alpha}$, C_0, C_1 and $C(\cdots)$ to denote constants depending only on T, β, γ, δ, $k_{j,m}$, $f_{j,m}(0,0)$, $\underline{d}_j$, $\underline{e}_j$ $(j,m=1,2)$, $\bar{d}_1$, $\bar{e}_2$, $\|u_0\|_{W^1_{r_0}(0,l)}$, $\|v_0\|_{W^1_{r_0}(0,l)}$ and the quantities appearing in parentheses, independent of ε. The same letter C will be used to denote different constants depending on the same set of arguments.

Lemma 3.2 *We have*

$$\max_{\bar{Q}_T} v_\varepsilon(x,t)\le C_0:=\max\Big\{\max_{x\in[0,l]} v_0, \max_{i=1,2} f_{2,i}(0,0)/\bar{e}_2\Big\}, \tag{3.6}$$

$$\max_{t\in[0,T]}\big\|u_\varepsilon(\cdot,t)\big\|_{L^1(0,l)}+\|u_\varepsilon\|_{L^2(Q_T)}\le C, \tag{3.7}$$

$$\|v_\varepsilon\|_{V_2(Q_T)}\le C. \tag{3.8}$$

Proof By using the maximum principle (see [17]) and (3.2)-(3.4), the proofs similar to those of Lemmas 2.1, 2.2, 2.4 in [6] imply (3.6)-(3.8). □

Lemma 3.3 *The following estimates hold*:

$$\sup_{t\in[0,T]}\big\|v_{\varepsilon x}(\cdot,t)\big\|_{L^2(0,l)}+\|v_{\varepsilon t}\|_{L^2(Q_T)}\le C, \tag{3.9}$$

$$\|v_{\varepsilon x}\|_{L^6(Q_T)}+\|v_{\varepsilon x}\|_{L^{\infty,4}(Q_T)}\le C. \tag{3.10}$$

Proof Let $w_\varepsilon=(1+\delta v_\varepsilon)v_\varepsilon$. Then

$$w_{\varepsilon t}=(1+2\delta v_\varepsilon)v_{\varepsilon t},\qquad w_{\varepsilon x}=(1+2\delta v_\varepsilon)v_{\varepsilon x}, \tag{3.11}$$

and

$$w_{\varepsilon t}=(1+2\delta v_\varepsilon)\big[\big(k_{2\varepsilon}(x)w_{\varepsilon x}\big)_x+v_\varepsilon f_{2\varepsilon}(x,u_\varepsilon,v_\varepsilon)\big]\quad\big((x,t)\in Q_T\big). \tag{3.12}$$

Recall that $v_\varepsilon(\cdot,t)\in C([0,\infty),W^1_{r_0}(0,l))\cap C^\infty((0,\infty),C^\infty(0,l))$. Multiplying (3.12) by $(k_{2\varepsilon}(x)w_{\varepsilon x})_x$, integrating it over Q_t and then using (3.4), (3.6), (3.7), and Cauchy's inequality, we see that, for any $\vartheta>0$,

$$\begin{aligned}&\frac{1}{2}\int_0^l k_{2\varepsilon}(x)w^2_{\varepsilon x}(x,t)\,\mathrm{d}x+\iint_{Q_t}(1+2\delta v_\varepsilon)\big(k_{2\varepsilon}(x)w_{\varepsilon x}\big)_x^2\,\mathrm{d}x\,\mathrm{d}t\\&\quad=\frac{1}{2}\int_0^l k_{2\varepsilon}(x)w^2_{\varepsilon x}(x,0)\,\mathrm{d}x-\iint_{Q_t}(1+2\delta v_\varepsilon)\big(k_{2\varepsilon}(x)w_{\varepsilon x}\big)_x v_\varepsilon f_{2\varepsilon}(x,u_\varepsilon,v_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$\le C\int_0^l v_{0x}^2\,\mathrm{d}x + \vartheta\iint_{Q_t}\big(k_{2\varepsilon}(x)w_{\varepsilon x}\big)_x^2\,\mathrm{d}x\,\mathrm{d}t + C(\vartheta)\iint_{Q_t}\big(1+u_\varepsilon^2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C(\vartheta) + \vartheta\iint_{Q_t}\big(k_{2\varepsilon}(x)w_{\varepsilon x}\big)_x^2\,\mathrm{d}x\,\mathrm{d}t.$$

We choose $\vartheta = 1/2$ and employ (3.12) to find

$$\sup_{t\in[0,T]}\big\|w_{\varepsilon x}(\cdot,t)\big\|_{L^2(0,l)} + \big\|\big(k_{2\varepsilon}(x)w_{\varepsilon x}\big)_x\big\|_{L^2(Q_T)} + \|w_{\varepsilon t}\|_{L^2(Q_T)} \le C. \tag{3.13}$$

Furthermore, from (3.13), (3.11), and (3.8) it follows that (3.9) holds, and

$$\big\|k_{2\varepsilon}(x)w_{\varepsilon x}\big\|_{V_2(Q_T)} \le C,$$

and from Chapter II, equation (3.8) in [18] that

$$\|w_{\varepsilon x}\|_{L^6(Q_T)} + \|w_{\varepsilon x}\|_{L^{\infty,4}(Q_T)} \le C.$$

Combining this inequality and (3.11) leads us to estimate (3.10). □

Based on Lemma 3.3, let us estimate $\max_{Q_T} u_\varepsilon$.

Lemma 3.4 *We have*

$$u_\varepsilon(x,t) \le C_1 \quad \big((x,t)\in Q_T\big). \tag{3.14}$$

Proof Choose $\hat\sigma > \max\{\max_{x\in[0,l]} u_0(x), 1\}$, and let

$$A_\sigma(t) := \big\{x : u_\varepsilon > \sigma, x\in[0,l]\big\}, \qquad u_\varepsilon^{(\sigma)} := \max\{u_\varepsilon - \sigma; 0\}, \quad \sigma \ge \hat\sigma.$$

Multiplying the first equation of (3.5) by $u_\varepsilon^{(\sigma)}$ and then integrating by parts over Q_{t_1}, we see from (3.2), (3.4), and Cauchy's inequality that, for any $\vartheta > 0$,

$$\frac{1}{2}\int_0^l\big[u_\varepsilon^{(\sigma)}(x,t_1)\big]^2\,\mathrm{d}x + \int_0^{t_1}\int_{A_\sigma(t)} k_{1\varepsilon}(x)(1+2\beta u_\varepsilon+\gamma v_\varepsilon)u_{\varepsilon x}^2\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^{t_1}\int_{A_\sigma(t)} k_{1\varepsilon}(x)\gamma u_\varepsilon u_{\varepsilon x}v_{\varepsilon x}\,\mathrm{d}x\,\mathrm{d}t + \int_0^{t_1}\int_{A_\sigma(t)} u_\varepsilon u_\varepsilon^{(\sigma)} f_{1\varepsilon}(x,u_\varepsilon,v_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C\int_0^{t_1}\int_{A_\sigma(t)} u_\varepsilon|u_{\varepsilon x}||v_{\varepsilon x}|\,\mathrm{d}x\,\mathrm{d}t + \max_{i=1,2} f_{1,i}(0,0)\int_0^{t_1}\int_{A_\sigma(t)} u_\varepsilon(u_\varepsilon-\sigma)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le \vartheta\int_0^{t_1}\int_{A_\sigma(t)} u_{\varepsilon x}^2\,\mathrm{d}x\,\mathrm{d}t + C(\vartheta)\int_0^{t_1}\int_{A_\sigma(t)}\big(v_{\varepsilon x}^2+1\big)\big[(u_\varepsilon-\sigma)^2+\sigma^2\big]\,\mathrm{d}x\,\mathrm{d}t.$$

Setting $\vartheta = \mu_0/2$, we have

$$\big\|u_\varepsilon^{(\sigma)}\big\|_{V_2(Q_{t_1})}^2 \le C\big\|\mathcal{D}(x,t)\big\|_{L^{\bar s,\bar r}(Q_{t_1}(\sigma))}\big\|(u_\varepsilon-\sigma)^2+\sigma^2\big\|_{L^{\bar s/(\bar s-1),\bar r/(\bar r-1)}(Q_{t_1}(\sigma))} \tag{3.15}$$

for $\bar s = \bar r = 2$, where $\mathcal{D}(x,t) := v_{\varepsilon x}^2 + 1$, $Q_{t_1}(\sigma) := \{(x,t) : x\in A_\sigma(t), t\in(0,t_1]\}$. Thus $1/\bar r + 1/(2\bar s) = 1 - \chi_1$ for $\chi_1 = 1/4$. In view of (3.10), we find that (3.15) has the same property as [18], Chapter III, equation (7.8). By using Chapter II, Theorem 6.1 and Remark 6.2 in [18], the proof similar to that of Chapter III, equation (7.15) in [18] leads to (3.14). □

3.3 Uniform Hölder estimates

In this subsection, we establish uniform Hölder estimates of u_ε, v_ε and their derivatives.

