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# Monotonic iteration positive symmetric solutions to a boundary value problem with $p$-Laplacian and Stieltjes integral boundary conditions

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**Abstract**

This paper investigates the existence of monotonic iteration positive symmetric solutions to a boundary value problem with $p$-Laplacian and Stieltjes integral boundary conditions. The main tool is a monotone iterative technique. Meanwhile, an example is worked out to demonstrate the main results.

**MSC:** 34K10; 34B15

**Keywords:** Stieltjes integral boundary conditions; successive iteration; completely continuous

## 1 Introduction

One of the most used elements in structures such as aircraft, buildings, ships, and bridges is the elastic beam. The deformation of an elastic beam in equilibrium state whose two ends are simply supported can be described by a fourth-order ordinary differential equation boundary value problem. Fourth-order equations arise in a variety of different areas of applied mathematics and physics and so on. To identify a few and for more on the applications of the fourth-order boundary value problems, we refer the reader to [1–4] and related topics.

With the rapid development of science and technology, a series of boundary value problems with integral boundary conditions appeared in various industries and fields, for example, thermal conduction, chemical engineering, semiconductors, underground water flow, hydrodynamics, thermoelasticity, and related topics; one may refer to [5–9].

Recently, because of the wide application, boundary value problems with integral boundary conditions have attracted many authors' attention. It is to be observed that such problems include two-point, three-point, multi-point and nonlocal boundary value problems as special cases, hence they are more general.

The existence of multiple solutions to some boundary value problems involving integral boundary conditions has recently been studied by many authors, for instance, Boucherif in [10] obtain the existence of at least one positive solution for a two-order boundary value problem, by using Krasnoselskii's fixed point theorem of cone. In [11], the authors considered a class of boundary value problems using a fixed-point theorem of cone expansion

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and compression of norm type. In [12], Ma concerned with the existence of at least one symmetric positive solution to a boundary value problem, by the application of the fixed point index in cones.

Motivated by the work mentioned above, in this paper, we study the existence of monotonic iteration positive symmetric solutions to the following boundary value problem with $p$-Laplacian and Stieltjes integral boundary conditions:

$$\begin{cases} (\phi_p(u''(t)))''(t) = \lambda q(t)f(t,u(t),u'(t),u''(t),u'''(t)), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 u(s)\,dg(s), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = \int_0^1 \phi_p(u''(s))\,dh(s), \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\phi_p(s) = |s|^{p-2}s$ with $p > 1$ is a $p$-Laplacian operator and $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

The main features of this paper are as follows. Comparing with [1–4], we discuss problem (1.1) involving integral boundary conditions. It includes two-point, three-point, multipoint, and nonlocal boundary value problems as special cases, so it is more general. Comparing with [8, 9, 13], we consider the $p$-Laplacian operator which is nonlinear. In addition, we consider the symmetric positive solutions to (1.1) by the application of a monotone iterative technique. To our knowledge, there is no paper in the literature dealing with the fourth-order boundary value problems with $p$-Laplacian and Stieltjes integral boundary conditions via a monotone iterative technique, especially when the nonlinear term is involved explicitly with all the lower derivatives of $u$. Hence we improve and generalize some results in the literature.

In our work, we obtain not only the existence of positive symmetric solutions to the problems we are concerned with, but we also construct some successive iterative schemes whose starting point is a known constant function or a simple quartic function for approximating the solutions. We emphasize that the construction of the monotone iterative schemes does not require the existence of lower and upper solutions for the boundary value problems we will study.

Finally, an example is given to illustrate the applicability of our results. We remark that knowledge of how to find the solutions is probably most important from a numerical and application standpoint.

## 2 Preliminaries

Now, we present the necessary definitions from the theory of cones in Banach spaces.

**Definition 2.1** Let $E$ be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that:

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$, and
(ii) $u, -u \in P$ implies $u = 0$.

**Definition 2.2** The map $\alpha$ is said to be concave on $[0, 1]$, if

$$\alpha\bigl(tu + (1-t)v\bigr) \geq t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in [0, 1]$ and $t \in [0, 1]$.

**Definition 2.3** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Definition 2.4** For $t \in [0,1]$, a function $u$ is said to be symmetric on $[0,1]$ if $u(t) = u(1-t)$.

