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Monotone iterative technique for causal differential equations with upper and lower solutions in the reversed order

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Abstract

In this paper, we use monotone iterative technique in the presence of (coupled) upper and lower solutions in the reversed order to discuss the existence of extremal solutions (quasi-solutions) for causal differential equations with nonlinear boundary conditions. Two examples are provided to illustrate the efficiency of the obtained results.

MSC: 39A10; 34B37

Keywords: monotone iterative technique; upper and lower solutions; reversed order; causal operators

1 Introduction

Causal differential equations are recognized as an excellent model for real world problems; compared with the traditional model [1], one has wider real-time applications in a variety of disciplines. Its theory also has the powerful quality of unifying ordinary differential equations, integro differential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral equations. For more information, the reader can refer to the monograph by Lakshmikantham [2] and to [3–5].

The monotone iterative technique is an effective and important tool to prove existence results for initial and boundary value problems [6] and nonlinear boundary value problems [7–10]. In the last decades, this method has been extended to causal differential equations; see for example [11–13]. It is important to indicate that this method combined with the method of upper and lower solutions is an interesting and powerful mechanism that offers the theoretical as well as constructive existence results for nonlinear problems in a closed set, generated by the lower and upper solutions. Since recently, there are numerous results in studying boundary value problems for ordinary differential equations in the presence of a lower solution α and an upper solution β with $\alpha \leq \beta$. But in many cases, the upper and lower solutions occur in the reversed order, that is, $\beta \leq \alpha$. However, only a few works discuss the existence results for the non-ordered case [14–16]. In this paper, we consider the following casual differential equations under the assumption of the existing upper and lower solutions in the reversed order, which is different from the classical lower and upper solutions used in [17, 18]. The type of the equation is as follows:

$$\begin{cases} u'(t) = (Qu)(t), & t \in J, \\ g(u(0), u(T)) = 0, \end{cases} \tag{1}$$

where $J = [0, T]$, $T > 0$, $E = C(J, \mathbb{R})$, $Q \in C(E, E)$ is a causal operator, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Our boundary conditions is given by a nonlinear function, and more general than ones given before. This paper is organized as follows. In Section 2, we prove a new comparison principle. In Section 3, we show the existence and uniqueness of the solutions for the linear problem of (1). Then by using the monotone iterative technique coupled with the upper and lower solutions in the reversed order, we obtain the existence of extremal solutions for problem (1). In Section 4, using the notion of coupled upper and lower solutions in the reversed order, the existence of coupled minimal and maximal quasi-solutions for (1) is established. Finally, two examples are given to illustrate our results.

2 Comparison results

In this section, we present a definition and a lemma which help to prove our main results.

Put $E = C(J, \mathbb{R})$, $J = [0, T]$, and $\Omega = E \cap C^1(J, \mathbb{R})$ are Banach spaces with the respective norms:

$$\|y\| = \max_{t \in J} |y(t)|.$$

A function $u \in \Omega$ is called a solution of (1) if it satisfies (1).

Definition 2.1 Suppose that $Q \in C(E, E)$, then Q is said to be a causal map or a nonanticipative map if $u(s) = v(s)$, $t_0 \leq s \leq t \leq T$, where $u, v \in E$, then

$$(Qu)(s) = (Qv)(s), \quad t_0 \leq s \leq t.$$

Lemma 2.1 *Let* $m \in \Omega$ *and*

$$\begin{cases} m'(t) \geq M(t)m(t) + (\mathcal{L}m)(t), & t \in J = [0, T], \\ \lambda m(0) \geq m(T), \end{cases} \tag{2}$$

where $M \in C(\mathbb{R}, \mathbb{R}^+)$ *and* $\mathcal{L} \in C(E, E)$ *is a positive linear operator.*

In addition, we assume that

$$\int_0^T \big(M(t) + (\mathcal{L}1)(t)\big)\, dt \leq \frac{\lambda}{\lambda + 1}, \quad 0 < \lambda \leq 1. \tag{3}$$

Then $m(t) \leq 0$ *for* $t \in J$.

Proof Suppose that $m(t) \leq 0, t \in J$ is not true, then we have the following two cases:

Case 1: there exists $\bar{t} \in J$ such that $m(\bar{t}) > 0$ and $m(t) \leq 0$ for all $t \in J \backslash \{\bar{t}\}$.