Lemma 3.5 *For any open interval $I' \subset\subset (0,l)$, there exists $\alpha' = \alpha'(d') \in (0,1)$ such that*

$$\|u_\varepsilon, v_\varepsilon\|_{C^{\alpha'}(\bar{I}'\times[0,T])} \le C(d'). \tag{3.16}$$

Here and below, $d' := \min\{\mathrm{dist}(0,I'), \mathrm{dist}(l,I')\}$.

Proof It is obvious that u_ε is a bounded generalized solution of equation $u_t - (\bar{a}_\varepsilon(x,t,u_x))_x + a_\varepsilon(x,t) = 0$ and v_ε is a bounded generalized solution of equation $v_t - (\bar{b}_\varepsilon(x,t,v_x))_x + b_\varepsilon(x,t) = 0$ in the sense of Chapter V, Section 1 in [18], where

$$\begin{aligned}
&\bar{a}_\varepsilon(x,t,p) = k_{1\varepsilon}(x)\big[1 + 2\beta u_\varepsilon(x,t) + \gamma v_\varepsilon(x,t)\big]p + k_{1\varepsilon}(x)\gamma v_{\varepsilon x}(x,t)u_\varepsilon(x,t),\\
&a_\varepsilon(x,t) = -u_\varepsilon(x,t)f_{1\varepsilon}\big(x, u_\varepsilon(x,t), v_\varepsilon(x,t)\big),\\
&\bar{b}_\varepsilon(x,t,q) = k_{2\varepsilon}(x)\big(1 + 2\delta v_\varepsilon(x,t)\big)q,\\
&b_\varepsilon(x,t) = -v_\varepsilon(x,t)f_{2\varepsilon}\big(x, u_\varepsilon(x,t), v_\varepsilon(x,t)\big).
\end{aligned}$$

By (3.2), (3.6), and (3.14), we get

$$\begin{aligned}
&\bar{a}_\varepsilon(x,t,p)p \ge \frac{\mu_0}{2}p^2 - C\varphi_0(x,t),\\
&\big|\bar{a}_\varepsilon(x,t,p)\big| \le C|p| + C\varphi_1(x,t) \quad \big((x,t,p) \in Q_T \times \mathbb{R}\big),\\
&\bar{b}_\varepsilon(x,t,q)q \ge \mu_0 q^2, \qquad \big|\bar{b}_\varepsilon(x,t,q)\big| \le C|q| \quad \big((x,t,q) \in Q_T \times \mathbb{R}\big),\\
&\big|a_\varepsilon(x,t)\big| + \big|b_\varepsilon(x,t)\big| \le C \quad \big((x,t) \in Q_T\big),
\end{aligned}$$

where $\varphi_0(x,t) := v_{\varepsilon x}^2(x,t)$ and $\varphi_1(x,t) := |v_{\varepsilon x}(x,t)|$. Choosing $s = r = 2$, we find from (3.10) that

$$\|\varphi_0\|_{L^{s,r}(Q_T)} \le C, \qquad \|\varphi_1\|_{L^{2s,2r}(Q_T)} \le C.$$

In view of $u_0(x), v_0(x) \in C^{1/2}([0,l])$, by employing Chapter V, Section 1, Theorem 1.1 in [18], we obtain (3.16). □

We next give the Hölder estimates of derivatives.

Lemma 3.6 *For any fixed $i \in \{1,2\}$ and for any $I'' \subset\subset I^{(i)}$, $\tau \in (0,T)$, there exists $\alpha'' = \alpha''(d'',\tau) \in (0,1)$ such that*

$$\|u_\varepsilon, v_\varepsilon\|_{C^{2+\alpha'',1+\alpha''/2}(\bar{I}''\times[\tau,T])} \le C(d'',\tau). \tag{3.17}$$

Hereafter, $d'' := \mathrm{dist}(\partial I^{(i)}, I'')$.

Proof Choose open intervals I''_j and positive numbers τ_j, $j = 1,\dots,4$, such that $I'' \subset\subset I''_4 \subset\subset \cdots \subset\subset I''_1 \subset\subset I^{(i)}$ and $\tau_1 < \cdots < \tau_4 < \tau$. By (3.1), for small enough ε, we have

$$k_{j\varepsilon}(x) = k_{j,i}, \qquad f_{j\varepsilon}(x,u,v) = f_{j,i}(u,v) \quad \bigl(x \in I''_1\bigr), j = 1,2.$$

We first derive uniform Hölder estimates of derivatives of v_ε. Set

$$\bar{B}_\varepsilon = \bar{B}_\varepsilon(x,t,v,q) = k_{2,i}(1+2\delta v)q, \qquad B_\varepsilon = B_\varepsilon(x,t,v) = vf_{2,i}\bigl(u_\varepsilon(x,t),v\bigr).$$

Then v_ε is a solution of equation

$$v_t = \bigl(\bar{B}_\varepsilon(x,t,v,v_x)\bigr)_x + B_\varepsilon(x,t,v) \quad \bigl((x,t) \in I''_1 \times (\tau_1,T]\bigr).$$

Note that

$$\bar{B}_{\varepsilon q}(x,t,v,q) = k_{2,i}(1+2\delta v), \qquad \bar{B}_{\varepsilon v} = 2k_{2,i}\delta q, \qquad \bar{B}_{\varepsilon x} = 0.$$

Thus when $(x,t,v,q) \in I''_1 \times (\tau_1,T] \times [0,C_0] \times \mathbb{R}$,

$$\mu_0 \le \bar{B}_{\varepsilon q}(x,t,v,q) \le C, \qquad |\bar{B}_{\varepsilon v}| \le C|q|, \qquad \bigl|B_\varepsilon(x,t,v)\bigr| \le C.$$

According to Chapter V, Theorem 3.1 in [18], it follows that, for some $\alpha''_1 = \alpha''_1(d'',\tau) \in (0,1)$,

$$\|v_{\varepsilon x}\|_{C^{\alpha''_1}(\bar{I}''_2 \times [\tau_2,T])} \le C\bigl(d'',\tau\bigr). \tag{3.18}$$

In addition, v_ε is a solution of the linear equation

$$v_t - \bar{h}_\varepsilon(x,t)v_{xx} + \hat{h}_\varepsilon(x,t)v_x = h_\varepsilon(x,t) \quad \bigl((x,t) \in I''_2 \times (\tau_2,T]\bigr),$$

where $\bar{h}_\varepsilon(x,t) = k_{2,i}(1+2\delta v_\varepsilon(x,t))$, $\hat{h}_\varepsilon(x,t) = -2k_{2,i}\delta v_{\varepsilon x}$ and $h_\varepsilon(x,t) = v_\varepsilon f_{2,i}(u_\varepsilon,v_\varepsilon)$. Estimates (3.16), (3.18) imply that, for some $\alpha''_2 = \alpha''_2(d'',\tau) \in (0,1)$,

$$\|\hat{h}_\varepsilon, \hat{h}_\varepsilon, h_\varepsilon\|_{C^{\alpha''_2}(\bar{I}''_2 \times [\tau_2,T])} \le C\bigl(d'',\tau\bigr).$$

The Schauder estimate for linear parabolic equation further yields

$$\|v_\varepsilon\|_{C^{2+\alpha''_2,1+\alpha''_2/2}(\bar{I}''_3 \times [\tau_3,T])} \le C\bigl(d'',\tau\bigr). \tag{3.19}$$

We next use (3.19) to show the uniform Hölder estimates of derivatives of u_ε. Let

$$\begin{aligned}
\bar{A}_\varepsilon &= \bar{A}_\varepsilon(x,t,u,p) := k_{1,i}\bigl(1+2\beta u+\gamma v_\varepsilon(x,t)\bigr)p + k_{1,i}\gamma u v_{\varepsilon x}(x,t),\\
A_\varepsilon &= A_\varepsilon(x,t,u) := uf_{1,i}\bigl(u,v_\varepsilon(x,t)\bigr).
\end{aligned}$$

Thus $u_\varepsilon = u_\varepsilon(x,t)$ is a solution of the following equation:

$$u_t = \bigl(\bar{A}_\varepsilon(x,t,u,u_x)\bigr)_x + A_\varepsilon(x,t,u) \quad \bigl((x,t) \in I''_3 \times (\tau_3,T]\bigr).$$

Since

$$\bar{A}_{\varepsilon p} = k_{1,i}\big(1 + 2\beta u + \gamma v_\varepsilon(x,t)\big), \qquad \bar{A}_{\varepsilon u} = 2k_{1,i}\beta p + k_{1,i}\gamma v_{\varepsilon x}(x,t),$$
$$\bar{A}_{\varepsilon x} = k_{1,i}\gamma v_{\varepsilon x}(x,t)p + k_{1,i}\gamma u v_{\varepsilon xx}(x,t),$$

then we see from (3.6), (3.14), and (3.19) that when $(x,t,u,p) \in I_3'' \times (\tau_3, T] \times [0, C_1] \times \mathbb{R}$,

$$\mu_0 \le \bar{A}_{\varepsilon p}(x,t,u,p) \le C, \qquad |\bar{A}_\varepsilon| + |\bar{A}_{\varepsilon u}| + |\bar{A}_{\varepsilon x}| \le C\big(d'',\tau\big)\big(|p| + 1\big),$$
$$\big|A_\varepsilon(x,t,u)\big| \le C.$$