Throughout, it is assumed that the following conditions hold:

($H_1$): $f(t,x,y,z,\delta) \in C([0,1] \times [0,+\infty) \times R^3, [0,+\infty))$, $f(t,x,y,z,\delta)$ is symmetric about $t$ on $[0,1]$ and $f(t,x,y,z,\delta) = f(t,x,-y,z,-\delta)$.

($H_2$): $q(t) \in C((0,1),[0,+\infty))$ and symmetric about $t$ on $[0,1]$, $q(t) \not\equiv 0$ on any subinterval of $(0,1)$, and $\int_0^1 q(t)\,dt < +\infty$.

($H_3$): $g(t), h(t) \in C([0,1],[0,+\infty))$ are bounded variation and symmetric about $t$ on $[0,1]$.

Let the Banach space $E = C^3[0,1]$ be endowed with the norm

$$\|u\| := \max\Bigl\{\max_{0\le t\le 1}\bigl|u(t)\bigr|, \max_{0\le t\le 1}\bigl|u'(t)\bigr|, \max_{0\le t\le 1}\bigl|u''(t)\bigr|, \max_{0\le t\le 1}\bigl|u'''(t)\bigr|\Bigr\}.$$

We define the cone $P \subset E$ by

$$P = \bigl\{u \in E \mid u(t) \ge 0, u \text{ is concave and symmetric about } t \text{ on } [0,1]\bigr\}.$$

**Lemma 2.1** *Let $y \in L^1[0,1]$, then the boundary value problem*

$$\begin{cases} \phi_p(u''(t)) = y(t), & 0 < t < 1, \\ u(0) = u(1) = \int_0^1 u(s)\,dg(s), \end{cases}$$

*has a unique solution*

$$u(t) = -\int_0^1 \int_0^1 G(t,s)\phi_q\bigl(y(s)\bigr)\,ds\,dg(t) - \int_0^1 G(t,s)\phi_q\bigl(y(s)\bigr)\,ds, \tag{2.1}$$

*where*

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2.2}$$

*Proof* The proof follows by routine calculations, so we omit it here. □

**Lemma 2.2** *For $u \in C^3[0,1]$ the boundary value problem*

$$\begin{cases} y''(t) = \lambda q(t) f(t,u(t),u'(t),u''(t),u'''(t)), & 0 < t < 1, \\ y(0) = y(1) = \int_0^1 y(s)\,dh(s), \end{cases} \tag{2.3}$$

*has a unique solution,*

$$\begin{aligned} y(t) = {} & -\int_0^1 \int_0^1 G(s,\tau)\lambda q(\tau) f\bigl(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\,dh(s) \\ & - \int_0^1 G(t,s)\lambda q(s) f\bigl(s,u(s),u'(s),u''(s),u'''(s)\bigr)\,ds, \end{aligned} \tag{2.4}$$

*where*

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

*Proof* The proof follows by routine calculations, so we omit it here. □

Define an operator $T : P \to E$ by

$$\begin{aligned}
(Tu)(t) &= u(t) \\
&= \int_0^1 \int_0^1 G(s,\tau)\phi_q\biggl[\int_0^1 \int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\bigl(\zeta,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\bigr)\,d\zeta\,dh(\tau) \\
&\quad + \int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\bigl(s,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\bigr)\,d\zeta\biggr]\,d\tau\,dg(s) \\
&\quad + \int_0^1 G(t,s)\phi_q\biggl[\int_0^1 \int_0^1 G(s,\tau)\lambda q(\tau)f\bigl(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\,dh(s) \\
&\quad + \int_0^1 G(s,\tau)\lambda q(\tau)f\bigl(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\biggr]\,ds; \qquad (2.5)
\end{aligned}$$

by Lemma 2.1 and Lemma 2.2, boundary value problem (1.1) has a solution $u = u(t)$ if and only if $u$ is a fixed point of $T$.

**Lemma 2.3** *Assume that* ($H_1$)-($H_3$) *hold. Then* $T : P \to P$ *defined by* (2.5) *is completely continuous.*

*Proof* By (2.5), we have, for each $u \in P$, $Tu \in C^3[0,1]$, which satisfies (1.1).