By (2), we know that $m'(t) \geq 0$ on J and $m(t)$ is nondecreasing on J, thus we have

$$\begin{aligned} m(t) &= m(0) + \int_0^t m'(s)\,ds \\ &\geq m(0) + \int_0^t \bigl(M(s)m(s) + (\mathcal{L}m)(s)\bigr)\,ds \\ &\geq m(0)\left(1 + \int_0^t \bigl(M(s) + (\mathcal{L}1)(s)\bigr)\,ds\right). \end{aligned}$$

Therefore,

$$\lambda m(0) \geq m(T) \geq m(0)\left(1 + \int_0^T \bigl(M(t) + (\mathcal{L}1)(t)\bigr)\,dt\right) > m(0),$$

so $\lambda > 1$, which is a contradiction.

Case 2: there exist t_* and t^* such that $m(t_*) < 0$ and $m(t^*) > 0$.

Let $\min_{t\in J} m(t) = -r$, $r > 0$. Without loss of generality, we suppose $m(t_*) = -r$. From (2) we get

$$\begin{aligned} m(t) &\geq m(0) + \int_0^t \bigl(M(s)m(s) + (\mathcal{L}m)(s)\bigr)\,ds \\ &\geq m(0) - r\int_0^t \bigl(M(s) + (\mathcal{L}1)(s)\bigr)\,ds. \end{aligned}$$

Set $t = t_*$, we have

$$-r \geq m(0) - r\int_0^T \bigl(M(s) + (\mathcal{L}1)(s)\bigr)\,ds,$$

thus, we obtain

$$m(0) \leq -r + r\int_0^T \bigl(M(s) + (\mathcal{L}1)(s)\bigr)\,ds.$$

On the other hand,

$$m(t) = m(T) - \int_t^T m'(s)\,ds.$$

Take $t = t^*$, we have

$$0 < m\bigl(t^*\bigr) = m(T) - \int_{t^*}^T m'(s)\,ds.$$

Then

$$m(T) > \int_{t^*}^T m'(s)\,ds \geq -r\int_0^T \bigl(M(s) + (\mathcal{L}1)(s)\bigr)\,ds.$$

Using the fact $\lambda m(0) \geq m(T)$, we get

$$-r\lambda + r\lambda\int_0^T \bigl(M(s) + (\mathcal{L}1)(s)\bigr)\,ds \geq \lambda m(0) \geq m(T) > -r\int_0^T \bigl(M(s) + (\mathcal{L}1)(s)\bigr)\,ds.$$

A contradiction is then elicited due to (3). Hence $m(t) \leq 0$, and this completes the proof. □

3 Extremal solutions

In this section, we shall establish the existence of extremal solutions of problem (1).

Definition 3.1 Functions $\alpha, \beta \in \Omega$ are called lower and upper solutions of (1) if

$$\begin{cases} \alpha'(t) \leq (Q\alpha)(t), & t \in J, \\ g(\alpha(0), \alpha(T)) \leq 0, \end{cases}$$

and

$$\begin{cases} \beta'(t) \geq (Q\beta)(t), & t \in J, \\ g(\beta(0), \beta(T)) \geq 0. \end{cases}$$

Now we state our theorems. First we discuss the existence of solutions for the following linear problem:

$$\begin{cases} u'(t) = M(t)u(t) + (\mathcal{L}u)(t) + \sigma_\eta(t), & t \in J, \\ g(\eta(0), \eta(T)) + M_1(u(0) - \eta(0)) - M_2(u(T) - \eta(T)) = 0, \end{cases} \tag{4}$$

where $\sigma_\eta(t) = (Q\eta)(t) - M(t)\eta(t) - (\mathcal{L}\eta)(t)$.