Again by Chapter V, Theorem 3.1 in [18] we see that, for some $\alpha_3'' = \alpha_3''(d'',\tau) \in (0,1)$,

$$\|u_{\varepsilon x}\|_{C^{\alpha_3'',\alpha_3''/2}(\bar{I}_4''\times[\tau_4,T])} \le C\big(d'',\tau\big). \tag{3.20}$$

Furthermore, u_ε is a solution of the linear equation

$$u_t - \bar{g}_\varepsilon(x,t)u_{xx} + \hat{g}_\varepsilon(x,t)u_x + \tilde{g}_\varepsilon(x,t)u = g_\varepsilon(x,t) \quad \big((x,t) \in I_4'' \times (\tau_4, T]\big),$$

where

$$\bar{g}_\varepsilon(x,t) := k_{1,i}\big[1 + 2\beta u_\varepsilon(x,t) + \gamma v_\varepsilon(x,t)\big],$$
$$\hat{g}_\varepsilon(x,t) := -2k_{1,i}\big[\beta u_{\varepsilon x}(x,t) + \gamma v_{\varepsilon x}(x,t)\big],$$
$$\tilde{g}_\varepsilon(x,t) := -k_{1,i}\gamma v_{\varepsilon xx}(x,t),$$
$$g_\varepsilon(x,t) := u_\varepsilon(x,t)f_{1,i}\big(u_\varepsilon(x,t), v_\varepsilon(x,t)\big).$$

Using (3.16), (3.19), and (3.20) leads to

$$\|\bar{g}_\varepsilon, \hat{g}_\varepsilon, \tilde{g}_\varepsilon, g_\varepsilon\|_{C^{\alpha_4''}(\bar{I}_4''\times[\tau_4,T])} \le C\big(d'',\tau\big)$$

for some $\alpha_4'' = \alpha_4''(d'',\tau) \in (0,1)$. Again by Schauder estimate we have

$$\|u_\varepsilon\|_{C^{2+\alpha_4'',1+\alpha_4''/2}(\bar{I}''\times[\tau,T])} \le C\big(d'',\tau\big),$$

which along with (3.19) yields (3.17). □

3.4 Estimates of $\|u_{\varepsilon_1} - u_{\varepsilon_2}, v_{\varepsilon_1} - v_{\varepsilon_2}\|_{V_2(Q_T)}$

We first give the following uniform estimates of derivatives of u_ε.

Lemma 3.7 *We have*

$$\sup_{t\in[0,T]}\big\|u_{\varepsilon x}(\cdot,t)\big\|_{L^2([0,l])} + \|u_{\varepsilon t}\|_{L^2(Q_T)} \le C, \tag{3.21}$$

$$\|u_{\varepsilon x}\|_{L^6(Q_T)} + \|u_{\varepsilon x}\|_{L^{\infty,4}(Q_T)} \le C. \tag{3.22}$$

Proof Let $z_\varepsilon = (1+\beta u_\varepsilon + \gamma v_\varepsilon)u_\varepsilon$. Then

$$z_{\varepsilon t} = (1+2\beta u_\varepsilon + \gamma v_\varepsilon)u_{\varepsilon t} + \gamma u_\varepsilon v_{\varepsilon t}, \qquad z_{\varepsilon x} = (1+2\beta u_\varepsilon + \gamma v_\varepsilon)u_{\varepsilon x} + \gamma u_\varepsilon v_{\varepsilon x}, \tag{3.23}$$

$$z_{\varepsilon t} = (1+2\beta u_\varepsilon + \gamma v_\varepsilon)\bigl[\bigl(k_{1\varepsilon}(x)z_{\varepsilon x}\bigr)_x + u_\varepsilon f_{1\varepsilon}(x,u_\varepsilon,v_\varepsilon)\bigr] + \gamma u_\varepsilon v_{\varepsilon t} \quad \bigl((x,t)\in Q_T\bigr). \tag{3.24}$$

Multiplying (3.24) by $(k_{1\varepsilon}(x)z_{\varepsilon x})_x$ and then integrating it over Q_t, from (3.4), (3.9), (3.14), and Cauchy's inequality we see that, for any $\vartheta > 0$,

$$\begin{aligned}
&\frac{1}{2}\int_0^l k_{1\varepsilon}(x)z_{\varepsilon x}^2(x,t)\,\mathrm{d}x + \iint_{Q_t}(1+2\beta u_\varepsilon + \gamma v_\varepsilon)\bigl(k_{1\varepsilon}(x)z_{\varepsilon x}\bigr)_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \frac{1}{2}\int_0^l k_{1\varepsilon}(x)z_{\varepsilon x}^2(x,0)\,\mathrm{d}x - \iint_{Q_t}\bigl[(1+2\beta u_\varepsilon + \gamma v_\varepsilon)u_\varepsilon f_{1\varepsilon}(x,u_\varepsilon,v_\varepsilon) + \gamma u_\varepsilon v_{\varepsilon t}\bigr] \\
&\qquad \times \bigl(k_{1\varepsilon}(x)z_{\varepsilon x}\bigr)_x\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C(\vartheta) + \vartheta\iint_{Q_t}\bigl(k_{1\varepsilon}(x)z_{\varepsilon x}\bigr)_x^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Choosing $\vartheta = 1/2$ and using (3.24) and (3.9), we further get

$$\bigl\|k_{1\varepsilon}(x)z_{\varepsilon x}\bigr\|_{V_2(Q_T)} + \|z_{\varepsilon t}\|_{L^2(Q_T)} \le C.$$

As in the proof of Lemma 3.3, from (3.23), (3.24), (3.9), (3.10), and Chapter II, equation (3.8) in [18] we conclude that (3.21) and (3.22) hold. □

Lemma 3.8 *For any* $\varepsilon_1, \varepsilon_2 > 0$, *we have*

$$\|u_{\varepsilon_1} - u_{\varepsilon_2}, v_{\varepsilon_1} - v_{\varepsilon_2}\|_{V_2(Q_T)} \le C(\varepsilon_1 + \varepsilon_2)^{1/4}. \tag{3.25}$$

Proof Let

$$\bar{u} = u_{\varepsilon_1} - u_{\varepsilon_2}, \qquad \bar{v} = v_{\varepsilon_1} - v_{\varepsilon_2}, \qquad \bar{\mathbf{w}} = (\bar{u},\bar{v}).$$

Then it follows from the second equation of (3.5) that

$$\begin{aligned}
&\bar{v}_t - \bigl\{k_{2\varepsilon_1}(x)(1+2\delta v_{\varepsilon_1})\bar{v}_x\bigr\}_x - \bigl\{\bigl[k_{2\varepsilon_1}(x)(1+2\delta v_{\varepsilon_1}) - k_{2\varepsilon_2}(x)(1+2\delta v_{\varepsilon_2})\bigr]v_{\varepsilon_2 x}\bigr\}_x \\
&\quad = J_1 := \bar{v}f_{2\varepsilon_1}(x,u_{\varepsilon_1},v_{\varepsilon_1}) + v_{\varepsilon_2}\bigl[f_{2\varepsilon_1}(x,u_{\varepsilon_1},v_{\varepsilon_1}) - f_{2\varepsilon_2}(x,u_{\varepsilon_2},v_{\varepsilon_2})\bigr].
\end{aligned}$$

Multiplying this equation by $\bar{v}$ and integrating by parts over Q_t, from (3.6), (3.14), and Cauchy's inequality we find that, for any $\vartheta > 0$,

$$\begin{aligned}
&\frac{1}{2}\int_0^l \bar{v}^2(x,t)\,\mathrm{d}x + \iint_{Q_t} k_{2\varepsilon_1}(x)(1+2\delta v_{\varepsilon_1})\bar{v}_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \iint_{Q_t}\bigl[k_{2\varepsilon_2}(x)(1+2\delta v_{\varepsilon_2}) - k_{2\varepsilon_1}(x)(1+2\delta v_{\varepsilon_1})\bigr]v_{\varepsilon_2 x}\bar{v}_x\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_t}\bar{v}J_1\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \vartheta\iint_{Q_t}\bar{v}_x^2\,\mathrm{d}x\,\mathrm{d}t + C(\vartheta)\iint_{Q_t}\psi^2\bigl[\bigl(k_{2\varepsilon_1}(x) - k_{2\varepsilon_2}(x)\bigr)^2 + \bar{u}^2 + \bar{v}^2\bigr]\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where $\psi := |u_{\varepsilon_2 x}| + |u_{\varepsilon_1 x}| + |v_{\varepsilon_1 x}| + |v_{\varepsilon_2 x}| + 1$. Taking $\vartheta = \mu_0/2$, we get

$$\begin{aligned}
&\int_0^l \bar{v}^2(x,t)\,\mathrm{d}x + \iint_{Q_t} \bar{v}_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\iint_{Q_t} \psi^2\big[\big(k_{2\varepsilon_1}(x) - k_{2\varepsilon_2}(x)\big)^2 + \bar{u}^2 + \bar{v}^2\big]\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.26)
\end{aligned}$$