Since

$$\begin{aligned}
(Tu)'(t) &= \int_t^1 (1-s)\phi_q\biggl[\int_0^1 \int_0^1 G(s,\tau)\lambda q(\tau)f\bigl(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\,dh(s) \\
&\quad + \int_0^1 G(s,\tau)\lambda q(\tau)f\bigl(s,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\biggr]\,ds \\
&\quad - \int_0^t s\phi_q\biggl[\int_0^1 \int_0^1 G(s,\tau)\lambda q(\tau)f\bigl(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\,dh(s) \\
&\quad + \int_0^1 G(s,\tau)\lambda q(\tau)f\bigl(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\biggr]\,ds
\end{aligned}$$

and

$$\begin{aligned}
(Tu)''(t) &= -\phi_q\biggl[\int_0^1 \int_0^1 G(t,\tau)\lambda q(\tau)f\bigl(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\,dh(t) \\
&\quad + \int_0^1 G(t,\tau)\lambda q(\tau)f\bigl(s,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\bigr)\,d\tau\biggr],
\end{aligned}$$

we have $(Tu)''(t) \le 0$ for $0 \le t \le 1$.

On the other hand, by (2.5), we obtain

$$\begin{aligned}
(Tu)(0) &= (Tu)(1) \\
&= \int_0^1 \int_0^1 G(s,\tau)\phi_q\biggl[\int_0^1 \int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\bigl(\zeta,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\bigr)\,d\zeta\,dh(\tau) \\
&\quad + \int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\bigl(s,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\bigr)\,d\zeta\biggr]\,d\tau\,dg(s)
\end{aligned}$$

$$+\int_0^1 G(0,s)\phi_q\left[\int_0^1\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\,dh(s)\right.$$
$$\left.+\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\right]ds\geq 0.$$

Then it follows that $Tu$ is concave and nonnegative on $[0,1]$.

Moreover, we show that $Tu$ is symmetric about $t$ on $[0,1]$.

In fact, from $(\mathrm{H}_1)$-$(\mathrm{H}_3)$, we get

$$\begin{aligned}
&(Tu)(1-t)\\
&\quad=\int_0^1\int_0^1 G(s,\tau)\phi_q\left[\int_0^1\int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\big(\zeta,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\big)\,d\zeta\,dh(\tau)\right.\\
&\qquad\left.+\int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\big(s,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\big)\,d\zeta\right]d\tau\,dg(s)\\
&\qquad+\int_0^1 G(1-t,s)\phi_q\left[\int_0^1\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\,dh(s)\right.\\
&\qquad\left.+\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\right]ds\\
&\quad=-\int_0^1\int_0^1 G(1-s,1-\tau)\phi_q\left[\int_0^1\int_0^1 G(1-\tau,\zeta)\lambda q(\zeta)\right.\\
&\qquad\cdot f\big(\zeta,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\big)\,d\zeta\,dh(1-\tau)\\
&\qquad\left.+\int_0^1 G(1-\tau,\zeta)\lambda q(\zeta)f\big(s,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\big)\,d\zeta\right]d\tau\,dg(1-s)\\
&\qquad+\int_0^1 G(1-t,1-s)\phi_q\left[\int_0^1\int_0^1 G(1-s,1-\tau)\lambda q(1-\tau)\right.\\
&\qquad\cdot f\big(1-\tau,u(1-\tau),u'(1-\tau),u''(1-\tau),u'''(1-\tau)\big)\,d\tau\,dh(1-s)\\
&\qquad+\int_0^1 G(1-s,1-\tau)\lambda q(1-\tau)\\
&\qquad\left.\cdot f\big(1-\tau,u(1-\tau),u'(1-\tau),u''(1-\tau),u'''(1-\tau)\big)\,d\tau\right]ds\\
&\quad=\int_0^1\int_0^1 G(s,\tau)\phi_q\left[\int_0^1\int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\big(\zeta,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\big)\,d\zeta\,dh(\tau)\right.\\
&\qquad\left.+\int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\big(s,u(\zeta),u'(\zeta),u''(\zeta),u'''(\zeta)\big)\,d\zeta\right]d\tau\,dg(s)\\
&\qquad+\int_0^1 G(t,s)\phi_q\left[\int_0^1\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\,dh(s)\right.\\
&\qquad\left.+\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\right]ds\\
&\quad=(Tu)(t).
\end{aligned}$$

Therefore, $Tu$ is symmetric about $t$ on $[0,1]$. Thus, $T:P\to P$.