Theorem 3.1 *A function $u \in \Omega$ is a solution of* (4) *if and only if u is a solution of the following integral equation*:

$$u(t) = \frac{B_\eta e^{\int_0^t M(s)\,ds}}{M_1 - M_2 e^{\int_0^T M(s)\,ds}} + \int_0^T G(t,s)\big(\sigma_\eta(s) + (\mathcal{L}u)(s)\big)\,ds, \tag{5}$$

where $B_\eta = -g(\eta(0), \eta(T)) + M_1\eta(0) - M_2\eta(T)$, $M \in C(\mathbb{R}, \mathbb{R}^+)$, M_1, M_2 are constants satisfying $M_1 \neq M_2 e^{\int_0^T M(s)\,ds}$ and

$$G(t,s) = \frac{1}{M_1 - M_2 e^{\int_0^T M(s)\,ds}} \begin{cases} M_1 e^{\int_s^t M(r)\,dr}, & 0 \leq s < t \leq T, \\ M_2 e^{\int_s^T M(r)\,dr} e^{\int_s^t M(r)\,dr}, & 0 \leq t \leq s \leq T. \end{cases}$$

Proof Assume $u \in \Omega$ is a solution of (4). Set $u(t) = v(t)e^{\int_0^t M(s)\,ds}$, we see that $v(t)$ satisfies

$$\begin{cases} v'(t) = (\sigma_\eta(t) + (\mathcal{L}u)(t))e^{\int_0^t -M(s)\,ds}, \\ v(0) = \frac{B_\eta}{M_1} + \frac{M_2}{M_1} v(T) e^{\int_0^T M(s)\,ds}. \end{cases} \tag{6}$$

By using (6), we have

$$v(t) = v(0) + \int_0^t \big(\sigma_\eta(s) + (\mathcal{L}u)(s)\big)e^{\int_0^s -M(r)\,dr}\,ds. \tag{7}$$

If we set $t = T$ in (7), then we get

$$v(T) = v(0) + \int_0^T \big(\sigma_\eta(s) + (\mathcal{L}u)(s)\big)e^{\int_0^s -M(r)\,dr}\,ds. \tag{8}$$

From the boundary condition $v(T) = \frac{M_1 v(0) - B_\eta}{M_2 e^{\int_0^T M(t)\,dt}}$, we obtain

$$v(0) = \frac{B_\eta}{M_1 - M_2 e^{\int_0^T M(t)\,dt}} + \frac{M_2 e^{\int_0^T M(t)\,dt}}{M_1 - M_2 e^{\int_0^T M(t)\,dt}} \int_0^T \bigl(\sigma_\eta(s) + (\mathcal{L}u)(s)\bigr) e^{\int_0^s -M(r)\,dr}\,ds. \tag{9}$$

Substituting (9) into (7) and using $v(t) = u(t)e^{\int_0^t -M(s)\,ds}$, $t \in J$, we have

$$\begin{aligned} u(t) = {} & \frac{B_\eta e^{\int_0^t M(s)\,ds}}{M_1 - M_2 e^{\int_0^T M(t)\,dt}} + \frac{M_1}{M_1 - M_2 e^{\int_0^T M(t)\,dt}} \int_0^T \bigl(\sigma_\eta(s) + (\mathcal{L}u)(s)\bigr) e^{\int_s^t M(r)\,dr}\,ds \\ & + \frac{M_2 e^{\int_0^T M(t)\,dt}}{M_1 - M_2 e^{\int_0^T M(t)\,dt}} \int_0^T \bigl(\sigma_\eta(s) + (\mathcal{L}u)(s)\bigr) e^{\int_s^t M(r)\,dr}\,ds. \end{aligned}$$

Let

$$G(t,s) = \frac{1}{M_1 - M_2 e^{\int_0^T M(t)\,dt}} \begin{cases} M_1 e^{\int_s^t M(r)\,dr}, & 0 \le s < t \le T, \\ M_2 e^{\int_0^T M(t)\,dt} e^{\int_s^t M(r)\,dr}, & 0 \le t \le s \le T, \end{cases}$$

we see that u is a solution of (5). The proof is complete. □

In the following paper, we denote $\xi = \|G(t,s)\| = \max\{|\frac{M_1 e^{\int_0^T M(t)\,dt}}{M_1 - M_2 e^{\int_0^T M(t)\,dt}}|, |\frac{M_2 e^{\int_0^T M(t)\,dt}}{M_1 - M_2 e^{\int_0^T M(t)\,dt}}|\}$.