Similarity, we see from the first equation of (3.5) that

$$\begin{aligned}
\bar{u}_t &= \big\{k_{1\varepsilon_1}(x)(1 + 2\beta u_{\varepsilon_1} + \gamma v_{\varepsilon_1})\bar{u}_x\big\}_x \\
&\quad + \big\{\big[k_{1\varepsilon_1}(x)(1 + 2\beta u_{\varepsilon_1} + \gamma v_{\varepsilon_1}) - k_{1\varepsilon_2}(x)(1 + 2\beta u_{\varepsilon_2} + \gamma v_{\varepsilon_2})\big]u_{\varepsilon_2 x}\big\}_x \\
&\quad + \big\{k_{1\varepsilon_1}(x)\gamma u_{\varepsilon_1} v_{\varepsilon_1 x} - k_{1\varepsilon_2}(x)\gamma u_{\varepsilon_2} v_{\varepsilon_2 x}\big\}_x \\
&\quad + \bar{u} f_{1\varepsilon_1}(x, u_{\varepsilon_1}, v_{\varepsilon_1}) + u_{\varepsilon_2}\big[f_{1\varepsilon_1}(x, u_{\varepsilon_1}, v_{\varepsilon_1}) - f_{1\varepsilon_2}(x, u_{\varepsilon_2}, v_{\varepsilon_2})\big].
\end{aligned}$$

As we have done in the derivation of (3.26), by a direct computation we have

$$\begin{aligned}
&\int_0^l \bar{u}^2(x,t)\,\mathrm{d}x + \iint_{Q_t} \bar{u}_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\iint_{Q_t} \bar{v}_x^2\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_t} \psi^2\big[\big(k_{1\varepsilon_1}(x) - k_{1\varepsilon_2}(x)\big)^2 + \bar{u}^2 + \bar{v}^2\big]\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.27)
\end{aligned}$$

Combining (3.26) and (3.27) leads us to the inequality

$$\begin{aligned}
&\int_0^l |\bar{\mathbf{w}}|^2(x,t)\,\mathrm{d}x + \iint_{Q_t} |\bar{\mathbf{w}}_x|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\iint_{Q_t} \psi^2 \sum_{j=1}^{2}\big(k_{j\varepsilon_1}(x) - k_{j\varepsilon_2}(x)\big)^2\,\mathrm{d}x\,\mathrm{d}t + C\iint_{Q_t} \psi^2|\bar{\mathbf{w}}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le C\int_0^t \int_{l_0}^{l_0+\varepsilon_1+\varepsilon_2} \psi^2\,\mathrm{d}x\,\mathrm{d}t + Cy(t) \\
&\quad \le C(\varepsilon_1 + \varepsilon_2)^{1/2}\|\psi\|_{L^4(Q_T)}^2 + Cy(t), \qquad (3.28)
\end{aligned}$$

where $y(t) := \int_0^t \{\|\psi(\cdot,t)\|_{L^\infty(0,l)}^2 \int_0^l |\bar{\mathbf{w}}|^2\,\mathrm{d}x\}\,\mathrm{d}t$. In view of (3.10) and (3.22), it follows from (3.28) that

$$\mathrm{d}y(t)/\mathrm{d}t \le C\big\|\psi(\cdot,t)\big\|_{L^\infty(0,l)}^2 y(t) + C(\varepsilon_1 + \varepsilon_2)^{1/2}\big\|\psi(\cdot,t)\big\|_{L^\infty(0,l)}^2.$$

Moreover, Gronwall's inequality leads to

$$\begin{aligned}
y(t) &\le C(\varepsilon_1 + \varepsilon_2)^{1/2}\int_0^t \big\|\psi(\cdot,t)\big\|_{L^\infty(0,l)}^2\,\mathrm{d}t\,\exp\bigg\{C\int_0^t \big\|\psi(\cdot,t)\big\|_{L^\infty(0,l)}^2\,\mathrm{d}t\bigg\} \\
&\le C(\varepsilon_1 + \varepsilon_2)^{1/2},
\end{aligned}$$

which along with (3.28) implies (3.25). □

3.5 Uniform estimates of $\|u_{\varepsilon xt}, v_{\varepsilon xt}\|_{L^2(I'\times(\tau,T])}$

To get the regularity of the limit function of sequence $\{(u_\varepsilon, v_\varepsilon)\}$, we need to derive the uniform estimates of $\|u_{\varepsilon xt}, v_{\varepsilon xt}\|_{L^2(I'\times(\tau,T])}$.

Lemma 3.9 *For any open interval $I' \subset\subset (0,l)$ and any number $\tau \in (0,T)$, we have*

$$\sup_{[\tau,T]}\big\|u_{\varepsilon t}(\cdot,t)\big\|_{L^2(I')} + \|u_{\varepsilon xt}\|_{L^2(I'\times(\tau,T])} + \|u_{\varepsilon tx}\|_{L^2(I'\times(\tau,T])} \le C\big(d',\tau\big), \tag{3.29}$$

$$\sup_{[\tau,T]}\big\|v_{\varepsilon t}(\cdot,t)\big\|_{L^2(I')} + \|v_{\varepsilon xt}\|_{L^2(I'\times(\tau,T]))} + \|v_{\varepsilon tx}\|_{L^2(I'\times(\tau,T])} \le C\big(d',\tau\big). \tag{3.30}$$

Proof Choose open intervals I'_j and positive number τ_j, $j=1,2,3$, such that $I' \subset\subset I'_3 \subset\subset I'_2 \subset\subset I'_1 \subset\subset (0,l)$ and $\tau_1 < \tau_2 < \tau_3 < \tau$. Differentiating the second equation of (3.5) with respect to t and noting that $v_{\varepsilon xt} = v_{\varepsilon tx}$ we get

$$\begin{aligned} v_{\varepsilon tt} &= \big[k_{2\varepsilon}(x)(1+2\delta v_\varepsilon)v_{\varepsilon tx}\big]_x + \big[2k_{2\varepsilon}(x)\delta v_{\varepsilon t}v_{\varepsilon x}\big]_x \\ &\quad + \big[v_\varepsilon f_{2\varepsilon}(x,u_\varepsilon,v_\varepsilon)\big]_t \quad \big((x,t)\in Q_T\big). \end{aligned} \tag{3.31}$$

Let $\lambda = \lambda(x,t)$ be an arbitrary smooth function taking values in $[0,1]$ such that $\lambda = 1$ for $(x,t) \in I'_2 \times (\tau_2, T]$, $\lambda = 0$ for $x \notin I'_1$ or $t \le \tau_1$, and $|\lambda_x| + |\lambda_t| \le C/d' + C/\tau$ for all (x,t). Multiplying equation (3.31) by $v_{\varepsilon t}\lambda^2$ and then integrating by parts over $I'_1 \times (\tau_1, t_1]$ for $t_1 \in (\tau, T]$, from (3.6), (3.9), (3.14), (3.21), and Cauchy's inequality we deduce that, for any $\vartheta > 0$,

$$\begin{aligned} &\frac{1}{2}\int_{I'_1} v_{\varepsilon t}^2(x,t_1)\lambda^2(t_1)\,\mathrm{d}x + \int_{\tau_1}^{t_1}\int_{I'_1} k_{2\varepsilon}(x)(1+2\delta v_\varepsilon)v_{\varepsilon tx}^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{\tau_1}^{t_1}\int_{I'_1}\big\{-2k_{2\varepsilon}(x)\delta v_{\varepsilon t}v_{\varepsilon x}v_{\varepsilon tx}\lambda^2 + \big[v_\varepsilon f_{2\varepsilon}(x,u_\varepsilon,v_\varepsilon)\big]_t v_{\varepsilon t}\lambda^2 \\ &\qquad + v_{\varepsilon t}^2\lambda\lambda_t - 2k_{2\varepsilon}(x)\big[(1+2\delta v_\varepsilon)v_{\varepsilon tx}v_{\varepsilon t} + 2\delta v_{\varepsilon t}^2 v_{\varepsilon x}\big]\lambda\lambda_x\big\}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le \vartheta\int_{\tau_1}^{t_1}\int_{I'_1} v_{\varepsilon tx}^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t + C(\vartheta)\int_{\tau_1}^{t_1}\int_{I'_1} v_{\varepsilon x}^2 v_{\varepsilon t}^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t + C\big(d',\tau\big). \end{aligned}$$

Choosing $\vartheta = \mu_0/4$, we have

$$\begin{aligned} &\int_{I'_1} v_{\varepsilon t}^2(x,t_1)\lambda^2(t_1)\,\mathrm{d}x + \int_{\tau_1}^{t_1}\int_{I'_1} v_{\varepsilon tx}^2\lambda^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le C\big(d',\tau\big) + C\int_{\tau_1}^{t_1}\left\{\big\|v_{\varepsilon x}(\cdot,t)\big\|_{L^\infty(0,l)}^2\int_{I'_1} v_{\varepsilon t}^2\lambda^2\,\mathrm{d}x\right\}\mathrm{d}t. \end{aligned}$$

As in the proof of Lemma 3.8, employing Gronwall's inequality and (3.10), from this inequality we get (3.30) and the uniform estimate of $\|v_{\varepsilon t}\|_{V_2(I'_2\times(\tau_2,T])}$. Chapter II, equation (3.8) in [18] further implies that

$$\|v_{\varepsilon t}\|_{L^6(I'_2\times(\tau_2,T])} \le C\big(d',\tau\big). \tag{3.32}$$