It is obvious that $T$ is continuous. And the Arzela-Ascoli theorem guarantees that $T$ is compact. Then we see that $T$ is completely continuous. □

## 3 Existence of monotonic iteration positive symmetric solutions to (1.1)

For notational convenience, we denote

$$Q=\int_0^1 q(s)\,ds,\qquad A=\max\{1,4(q-1)\}\left(\frac{1}{4}\lambda Q\right)^{q-1}.$$

**Theorem 3.1** *Assume that* $(\mathrm{H}_1)$-$(\mathrm{H}_3)$ *hold, and there exists* $a>0$, *such that*

$(\mathrm{H}_4)$: $f(t,x_1,y_1,z_1,\delta_1)\le f(t,x_2,y_2,z_2,\delta_2)$ *for any* $0\le t\le 1$, $0\le x_1\le x_2\le a$, $0\le |y_1|\le |y_2|\le a$, $-a\le z_2\le z_1\le 0$, $0\le|\delta_1|\le|\delta_2|\le a$;

$(\mathrm{H}_5)$: $\max_{0\le t\le 1} f(t,a,a,-a,a)\le (\frac{a}{A})^{p-1}$;

$(\mathrm{H}_6)$: $f(t,0,0,0,0)\not\equiv 0$ *for* $0\le t\le 1$.

*Then the boundary value problem* (1.1) *has at least one positive symmetric concave solution* $w^*$ *or* $v^*$, *such that*

$$\begin{aligned}
&0\le w^*\le a,\qquad 0\le\left|\left(w^*\right)'\right|\le a,\\
&-a\le\left(w^*\right)''\le 0,\qquad 0\le\left|\left(w^*\right)'''\right|\le a,\\
&w^*=\lim_{n\to\infty}w_n=\lim_{n\to\infty}T^n w_0,\\
&\left(w^*\right)'=\lim_{n\to\infty}(w_n)'=\lim_{n\to\infty}\left(T^n w_0\right)',\\
&\left(w^*\right)''=\lim_{n\to\infty}(w_n)''=\lim_{n\to\infty}\left(T^n w_0\right)'',\\
&\left(w^*\right)'''=\lim_{n\to\infty}(w_n)'''=\lim_{n\to\infty}\left(T^n w_0\right)''',\\
&\quad \text{where } w_0(t)=a\left(\frac{4}{3}t^4-\frac{8}{3}t^3+\frac{4}{3}t+\frac{3}{8}\right),0\le t\le 1,
\end{aligned}$$

*and*

$$\begin{aligned}
&0\le v^*\le a,\qquad 0\le\left|\left(v^*\right)'\right|\le a,\\
&-a\le\left(v^*\right)''\le 0,\qquad 0\le\left|\left(v^*\right)'''\right|\le a,\\
&v^*=\lim_{n\to\infty}v_n=\lim_{n\to\infty}T^n v_0,\\
&\left(v^*\right)'=\lim_{n\to\infty}(v_n)'=\lim_{n\to\infty}\left(T^n v_0\right)',\\
&\left(v^*\right)''=\lim_{n\to\infty}(v_n)''=\lim_{n\to\infty}\left(T^n v_0\right)'',\\
&\left(v^*\right)'''=\lim_{n\to\infty}(v_n)'''=\lim_{n\to\infty}\left(T^n v_0\right)''',\\
&\quad \text{where } v_0(t)=0,0\le t\le 1,
\end{aligned}$$

*where* $(Tu)(t)$ *is defined by* (2.5).

The successive iterative scheme in the theorem is $w_0(t)=a(\frac{4}{3}t^4-\frac{8}{3}t^3+\frac{4}{3}t+\frac{3}{8})$, $w_{n+1}=Tw_n=T^n w_0$, $n=0,1,2,\ldots$, which starts off with a known simple quartic function or $v_0(t)=0$, $v_{n+1}=Tv_n=T^n v_0$, $n=0,1,2,\ldots$, which starts off with the zero function.

*Proof* Now in order to investigate the properties of the operator $T$, let us denote $\overline{P_a} = \{u \in P \mid \|u\| \leq a\}$. In the following, we will prove that $T : \overline{P_a} \to \overline{P_a}$ firstly.