Theorem 3.2 *Assume that $M \in C(\mathbb{R}, \mathbb{R}^+)$, $M_1 \neq M_2 e^{\int_0^T M(s)\,ds}$, and*

$$\xi \|\mathcal{L}\| T < 1. \tag{10}$$

Then problem (4) *has a unique solution.*

Proof By Theorem 3.1, we know that $u \in \Omega$ is the solution of (4) if and only if u is a solution of the integral equation (5). Now we prove (5) has a unique solution $u \in \Omega$. Define an operator $F : \Omega \to \Omega$ by

$$(Fu)(t) = \frac{B_\eta e^{\int_0^t M(s)\,ds}}{M_1 - M_2 e^{\int_0^T M(s)\,ds}} + \int_0^T G(t,s)\bigl(\sigma_\eta(s) + (\mathcal{L}u)(s)\bigr)\,ds.$$

For any $u_1, u_2 \in \Omega$, we have

$$|Fu_1 - Fu_2| \le \int_0^T \bigl|G(t,s)\bigr| \bigl|\bigl(\mathcal{L}(u_1 - u_2)\bigr)(t)\bigr| \le \xi T \|\mathcal{L}\| \|u_1 - u_2\|.$$

Hence, $\|Fu_1 - Fu_2\| = \max_{t\in J} |Fu_1(t) - Fu_2(t)| = \tau \|u_1 - u_2\|$, where

$$\tau = \xi T \|\mathcal{L}\|.$$

By (10) and the Banach contraction principle, F has a unique fixed point. It is clear that this fixed point is the solution of (5). The proof is complete. □

Theorem 3.3 *Let* (3), (10) *hold and* $Q \in C[E,E]$. *In addition, we assume that*

(H_1) *the functions* $\alpha, \beta \in \Omega$ *are lower and upper solutions of problem* (1), *respectively, such that* $\beta \leq \alpha$;

(H_2) $\mathcal{L} \in C(E,E)$ *is a positive linear operator and* $M \in C(\mathbb{R}, \mathbb{R}^+)$ *such that*

$$(Qu)(t) - (Qv)(t) \leq M(t)\big(u(t) - v(t)\big) + \big(\mathcal{L}(u-v)\big)(t), \quad \text{for } \beta \leq v \leq u \leq \alpha;$$

(H_3) *there exist* $M_2 \geq M_1 > 0$ *satisfying* $M_1 \neq M_2 e^{\int_0^T M(s)\,ds}$, *and*

$$g(\bar{u}, \bar{v}) - g(u,v) \geq M_1(\bar{u} - \bar{v}) - M_2(u - v),$$

whenever $\beta(0) \leq u(0) \leq \bar{u}(0) \leq \alpha(0), \beta(T) \leq v(T) \leq \bar{v}(T) \leq \alpha(T)$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$ *with* $\alpha_0 = \alpha, \beta_0 = \beta$, *which converge to the extremal solutions of problem* (1) *in the sector* $[\beta, \alpha] = \{u \in C^1(J, \mathbb{R}) : \beta(t) \leq u(t) \leq \alpha(t), t \in J\}$.

Proof First, we construct two sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ which satisfy the following equations:

$$\begin{cases} \alpha_n'(t) = M(t)\alpha_n(t) + (\mathcal{L}\alpha_n)(t) + (Q\alpha_{n-1})(t) - M(t)\alpha_{n-1}(t) - (\mathcal{L}\alpha_{n-1})(t), \\ g(\alpha_{n-1}(0), \alpha_{n-1}(T)) + M_1(\alpha_n(0) - \alpha_{n-1}(0)) - M_2(\alpha_n(T) - \alpha_{n-1}(T)) = 0, \end{cases} \tag{11}$$

and

$$\begin{cases} \beta_n'(t) = M(t)\beta_n(t) + (\mathcal{L}\beta_n)(t) + (Q\beta_{n-1})(t) - M(t)\beta_{n-1}(t) - (\mathcal{L}\beta_{n-1})(t), \\ g(\beta_{n-1}(0), \beta_{n-1}(T)) + M_1(\beta_n(0) - \beta_{n-1}(0)) - M_2(\beta_n(T) - \beta_{n-1}(T)) = 0, \end{cases} \tag{12}$$

for $n = 1, 2, \ldots$, where $\alpha_0 = \alpha$, $\beta_0 = \beta$.