Based on (3.30) and (3.32), we next show that (3.29) holds. Let $\xi = \xi(x,t)$ be a smooth function with values between 0 and 1 such that $\xi = 1$ for $(x,t) \in I'_3 \times (\tau_3, T]$, $\xi = 0$ for $x \notin I'_2$

or $t \le \tau_2$, and $|\xi_x| + |\xi_t| \le C/d' + C/\tau$ for all (x,t). Similarly, differentiating the first equation of (3.5) with respect to t, multiplying the resulting equation by $u_{\varepsilon t}\xi^2$ and integrating it over $I_2' \times (\tau_2, T]$, we find from (3.6), (3.9), (3.14), (3.21), and Cauchy's inequality that, for any $\vartheta > 0$,

$$\begin{aligned}
&\frac{1}{2}\int_{I_2'} u_{\varepsilon t}^2(x,t_1)\xi^2(t_1)\,\mathrm{d}x + \int_{\tau_2}^{t_1}\int_{I_2'} k_{1\varepsilon}(x)(1+2\beta u_\varepsilon + \gamma v_\varepsilon)u_{\varepsilon tx}^2\xi^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_{\tau_2}^{t_1}\int_{I_2'} \bigl\{\bigl[u_\varepsilon f_{1\varepsilon}(x,u_\varepsilon,v_\varepsilon)\bigr]_t u_{\varepsilon t}\xi^2 + u_{\varepsilon t}^2\xi\xi_t \\
&\qquad - k_{1\varepsilon}(x)\bigl[(2\beta u_{\varepsilon t} + \gamma v_{\varepsilon t})u_{\varepsilon x} + \gamma(u_{\varepsilon t}v_{\varepsilon x} + u_\varepsilon v_{\varepsilon tx})\bigr]u_{\varepsilon tx}\xi^2 \\
&\qquad - 2k_{1\varepsilon}(x)\bigl[(2\beta u_{\varepsilon t} + \gamma v_{\varepsilon t})u_{\varepsilon x} + \gamma(u_{\varepsilon t}v_{\varepsilon x} + u_\varepsilon v_{\varepsilon tx})\bigr]u_{\varepsilon t}\xi\xi_x\bigr\}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \vartheta\int_{\tau_2}^{t_1}\int_{I_2'} u_{\varepsilon tx}^2\xi^2\,\mathrm{d}x\,\mathrm{d}t + C(\vartheta)\int_{\tau_2}^{t_1}\int_{I_2'} v_{\varepsilon tx}^2\,\mathrm{d}x\,\mathrm{d}t + C\bigl(d',\tau,\vartheta\bigr) \\
&\qquad + C\bigl(d',\tau,\vartheta\bigr)\int_{\tau_2}^{t_1}\int_{I_2'}\bigl(u_{\varepsilon x}^4 + v_{\varepsilon t}^4\bigr)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + C\int_{\tau_2}^{t_1}\int_{I_2'} u_{\varepsilon t}^2\bigl(|u_{\varepsilon x}| + |v_{\varepsilon x}|\bigr)^2\xi^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Setting $\vartheta = \mu_0/4$ and using (3.10), (3.22), (3.30), and (3.32), we have

$$\begin{aligned}
&\int_{I_2'} u_{\varepsilon t}^2(x,t_1)\xi^2(t_1)\,\mathrm{d}x + \int_{\tau_2}^{t_1}\int_{I_2'} u_{\varepsilon tx}^2\xi^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le C\bigl(d',\tau\bigr) + C\int_{\tau_2}^{t_1}\biggl\{\bigl\||u_{\varepsilon x}(\cdot,t)| + |v_{\varepsilon x}(\cdot,t)|\bigr\|_{L^\infty(0,l)}^2\int_{I_2'} v_{\varepsilon t}^2\xi^2\,\mathrm{d}x\biggr\}\,\mathrm{d}t.
\end{aligned}$$

Again by Gronwall's inequality we obtain (3.29). □

4 The proof of Theorem 2.1

To prove Theorem 2.1, let us discuss the behavior of vector sequence $\{(u_\varepsilon, v_\varepsilon)\}$ governed by (3.5) as $\varepsilon \to 0$.

Lemma 4.1 *The sequences $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ converge to nonnegative bounded functions u and v in $V_2(Q_T)$, respectively. Moreover, (u,v) possesses property* (i) *in Definition* 2.1, *and satisfies pointwise the equations in* (1.1) *on $Q_T^{(i)}$ $(i = 1,2)$ and the initial conditions on $\{(x,t) : x \in (0,l), t = 0\}$. For any $\eta \in W_2^{1,0}(Q_T)$, the following integral identities hold*:

$$\iint_{Q_t}\bigl[u_t\eta + k_1(x)\bigl((1+\beta u+\gamma v)u\bigr)_x\eta_x\bigr]\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_t} uf_1(x,u,v)\eta\,\mathrm{d}x\,\mathrm{d}t, \tag{4.1}$$

$$\iint_{Q_t}\bigl[v_t\eta + k_2(x)\bigl((1+\delta v)v\bigr)_x\eta_x\bigr]\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_t} vf_2(x,u,v)\eta\,\mathrm{d}x\,\mathrm{d}t. \tag{4.2}$$

Proof We see from estimate (3.25) that there exist functions $u, v \in V_2(Q_T)$ such that

$$u_\varepsilon \to u, \qquad v_\varepsilon \to v \quad \text{in } V_2(Q_T), \tag{4.3}$$

and from (3.9), (3.21) that there exist subsequences (hereafter we retain the same notations for them) $\{u_\varepsilon\}$, $\{v_\varepsilon\}$, such that

$$u_{\varepsilon t} \rightharpoonup u_t, \qquad v_{\varepsilon t} \rightharpoonup v_t \quad \text{weakly in } L^2(Q_T). \tag{4.4}$$

In addition, it follows from (3.6), (3.14), (3.16), and the Arzela-Ascoli theorem that, for any given open interval $I' \subset\subset (0,l)$, there exist subsequences $\{u_\varepsilon\}$, $\{v_\varepsilon\}$, such that $\{u_\varepsilon\}$, $\{v_\varepsilon\}$ converge to u, v in $C(\bar{I}' \times [0,T])$, respectively, and from (3.17) that, for any given $i \in \{1,2\}$, $I'' \subset\subset Q_T^{(i)}$ and $\tau \in (0,T)$, there exist subsequences $\{u_\varepsilon\}$, $\{v_\varepsilon\}$, such that $\{u_\varepsilon\}$, $\{v_\varepsilon\}$ converge to u, v in $C^{2,1}(\bar{I}'' \times [\tau,T])$, respectively. Then u, v are in $W_2^{1,1}(Q_T) \cap C^{\alpha'}(\bar{I}' \times [0,T]) \cap C^{2,1}(Q_T^{(i)}) \cap C^{2+\alpha'',1+\alpha''/2}(\bar{I}'' \times [\tau,T])$. Furthermore, (3.5) implies that (u,v) satisfies pointwise the equations in (1.1) on $Q_T^{(i)}$ $(i=1,2)$ and the initial conditions on $\{(x,t) : x \in (0,l), t=0\}$. By (3.6), (3.9), (3.14), and (3.21) we have

$$(0,0) \le (u,v) \le (C_1,C_0) \quad \bigl((x,t) \in Q_T\bigr), \tag{4.5}$$

$$\sup_{t\in[0,T]} \bigl\| u_x(\cdot,t), v_x(\cdot,t)\bigr\|_{L^2(0,l)}, \qquad \|u_t,v_t\|_{L^2(Q_T)} \le C. \tag{4.6}$$

For any $\eta \in W_2^{1,0}(Q_T)$, multiply the equations in (3.5) by η and then integrate by parts over Q_t to get

$$\begin{aligned} &\iint_{Q_t} \bigl[u_{\varepsilon t}\eta + k_{1\varepsilon}(x)\bigl((1+\beta u_\varepsilon + \gamma v)u_\varepsilon\bigr)_x \eta_x\bigr]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \iint_{Q_t} u_\varepsilon f_{1\varepsilon}(x,u_\varepsilon,v_\varepsilon)\eta\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.7}$$

$$\iint_{Q_t} \bigl[v_{\varepsilon t}\eta + k_{2\varepsilon}(x)\bigl((1+\delta v_\varepsilon)v_\varepsilon\bigr)_x \eta_x\bigr]\,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_t} v_\varepsilon f_{2\varepsilon}(x,u_\varepsilon,v_\varepsilon)\eta\,\mathrm{d}x\,\mathrm{d}t. \tag{4.8}$$