If $u \in \overline{P_a}$, ($\mathrm{H}_4$), and ($\mathrm{H}_5$) implies that

$$\begin{aligned} 0 &\leq f\bigl(t, u(t), u'(t), u''(t), u'''(t)\bigr) \leq f(t, a, a, -a, a) \\ &\leq \max_{0\leq t\leq 1} f(t, a, a, -a, a) \leq \frac{a}{A} \quad \text{for } 0 \leq t \leq 1. \end{aligned}$$

Since

$$\begin{aligned} \max_{0\leq t\leq 1}\bigl|(Tu)(t)\bigr| &\leq \int_0^1 \int_0^1 \frac{1}{4}\phi_q\biggl[\int_0^1 \int_0^1 \frac{1}{4}\lambda q(\zeta) f\bigl(\zeta, u(\zeta), u'(\zeta), u''(\zeta), u'''(\zeta)\bigr)\,d\zeta\,dh(\tau) \\ &\quad + \int_0^1 \frac{1}{4}\lambda q(\zeta) f\bigl(\zeta, u(\zeta), u'(\zeta), u''(\zeta), u'''(\zeta)\bigr)\,d\zeta\biggr]\,d\tau\,dg(s) \\ &\quad + \int_0^1 \frac{1}{4}\phi_q\biggl[\int_0^1 \int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\,dh(s) \\ &\quad + \int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\biggr]\,ds \\ &= \frac{1}{4}\phi_q\biggl[\int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\biggr] \\ &\leq \frac{a}{A}\frac{1}{4}\biggl(\frac{1}{4}\lambda Q\biggr)^{q-1} < a, \\ \max_{0\leq t\leq 1}\bigl|(Tu)'(t)\bigr| &\leq \int_t^1 (1-s)\phi_q\biggl[\int_0^1 \int_0^1 G(s,\tau)\lambda q(\tau) \\ &\quad \cdot f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\,dh(s) \\ &\quad + \int_0^1 G(s,\tau)\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\biggr]\,ds \\ &\leq \int_t^1 (1-s)\phi_q\biggl[\int_0^1 \int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\,dh(s) \\ &\quad + \int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\biggr]\,ds \\ &\leq \int_t^1 (1-s)\phi_q\biggl[\int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\biggr]\,ds \\ &\leq \frac{a}{A}\frac{1}{2}\biggl(\frac{1}{4}\lambda Q\biggr)^{q-1} < a, \\ \max_{0\leq t\leq 1}\bigl|(Tu)''(t)\bigr| &\leq \phi_q\biggl[\int_0^1 \int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\,dh(t) \\ &\quad + \int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\biggr] \\ &\leq \phi_q\biggl[\int_0^1 \frac{1}{4}\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\,d\tau\biggr] \\ &\leq \frac{a}{A}\biggl(\frac{1}{4}\lambda Q\biggr)^{q-1} \leq a, \end{aligned}$$

and

$$\begin{aligned}
\max_{0\le t\le 1}\left|(Tu)'''(t)\right| &\le (q-1)\left[\int_0^1\int_0^1 \frac{1}{4}\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\,dh(t)\right.\\
&\quad\left.+\int_0^1 \frac{1}{4}\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\right]^{q-2}\\
&\quad\cdot\int_t^1 (1-\tau)\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\\
&\le (q-1)\left[\int_0^1 \frac{1}{4}\lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\right]^{q-2}\\
&\quad\cdot\int_0^1 \lambda q(\tau)f\big(\tau,u(\tau),u'(\tau),u''(\tau),u'''(\tau)\big)\,d\tau\\
&\le \frac{a}{A}4(q-1)\left(\frac{1}{4}\lambda Q\right)^{q-1}\le a,
\end{aligned}$$

we get $T:\overline{P_a}\to\overline{P_a}$.

Let $w_0(t)=a(\frac{4}{3}t^4-\frac{8}{3}t^3+\frac{4}{3}t+\frac{3}{8})$, $0\le t\le 1$, then $w_0(t)\in\overline{P_a}$. Let $w_1=Tw_0$, then $w_1\in\overline{P_a}$. We denote $w_{n+1}=Tw_n$, $n=0,1,2,\ldots$. Then we have $w_n\subseteq\overline{P_a}$, $n=1,2,\ldots$. Since $T$ is completely continuous, we assert that $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Next, we investigate the convergence property of the iterative scheme; since

$$\begin{aligned}
w_1(t) &= Tw_0(t)\\
&= \int_0^1\int_0^1 G(s,\tau)\phi_q\left[\int_0^1\int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\big(\zeta,w_0(\zeta),w_0'(\zeta),w_0''(\zeta),w_0'''(\zeta)\big)\,d\zeta\,dh(\tau)\right.\\
&\quad\left.+\int_0^1 G(\tau,\zeta)\lambda q(\zeta)f\big(s,w_0(\zeta),w_0'(\zeta),w_0''(\zeta),w_0'''(\zeta)\big)\,d\zeta\right]d\tau\,dg(s)\\
&\quad+\int_0^1 G(t,s)\phi_q\left[\int_0^1\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,w_0(\tau),w_0'(\tau),w_0''(\tau),w_0'''(\tau)\big)\,d\tau\,dh(s)\right.\\
&\quad\left.+\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,w_0(\tau),w_0'(\tau),w_0''(\tau),w_0'''(\tau)\big)\,d\tau\right]ds\\
&\le \int_0^1 G(t,s)\phi_q\left[\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,w_0(\tau),w_0'(\tau),w_0''(\tau),w_0'''(\tau)\big)\,d\tau\right]ds\\
&\le a\left(\frac{4}{3}t^4-\frac{8}{3}t^3+\frac{4}{3}t+\frac{3}{8}\right),\quad 0\le t\le 1,
\end{aligned}$$

$$\begin{aligned}
\left|w_1'(t)\right| &= \left|(Tw_0)'(t)\right|\\
&= \left|\int_t^1 (1-s)\phi_q\left[\int_0^1\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,w_0(\tau),w_0'(\tau),w_0''(\tau),w_0'''(\tau)\big)\,d\tau\,dh(s)\right.\right.\\
&\quad\left.+\int_0^1 G(s,\tau)\lambda q(\tau)f\big(s,w_0(\tau),w_0'(\tau),w_0''(\tau),w_0'''(\tau)\big)\,d\tau\right]ds\\
&\quad-\int_0^t s\phi_q\left[\int_0^1\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,w_0(\tau),w_0'(\tau),w_0''(\tau),w_0'''(\tau)\big)\,d\tau\,dh(s)\right.\\
&\quad\left.\left.+\int_0^1 G(s,\tau)\lambda q(\tau)f\big(\tau,w_0(\tau),w_0'(\tau),w_0''(\tau),w_0'''(\tau)\big)\,d\tau\right]ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left| \int_t^1 (1-s)\phi_q \left[ \int_0^1 G(s,\tau)\lambda q(\tau) f\bigl(s, w_0(\tau), w_0'(\tau), w_0''(\tau), w_0'''(\tau)\bigr)\, d\tau \right] ds \right| \\
&\leq a\left(\frac{16}{3}t^3 - 8t^2 + \frac{4}{3}\right), \quad 0 \leq t \leq 1,
\end{aligned}
$$

$$
\begin{aligned}
w_1''(t) &= (Tw_0)''(t) \\
&= -\phi_q \left[ \int_0^1 \int_0^1 G(t,\tau)\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau\, dh(t) \right. \\
&\quad \left. + \int_0^1 G(t,\tau)\lambda q(\tau) f\bigl(s, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau \right] \\
&\geq -\phi_q \left[ \int_0^1 G(t,\tau)\lambda q(\tau) f\bigl(s, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau \right] \\
&\geq a\bigl(16t^2 - 16t\bigr), \quad 0 \leq t \leq 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\bigl|w_1'''(t)\bigr| &= \bigl|(Tw_0)'''(t)\bigr| \\
&\leq (q-1)\left[ \int_0^1 \int_0^1 G(t,\tau)\lambda q(\tau) f\bigl(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau\, dh(t) \right. \\
&\quad \left. + \int_0^1 G(t,\tau)\lambda q(\tau) f\bigl(s, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau \right]^{q-2} \\
&\quad \cdot \left( \int_t^1 (1-\tau)\lambda q(\tau) f\bigl(s, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau \right. \\
&\quad \left. - \int_0^t \tau\lambda q(\tau) f\bigl(s, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau \right) \\
&\leq (q-1)\left[ \int_0^1 G(t,\tau)\lambda q(\tau) f\bigl(s, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau \right]^{q-2} \\
&\quad \cdot \int_t^1 (1-\tau)\lambda q(\tau) f\bigl(s, u(\tau), u'(\tau), u''(\tau), u'''(\tau)\bigr)\, d\tau \\
&\leq \frac{a}{A}(q-1)\left(\frac{1}{4}\lambda Q\right)^{q-1} 16t \leq 32at, \quad 0 \leq t \leq 1.
\end{aligned}
$$