We next investigate the limit of each term in (4.7), (4.8). It follows from (3.1), (4.3), and (4.5) that

$$\begin{aligned} &\bigl\|u_\varepsilon f_{1\varepsilon}(x,u_\varepsilon,v_\varepsilon) - u f_1(x,u,v)\bigr\|^2_{L^2(Q_T)} \\ &\quad \le C\|u_\varepsilon - u\|^2_{L^2(Q_T)} + C\bigl\|f_{1\varepsilon}(x,u_\varepsilon,v_\varepsilon) - f_1(x,u,v)\bigr\|^2_{L^2(Q_T)} \\ &\quad \le C\bigl[\|u_\varepsilon - u\|^2_{L^2(Q_T)} + \bigl\|f_{1,1}(u_\varepsilon,v_\varepsilon) - f_{1,1}(u,v)\bigr\|^2_{L^2(Q_T^{(1)})} \\ &\qquad + \bigl\|f_{1,2}(u_\varepsilon,v_\varepsilon) - f_{1,2}(u,v)\bigr\|^2_{L^2((l_0+\varepsilon,l)\times(0,T])} \\ &\qquad + \bigl\|f_{1\varepsilon}(x,u_\varepsilon,v_\varepsilon) - f_1(x,u,v)\bigr\|^2_{L^2((l_0,l_0+\varepsilon)\times(0,T])}\bigr] \\ &\quad \le C\bigl(\|u_\varepsilon - u\|^2_{L^2(Q_T)} + \|v_\varepsilon - v\|^2_{L^2(Q_T)}\bigr) + C\varepsilon \\ &\quad \to 0 \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\sup_{t\in[0,T]} \bigl\|k_{1\varepsilon}(x)(1+2\beta u_\varepsilon + \gamma v_\varepsilon) - k_1(x)(1+2\beta u + \gamma v)\bigr\|^2_{L^2(0,l)} \\ &\quad \le C\bigl\|k_{1\varepsilon}(x) - k_1(x)\bigr\|^2_{L^2(l_0,l_0+\varepsilon)} \end{aligned}$$

$$
\begin{aligned}
&+ C \sup_{t\in[0,T]} \bigl[\bigl\| u_\varepsilon(\cdot,t) - u(\cdot,t) \bigr\|^2_{L^2(0,l)} + \bigl\| v_\varepsilon(\cdot,t) - v(\cdot,t) \bigr\|^2_{L^2(0,l)} \bigr] \\
&\to 0. \qquad (4.10)
\end{aligned}
$$

Furthermore, (3.22) and (4.10) yield

$$
\begin{aligned}
&\iint_{Q_T} \bigl| k_{1\varepsilon}(x)(1 + 2\beta u_\varepsilon + \gamma v_\varepsilon) - k_1(x)(1 + 2\beta u + \gamma v) \bigr|^2 u^2_{\varepsilon x}\, \mathrm{d}x\, \mathrm{d}t \\
&\quad \le \sup_{t\in[0,T]} \int_0^l \bigl| k_{1\varepsilon}(x)(1 + 2\beta u_\varepsilon + \gamma v_\varepsilon) - k_1(x)(1 + 2\beta u + \gamma v) \bigr|^2 \, \mathrm{d}x \\
&\qquad \times \int_0^T \bigl\| u_{\varepsilon x}(\cdot,t) \bigr\|^2_{L^\infty([0,l])} \, \mathrm{d}t \\
&\quad \to 0.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\bigl\| k_{1\varepsilon}(1 + 2\beta u_\varepsilon + \gamma v_\varepsilon) u_{\varepsilon x} - k_1(1 + 2\beta u + \gamma v) u_x \bigr\|^2_{L^2(Q_T)} \\
&\quad \le 2 \bigl\| \bigl[k_{1\varepsilon}(1 + 2\beta u_\varepsilon + \gamma v_\varepsilon) - k_1(1 + 2\beta u + \gamma v) \bigr] u_{\varepsilon x} \bigr\|^2_{L^2(Q_T)} + C \| u_{\varepsilon x} - u_x \|^2_{L^2(Q_T)} \\
&\quad \to 0. \qquad (4.11)
\end{aligned}
$$

Similar arguments show that

$$
v_\varepsilon f_{2\varepsilon}(x, u_\varepsilon, v_\varepsilon) \to v f_2(x,u,v), \qquad \gamma u_\varepsilon v_{\varepsilon x} \to \gamma u v_x \quad \text{in } L^2(Q_T), \qquad (4.12)
$$

$$
k_{2\varepsilon}(x)(1 + 2\delta v_\varepsilon) v_{\varepsilon x} \to k_2(x)(1 + 2\delta v) v_x \quad \text{in } L^2(Q_T). \qquad (4.13)
$$

Letting $\varepsilon \to 0$ and using (4.4) and (4.9)-(4.13), we find from (4.7), (4.8) that the integral identities (4.1), (4.2) hold. □

We next investigate the regularity of (u,v) in the neighborhood of inner boundary Γ_T.

Lemma 4.2 *For any $I' \subset\subset (0,l)$ and $\tau \in (0,T)$, $v_t \in C^{\bar\alpha'}(\bar I' \times [\tau,T])$ and $v_x \in C^{\bar\alpha'}((\bar I' \cap \bar I^{(i)}) \times [\tau,T])$ $(i = 1,2)$ for some $\bar\alpha' = \bar\alpha'(d',\tau) \in (0,1)$.*

Proof It follows from (3.29), (3.30), (4.3), and (4.6) that there exist subsequences $\{u_\varepsilon\}$, $\{v_\varepsilon\}$, such that

$$
u_{\varepsilon xt} \rightharpoonup u_{xt}, \qquad v_{\varepsilon xt} \rightharpoonup v_{xt} \quad \text{weakly in } L^2\bigl(I' \times (\tau,T]\bigr),
$$

and

$$
\sup_{[\tau,T]} \bigl\| u_t(\cdot,t), v_t(\cdot,t) \bigr\|_{L^2(I')}, \| u_{xt}, v_{xt} \|_{L^2(I' \times (\tau,T])} \le C\bigl(d',\tau\bigr). \qquad (4.14)
$$

Let $\tilde B := \tilde B(x,t,w,q) = k_2(x)(1 + 2\delta w)q$, $\hat B := \hat B(x,t,w) = -w f_2(x,u(x,t),w)$. Then $v(x,t)$ is a generalized solution of the single equation $w_t - (\tilde B(x,t,w,w_x))_x + \hat B(x,t,w) = 0$ in the sense of Definition 1.1 in [19]. We find that when $(x,t,w,q) \in (I' \cap I^{(i)}) \times (\tau, T] \times [0, C_0] \times \mathbb{R}$,

$$\begin{aligned}
&\frac{\partial \tilde{B}}{\partial q} = k_{2,i}(1+2\delta w), \qquad \frac{\partial \tilde{B}}{\partial w} = 2k_{2,i}\delta q, \qquad \frac{\partial \tilde{B}}{\partial x} = \frac{\partial \tilde{B}}{\partial t} = 0, \\
&\frac{\partial \hat{B}}{\partial w} = -f_{2,i}\bigl(u(x,t), w\bigr) - w\frac{\partial f_{2,i}(u(x,t),w)}{\partial w}, \\
&\hat{B}_x = -w\frac{\partial f_{2,i}(u,w)}{\partial u}\bigg|_{u=u(x,t)} u_x(x,t), \qquad \hat{B}_t = -w\frac{\partial f_{2,i}(u,w)}{\partial u}\bigg|_{u=u(x,t)} u_t(x,t).
\end{aligned}$$

Hence when $(x,t,w,q) \in (I' \cap I^{(i)}) \times (\tau, T] \times [0, C_0] \times \mathbb{R}$,

$$\begin{aligned}
&\bigl|\tilde{B}(x,t,w,q)\bigr| \le C\bigl(1+|q|\bigr), \qquad \mu_0 \le \tilde{B}_q \le C, \qquad |\tilde{B}_w| \le C|q|, \\
&\bigl|\hat{B}(x,t,w)\bigr| + |\hat{B}_w| \le C, \qquad |\hat{B}_x| \le C\psi_1(x,t), \qquad |\hat{B}_t| \le C\psi_1(x,t),
\end{aligned}$$

where $\psi_1(x,t) := |u_x(x,t)| + |u_t(x,t)|$. By (4.6) and (4.14), we have

$$\sup_{t\in[\tau,T]} \bigl\|\psi_1(\cdot,t)\bigr\|_{L^2(0,l)} \le C\bigl(d',\tau\bigr).$$

According to Theorem 1.1 in [19], there exists $\bar{\alpha}_1' = \bar{\alpha}_1'(d',\tau) \in (0,1)$ such that

$$\|v_t\|_{C^{\bar{\alpha}_1'}(\bar{I}'\times[\tau,T])} \le C\bigl(d',\tau\bigr). \tag{4.15}$$

Note that $v \in C^{\alpha'}(\bar{I}' \times [0,T])$ and $(k_{2,i}(1+2\delta v)v_x)_x = v_t - vf_{2,i}(u,v)$ on $Q_T^{(i)}$. We see that $(k_{2,i}(1+2\delta v)v_x)_x \in C^{\bar{\alpha}_2'}((\bar{I}' \cap \bar{I}^{(i)}) \times [\tau,T])$, and $k_{2,i}(1+2\delta v)v_x \in C^{1+\bar{\alpha}_2',\bar{\alpha}_2'}((\bar{I}' \cap \bar{I}^{(i)}) \times [\tau,T])$ for some $\bar{\alpha}_2' \in (0,1)$. Thus $v_x \in C^{\bar{\alpha}_3'}((\bar{I}' \cap \bar{I}^{(i)}) \times [\tau,T])$. Since $(k_{2,i}(1+2\delta v)v_x)_x = k_{2,i}(1+2\delta v)v_{xx} + 2k_{2,i}\delta v_x^2$ on $Q_T^{(i)}$,

$$\|v_x, v_{xx}\|_{C^{\bar{\alpha}_3'}(\bar{I}'\cap\bar{I}^{(i)}\times[\tau,T])} \le C\bigl(d',\tau\bigr). \tag{4.16}$$