We get

$$
\begin{aligned}
&w_2(t) = Tw_1(t) \leq Tw_0(t) = w_1(t), \\
&\bigl|w_2'(t)\bigr| = \bigl|(Tw_1)'(t)\bigr| \leq \bigl|(Tw_0)'(t)\bigr| = \bigl|w_1'(t)\bigr|, \\
&w_2''(t) = (Tw_1)''(t) \geq (Tw_0)''(t) = w_1''(t), \\
&\bigl|w_2'''(t)\bigr| = \bigl|(Tw_1)'''(t)\bigr| \leq \bigl|(Tw_0)'''(t)\bigr| = \bigl|w_1'''(t)\bigr|, \quad 0 \leq t \leq 1.
\end{aligned}
$$

By induction, the iterative scheme is clear, then

$$
\begin{aligned}
&w_{n+1} \leq w_n, \qquad \bigl|w_{n+1}'(t)\bigr| \leq \bigl|w_n'(t)\bigr|, \qquad w_{n+1}''(t) \geq w_n''(t), \\
&\bigl|w_{n+1}'''(t)\bigr| \leq \bigl|w_n'''(t)\bigr|, \quad 0 \leq t \leq 1, n = 0, 1, 2, \ldots.
\end{aligned}
$$

Thus, we see that there exists $w^* \in \overline{P_a}$ such that $w_n \to w^*$. Combining with the continuity of $T$ and $w_{n+1} = Tw_n$, we obtain $Tw^* = w^*$.

On the other hand, another way to approach this is to start off with the zero function. Let $v_0(t) = 0$, $0 \le t \le 1$, then $v_0(t) \in \overline{P_a}$. Let $v_1 = Tv_0$, then $v_1 \in \overline{P_a}$. We denote $v_{n+1} = Tv_n$, $n = 0,1,2,\ldots$. Then we have $v_n \subseteq \overline{P_a}$, $n = 1,2,\ldots$. Since $T$ is completely continuous, we assert that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set.

In a similar way, since $v_1 = Tv_0 \in \overline{P_a}$, we have $v_1(t) = Tv_0(t) \ge 0$, $|v_1'(t)| = |(Tv_0)'(t)| \ge 0$, $v_1''(t) = (Tv_0)''(t) \le 0$, $|v_1'''(t)| = |(Tv_0)'''(t)| \ge 0$, for $0 \le t \le 1$. Then $v_2(t) \ge v_1(t)$, $|v_2'(t)| \ge |v_1'(t)|$, $v_2''(t) \le v_1''(t)$, $|v_2'''(t)| \ge |v_1'''(t)|$, for $0 \le t \le 1$. By an induction argument similar to the above we easily obtain

$$\begin{aligned} &v_{n+1} \ge v_n, \qquad \left|v_{n+1}'(t)\right| \ge \left|v_n'(t)\right|, \qquad v_{n+1}''(t) \le v_n''(t), \\ &\left|v_{n+1}'''(t)\right| \ge \left|v_n'''(t)\right|, \quad 0 \le t \le 1, n = 0,1,2,\ldots. \end{aligned}$$

Hence there exists $v^* \in \overline{P_a}$ such that $v_n \to v^*$. Combining with the continuity of $T$ and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.

The assumption $(\mathrm{H}_6)$ indicates that $f(t,0,0,0,0) \not\equiv 0$, $0 \le t \le 1$, then the zero function is not the solution of (1.1). Thus we have $v^* > 0$, for $0 < t < 1$.

It is well known that each fixed point of $T$ in $P$ is a solution of (1.1). Hence, we assert that the boundary value problem (1.1) has at least one positive symmetric concave solution $w^*$ or $v^*$.

The proof is completed. □

**Remark 3.1** If $\lim_{n\to\infty} w_n \neq \lim_{n\to\infty} v_n$, then $w^*$ and $v^*$ are two positive symmetric concave solutions of the problem (1.1). And if $\lim_{n\to\infty} w_n = \lim_{n\to\infty} v_n$, then $w^* = v^*$ is a positive symmetric concave solution of the problem (1.1). Anyway, the problem (1.1) has at least one positive symmetric concave solution.

The following corollary follows easily.

**Corollary 3.1** *Assume that* $(\mathrm{H}_1)$-$(\mathrm{H}_4)$ *and* $(\mathrm{H}_6)$ *hold, and there exists* $a > 0$, *such that*

$(\mathrm{H}_7)$: $\underline{\lim}_{\ell\to+\infty} \max_{0\le t\le 1} \frac{f(t,\ell,a,-a,a)}{\ell^{p-1}} \le \frac{1}{A^{p-1}}$ (*particularly*, $\underline{\lim}_{\ell\to+\infty} \max_{0\le t\le 1} \frac{f(t,\ell,a,-a,a)}{\ell^{p-1}} = 0$).

*Then the boundary value problem* (1.1) *has at least one positive symmetric concave solution* $w^*$ *or* $v^*$, *such that the conclusion of Theorem* 3.1 *hold*.

## 4 Example

In what follows, we discuss an example and simulations. Our purpose is to illustrate the main results of the previous arguments.

**Example 4.1** Let $p = 3$, $q(t) = 1$, $h(t) = g(t) = t(1-t)$, we consider the following boundary value problem

$$\begin{cases} (\phi_p(u''(t)))''(t) = f(t,u(t),u'(t),u''(t),u'''(t)), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 u(s)\,d(s(1-s)), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = \int_0^1 \phi_p(u''(s))\,d(s(1-s)), \end{cases} \tag{4.1}$$

where $f(t,x,y,z,\delta) = t(t-1) + \frac{1}{4}x + \frac{1}{4}y^2 - \frac{1}{4}z + \frac{1}{4}\delta^2$. Choose $a = 2$, then we have $A = 1$.

So by Theorem 3.1, the boundary value problem (4.1) has at least one positive symmetric concave solution $w^*$ or $v^*$, such that

$$\begin{aligned}
&0 \le w^* \le 2, \qquad 0 \le \left|\left(w^*\right)'\right| \le 2,\\
&-2 \le \left(w^*\right)'' \le 0, \qquad 0 \le \left|\left(w^*\right)'''\right| \le 2,\\
&w^* = \lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0,\\
&\left(w^*\right)' = \lim_{n\to\infty} (w_n)' = \lim_{n\to\infty} \left(T^n w_0\right)',\\
&\left(w^*\right)'' = \lim_{n\to\infty} (w_n)'' = \lim_{n\to\infty} \left(T^n w_0\right)'',\\
&\left(w^*\right)''' = \lim_{n\to\infty} (w_n)''' = \lim_{n\to\infty} \left(T^n w_0\right)''',\\
&\quad \text{where } w_0(t) = \frac{8}{3}t^4 - \frac{16}{3}t^3 + \frac{8}{3}t + \frac{3}{4}, 0 \le t \le 1,
\end{aligned}$$

and

$$\begin{aligned}
&0 \le v^* \le 2, \qquad 0 \le \left|\left(v^*\right)'\right| \le 2,\\
&-2 \le \left(v^*\right)'' \le 0, \qquad 0 \le \left|\left(v^*\right)'''\right| \le 2,\\
&v^* = \lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0,\\
&\left(v^*\right)' = \lim_{n\to\infty} (v_n)' = \lim_{n\to\infty} \left(T^n v_0\right)',\\
&\left(v^*\right)'' = \lim_{n\to\infty} (v_n)'' = \lim_{n\to\infty} \left(T^n v_0\right)'',\\
&\left(v^*\right)''' = \lim_{n\to\infty} (v_n)''' = \lim_{n\to\infty} \left(T^n v_0\right)''',\\
&\quad \text{where } v_0(t) = 0, 0 \le t \le 1,
\end{aligned}$$

where $(Tu)(t)$ is defined by (2.5).

**Competing interests**
The author declares that they have no competing interests.

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