This completes the proof of Lemma 4.2. □

Lemma 4.3 *For any $I' \subset\subset (0,l)$ and $\tau \in (0,T)$, $u_t \in C^{\tilde{\alpha}'}(\bar{I}' \times [\tau,T])$ and $u_x \in C^{\tilde{\alpha}'}((\bar{I}' \cap \bar{I}^{(i)}) \times [\tau,T])$ $(i = 1,2)$ for some $\tilde{\alpha}' = \tilde{\alpha}'(d',\tau) \in (0,1)$. Furthermore, (u,v) satisfies pointwise the inner boundary conditions in* (1.1) *on Γ_T, and satisfies the homogeneous Neumann boundary conditions on S_T for almost all $t \in [0,T]$.*

Proof To investigate the regularity of u, we need to estimate $\sup_{t\in[\tau,T]} \|v_{xt}(\cdot,t)\|_{L^2(I')}$. Choose open intervals I_1', I_2', I_3' and positive numbers τ_1, τ_2, τ_3, such that $I' \subset\subset I_3' \subset\subset I_2' \subset\subset I_1' \subset\subset (0,l)$ and $\tau_1 < \tau_2 < \tau_3 < \tau$. According to Lemma 4.1, (u,v) satisfies pointwise the equations in (1.1) on $Q_T^{(i)}$ $(i = 1,2)$. Furthermore, from (4.2) in Lemma 4.1 and the regularity of v in Lemma 4.2 we conclude that v satisfies pointwise the inner boundary condition $[k_2(x)((1+\delta v)v)_x]^-(l_0,t) = [k_2(x)((1+\delta v)v)_x]^+(l_0,t)$ for $t \in (0,T]$.

Let $w = (1+\delta v)v$, $w_{(t)} := (w(x,t+\Delta t) - w(x,t))/\Delta t$. Then

$$\begin{cases} w_t = (1+2\delta v)v_t = (1+2\delta v)(k_2(x)w_x)_x + (1+2\delta v)vf_2(x,u,v), \\ w_x = (1+2\delta v)v_x, \qquad w_{xt} = (1+2\delta v)v_{xt} + 2\delta v_x v_t, \end{cases} \tag{4.17}$$

and

$$\begin{cases} -w_{t(t)} + \{(1+2\delta v)(k_2(x)w_x)_x\}_{(t)} = -\{(1+2\delta v)vf_2(x,u,v)\}_{(t)} \\ \quad ((x,t) \in Q_T^{(i)}), i = 1,2, \\ w_{(t)}^-(l_0,t) = w_{(t)}^+(l_0,t) \quad (t \in (0,T]), \\ [(k_2(x)w_x)_{(t)}]^-(l_0,t) = [(k_2(x)w_x)_{(t)}]^+(l_0,t) \quad (t \in (0,T]). \end{cases} \tag{4.18}$$

Let $\xi = \xi(x,t)$ be a smooth function taking values between 0 and 1 such that $\xi(x,t) = 1$ for $(x,t) \in I_2' \times (\tau_2, T]$, $\xi = 0$ for $x \notin I_1'$ or $t \le \tau_1$, and $|\xi_x| + |\xi_t| \le C(d',\tau)$ for all (x,t). Since $v_{\varepsilon xt} = v_{\varepsilon tx}$, then $v_{xt} = v_{tx}$ and $w_{xt} = w_{tx}$ on $I_1' \times (\tau_1, T]$. Multiplying the equations in (4.18) by $(k_2(x)w_{x(t)})_x\xi^2$ and integrating by parts over $I_1' \times (\tau_1, t)$, by direct computation we have

$$\begin{aligned} &\frac{1}{2}\int_{I_1'} k_2(x)w_{x(t)}^2\xi^2\,\mathrm{d}x + \int_{\tau_1}^t\int_{I_1'}\big[-k_2(x)w_{x(t)}^2\xi\xi_t - 2k_2(x)w_{t(t)}w_{x(t)}\xi\xi_x\big]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\tau_1}^t\int_{I_1'}\big\{\big(1+2\delta v(x,t+\Delta t)\big)\big(k_2(x)w_{x(t)}\big)_x^2 \\ &\quad + 2\delta v_{(t)}\big(k_2(x)w_x\big)_x\big(k_2(x)w_{x(t)}\big)_x\big\}\xi^2\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_{\tau_1}^t\int_{I_1'}\big[(1+2\delta v)vf_2(x,u,v)\big]_{(t)}\big(k_2(x)w_{x(t)}\big)_x\xi^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Employing Cauchy's inequality we see that, for any $\vartheta_1, \vartheta_2 > 0$,

$$\begin{aligned} &\frac{1}{2}\int_{I_1'} k_2(x)w_{x(t)}^2\xi^2\,\mathrm{d}x + \int_{\tau}^t\int_{I_1'}\big(1+2\delta v(x,t+\Delta t)\big)\big(k_2(x)w_{x(t)}\big)_x^2\xi^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le \vartheta_1\int_{\tau_1}^t\int_{I_1'}\big(k_2(x)w_{x(t)}\big)_x^2\xi^2\,\mathrm{d}x\,\mathrm{d}t + \vartheta_2\int_{\tau_1}^t\int_{I_1'} w_{t(t)}^2\xi^2\,\mathrm{d}x\,\mathrm{d}t + C\big(\vartheta_1,\vartheta_2,d',\tau\big)J_2, \end{aligned} \tag{4.19}$$

where $J_2 = \int_{\tau_1}^t\int_{I_1'}[w_{x(t)}^2 + v_{(t)}^2(k_2(x)w_x)_x^2 + u_{(t)}^2 + v_{(t)}^2]\,\mathrm{d}x\,\mathrm{d}t$. Moreover, it follows from the equations in (4.18) that

$$\int_{\tau_1}^t\int_{I_1'} w_{t(t)}^2\xi^2\,\mathrm{d}x\,\mathrm{d}t \le C^*\int_{\tau_1}^t\int_{I_1'}\big(k_2(x)w_{x(t)}\big)_x^2\xi^2\,\mathrm{d}x\,\mathrm{d}t + CJ_2, \tag{4.20}$$

and from (4.14)-(4.17) that

$$J_2 \le \int_{\tau_1}^t\int_{I_1'}\big[w_{xt}^2 + \big(k_2(x)w_x\big)_x^2 + u_t^2 + 1\big]\,\mathrm{d}x\,\mathrm{d}t \le C\big(d',\tau\big). \tag{4.21}$$

Choosing ϑ_1, ϑ_2, such that $\vartheta_1 + C^*\vartheta_2 = 1/2$, we find from (4.19)-(4.21) that

$$\int_{I_1'} w_{x(t)}^2\xi^2\,\mathrm{d}x + \int_{\tau_1}^t\int_{I_1'}\big(k_2(x)w_{x(t)}\big)_x^2\xi^2\,\mathrm{d}x\,\mathrm{d}t \le C\big(d',\tau\big).$$

Then

$$\sup_{t\in[\tau_2,T]}\big\|w_{xt}(\cdot,t)\big\|_{L^2(I_2')} \le C\big(d',\tau\big),$$

which along with (4.17) further implies that

$$\sup_{t\in[\tau_2,T]}\left\|v_{xt}(\cdot,t)\right\|_{L^2(I_2')}\le C\left(d',\tau\right). \tag{4.22}$$

We next use (4.22) to prove that u possesses property (ii) in Definition 2.1. It is obvious that $u(x,t)$ is a generalized solution of the single equation $z_t-(\tilde{A}(x,t,z,z_x))_x+\check{A}(x,t,z)=0$ in the sense of Definition 1.1 in [19], where

$$\begin{aligned}&\tilde{A}=\tilde{A}(x,t,z,p):=k_1(x)\bigl(1+2\beta z+v(x,t)\bigr)p+k_1(x)zv_x(x,t),\\&\check{A}=\check{A}(x,t,z):=-zf_1\bigl(x,z,v(x,t)\bigr).\end{aligned}$$

From (4.15) and (4.16) it follows that when $(x,t,z,p)\in(I_2'\cap I^{(i)})\times(\tau_2,T]\times[0,C_1]\times\mathbb{R}$,

$$\begin{aligned}&\left|\tilde{A}(x,t,z,p)\right|\le C\left(d',\tau\right)\left(1+|p|\right),\qquad \mu_0\le\tilde{A}_p\le C,\\&|\tilde{A}_z|\le C\left(d',\tau\right)\left(1+|p|\right),\qquad |\tilde{A}_x|\le C\left(d',\tau\right)\left(1+|p|\right),\\&|\tilde{A}_t|\le C|p|+C\phi_1,\qquad \left|\check{A}(x,t,z)\right|+|\check{A}_z|+|\check{A}_x|+|\check{A}_t|\le C\left(d',\tau\right),\end{aligned}$$

where $\phi_1:=|v_{xt}|$, and from (4.22) that

$$\sup_{t\in[\tau_2,T]}\left\|\phi_1(\cdot,t)\right\|_{L^2(I_2')}\le C\left(d',\tau\right).$$

Consequently, the conditions of Theorem 1.1 in [19] are fulfilled. Then Theorem 1.1 in [19] shows that

$$\|u_t\|_{C^{\tilde{\alpha}_1'}(\bar{I}_3'\times[\tau_3,T])}\le C(d,\tau). \tag{4.23}$$

Since $[k_{1,i}((1+\beta u+\gamma v)u)_x]_x=uf_{1,i}(u,v)-u_t$ on $(I_2'\cap I^{(i)})\times(\tau_2,T]$, then by (4.23) we get $[k_{1,i}((1+\beta u+\gamma v)u)_x]_x\in C^{\tilde{\alpha}_2'}((\bar{I}_3'\cap\bar{I}^{(i)})\times[\tau_3,T])$. Thus $(1+2\beta u+\gamma v)u_x+\gamma uv_x\in C^{1+\tilde{\alpha}_2',\tilde{\alpha}_2'}((\bar{I}'\cap\bar{I}^{(i)})\times[\tau,T])$. In view of $u,v\in C^{\alpha'}(\bar{I}'\times[0,T])$ and $v_x\in C^{\tilde{\alpha}_1'}((\bar{I}'\cap\bar{I}^{(i)})\times[\tau,T])$, we have $u_x\in C^{\tilde{\alpha}_3'}((\bar{I}'\cap\bar{I}^{(i)})\times[\tau,T])$. Since (u,v) satisfies pointwise the equations in (1.1) on $Q_T^{(i)}$ $(i=1,2)$, then we further use integral identities (4.1), (4.2), the regularity of (u,v) in the neighborhood of Γ_T and the compatibility conditions in (2.3) to conclude that (u,v) satisfies pointwise the inner boundary conditions in (1.1) on Γ_T, and it satisfies homogeneous Neumann boundary conditions on S_T for almost all t (see Chapter III, Section 13 in [18]). □

From Lemmas 4.1-4.3 we see that problem (1.1) has at least one solution.

Lemma 4.4 *The solution of problem* (1.1) *is unique.*

Proof Suppose that (u_1,v_1), (u_2,v_2) are solutions of (1.1). Let

$$\tilde{u}=u_1-u_2,\qquad \tilde{v}=v_1-v_2,\qquad \tilde{\mathbf{w}}=(\tilde{u},\tilde{v}).$$

Set $\eta = \tilde{v}$ in (4.2). Then by a subtraction of the two integral identities (4.2) for v_1, v_2, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^l \tilde{v}^2(x,t)\,\mathrm{d}x + \iint_{Q_t} k_2(x)(1+2\delta v_1)\tilde{v}_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \iint_{Q_t} \bigl\{-2\delta k_2(x)v_{2x}\tilde{v}\tilde{v}_x + \tilde{v}^2 f_2(x,u_1,v_1) \\
&\qquad + v_2\bigl[f_2(x,u_1,v_1) - f_2(x,u_2,v_2)\bigr]\tilde{v}\bigr\}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Using Cauchy's inequality we get

$$\int_0^l \tilde{v}^2(x,t)\,\mathrm{d}x + \iint_{Q_t} \tilde{v}_x^2\,\mathrm{d}x\,\mathrm{d}t \le C\iint_{Q_t} \bar{\psi}^2\bigl(\tilde{u}^2 + \tilde{v}^2\bigr)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.24}$$

where $\bar{\psi} := 1 + |u_{1x}| + |u_{2x}| + |v_{1x}| + |v_{2x}|$.

Similarly, set $\eta = \tilde{u}$ in (4.1). By a subtraction of the two integral identities (4.1) for u_1, u_2, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^l \tilde{u}^2(x,t)\,\mathrm{d}x + \iint_{Q_t} k_1(x)(1+2\beta u_1 + \gamma v_1)\tilde{u}_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\iint_{Q_t} \bigl\{\bigl[k_1(x)(2\beta\tilde{u} + \gamma\tilde{v})\bigr]u_{2x} + k_1(x)\gamma u_1 v_{1x} - k_1(x)\gamma u_2 v_{2x}\bigr\}\tilde{u}_x\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \iint_{Q_t} \bigl\{\tilde{u}f_1(x,u_1,v_1) + u_2\bigl[f_1(x,u_1,v_1) - f_1(x,u_2,v_2)\bigr]\bigr\}\tilde{u}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

which, together with Cauchy's inequality, implies that

$$\int_0^l \tilde{u}^2(x,t)\,\mathrm{d}x + \iint_{Q_t} \tilde{u}_x^2\,\mathrm{d}x\,\mathrm{d}t \le C\iint_{Q_t} \bar{\psi}^2\bigl(\tilde{u}^2 + \tilde{v}^2\bigr)\,\mathrm{d}x\,\mathrm{d}t + C\iint_{Q_t} \tilde{v}_x^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.25}$$

Furthermore, (4.24), (4.25) yield

$$\int_0^l |\tilde{\mathbf{w}}|^2(x,t)\,\mathrm{d}x + \iint_{Q_t} |\tilde{\mathbf{w}}_x|^2\,\mathrm{d}x\,\mathrm{d}t \le C\int_0^t \left\{\bigl\|\bar{\psi}(\cdot,t)\bigr\|^2_{L^\infty(0,l)}\int_0^l |\tilde{\mathbf{w}}|^2\,\mathrm{d}x\right\}\mathrm{d}t.$$

Again by using Gronwall's inequality we deduce that $\tilde{\mathbf{w}} = 0$. Then the solution of (1.1) is unique. □

By Lemmas 4.1-4.4 we complete the proof of Theorem 2.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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References

1. Tan, QJ: An elliptic cross-diffusion system describing two-species models on a bounded domain with different natural conditions. J. Math. Anal. Appl. **437**, 853-869 (2016)
2. Shigesada, N, Kawasaki, K, Teramoto, E: Spatial segregation of interacting species. J. Theor. Biol. **79**, 83-99 (1979)
3. Chen, X, Qi, Y, Wang, M: A strongly coupled predator-prey system with non-monotonic functional response. Nonlinear Anal. **67**, 1966-1979 (2007)
4. Li, H, Pang, PYH, Wang, M: Global solutions for a general strongly coupled prey-predator model. Appl. Math. Lett. **22**, 1508-1512 (2009)
5. Ling, Z, Zhang, L, Lin, Z: Turing pattern formation in a predator-prey system with cross diffusion. Appl. Math. Model. **38**, 5022-5032 (2014)
6. Lou, Y, Ni, WM, Wu, Y: On the global existence of a cross-diffusion system. Discrete Contin. Dyn. Syst. **4**, 193-203 (1998)
7. Lou, Y, Ni, WM: Diffusion, self-diffusion and cross-diffusion. J. Differ. Equ. **131**, 79-131 (1996)
8. Lou, Y, Ni, WM, Yotsutani, S: Pattern formation in a cross-diffusion system. Discrete Contin. Dyn. Syst. **35**, 1589-1607 (2015)
9. Ko, W, Ryu, K, Ahn, I: Coexistence of three competing species with non-negative cross-diffusion rate. J. Dyn. Control Syst. **20**, 229-240 (2014)
10. Ni, WM, Wu, Y, Xu, Q: The existence and stability of nontrivial steady states for S-K-T competition model with cross diffusion. Discrete Contin. Dyn. Syst. **34**, 5271-5298 (2014)
11. Pao, CV: Strongly coupled elliptic systems and applications to Lotka-Volterra models with cross-diffusion. Nonlinear Anal. **60**, 1197-1217 (2005)
12. Ryu, K, Ahn, I: Coexistence theorem of steady states for nonlinear self-cross diffusion systems with competitive dynamics. J. Math. Anal. Appl. **283**, 46-65 (2003)
13. Shim, SA: Uniform boundedness and convergence of solutions to cross-diffusion systems. J. Differ. Equ. **185**, 281-305 (2002)
14. Tuộc, PV: Global existence of solutions to Shigesada-Kawasaki-Teramoto cross-diffusion systems on domains of arbitrary dimensions. Proc. Am. Math. Soc. **135**, 3933-3941 (2007)
15. Tuoc, PV: On global existence of solutions to a cross-diffusion system. J. Math. Anal. Appl. **343**, 826-834 (2008)
16. Wu, Y, Zhao, Y: The existence and stability of traveling waves with transition layers for the S-K-T competition model with cross-diffusion. Sci. China Math. **53**, 1161-1184 (2010)
17. Protter, MH, Weinberger, HF: Maximum Principles in Differential Equations, 2nd edn. Springer, New York (1984)
18. Ladyzenskaya, OA: Linear and Quasilinear Equations of Parabolic Type. Am. Math. Soc., Providence (1968)
19. Tan, QJ, Leng, ZJ: The regularity of generalized solutions for the *n*-dimensional quasi-linear parabolic diffraction problems. Nonlinear Anal. **69**, 4624-4642 (2008)