It follows from Theorem 3.2 that both (11) and (12) have a unique solution, respectively. Then we complete the proof by four steps.

Step 1 We prove that $\beta_{n-1} \leq \beta_n$ and $\alpha_n \leq \alpha_{n-1}$, $n = 1, 2, \ldots$.

Set $m = \alpha_1 - \alpha$, $t \in J$. Employing (H_1), we have

$$\begin{aligned} m'(t) &= \alpha_1'(t) - \alpha'(t) \\ &\geq M(t)\alpha_1(t) + (\mathcal{L}\alpha_1)(t) + (Q\alpha)(t) - M(t)\alpha(t) - (\mathcal{L}\alpha)(t) - (Q\alpha)(t) \\ &= M(t)m(k) + (\mathcal{L}m)(t), \quad t \in J, \end{aligned}$$

and

$$m(0) = -\frac{1}{M_1} g\big(\alpha(0), \alpha(T)\big) + \frac{M_2}{M_1}\big(\alpha_1(T) - \alpha(T)\big) + \alpha(0) - \alpha(0) \geq \frac{M_2}{M_1} m(T).$$

From Lemma 2.1, we get $m(t) \leq 0, t \in J$, so $\alpha_1(t) \leq \alpha(t)$.

By mathematical induction, we obtain the sequence α_n is a non-increasing sequence. Analogously, we can show β_n is a nondecreasing sequence.

Step 2 We show that $\beta_1 \leq \alpha_1$ if $\beta \leq \alpha$.

Let $m = \beta_1 - \alpha_1$. Using (H_1), (H_2), and (H_3), we get

$$\begin{aligned} m'(t) &= \beta_1'(t) - \alpha_1'(t) \\ &= M(t)\beta_1(t) + (\mathcal{L}\beta_1)(t) + (Q\beta)(t) - M(t)\beta(t) - (\mathcal{L}\beta)(t) \\ &\quad - M(t)\alpha_1(t) - (\mathcal{L}\alpha_1)(t) - (Q\alpha)(t) + M(t)\alpha(t) + (\mathcal{L}\alpha)(t) \\ &\geq M(t)m(t) + (\mathcal{L}m)(t), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} m(0) &= \beta_1(0) - \alpha_1(0) \\ &= -\frac{1}{M_1} g\bigl(\beta(0), \beta(T)\bigr) + \frac{M_2}{M_1}\bigl(\beta_1(T) - \beta(T)\bigr) + \beta(0) \\ &\quad - \left[\frac{1}{M_1} g\bigl(\alpha(0), \alpha(T)\bigr) + \frac{M_2}{M_1}\bigl(\alpha_1(T) - \alpha(T)\bigr) + \alpha(0)\right] \\ &\geq \frac{M_2}{M_1} m(T). \end{aligned}$$

Then based on Lemma 2.1, we have $m \leq 0$, which implies $\beta_1 \leq \alpha_1$. By mathematical induction, we obtain $\beta_n \leq \alpha_n$, $n = 1, 2, \ldots$.

Step 3 We prove that there exists a solution of problem (1) that satisfies $\beta(t) \leq u(t) \leq \alpha(t)$ in J.

By the first two steps, we get

$$\beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \leq \cdots \leq \alpha_n \leq \cdots \leq \alpha_2 \leq \alpha_1 \leq \alpha_0, \tag{13}$$

and each α_n, β_n satisfies (11) and (12). It is easy to see that the sequence $\{\beta_n(t)\}$ is uniformly bounded and equicontinuous, employing the Ascoli-Arzela theorem, the nondecreasing sequences $\{\beta_n(t)\}$ converges pointwise to a function $u(t)$ that satisfies $\beta(t) \leq u(t) \leq \alpha(t)$. Therefore, there exists a solution $u(t)$ of problem (1) that satisfies $\beta(t) \leq u(t) \leq \alpha(t)$ in J.

Step 4 We prove that there exist extremal solutions of problem (1) in $[\beta, \alpha]$.

Apparently, from (13), there exist ρ and r such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$ and $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly on J. Clearly, $\rho(t), r(t)$ satisfy problem (1). Let $u(t)$ be any solution of (1) such that $\beta(t) \leq u(t) \leq \alpha(t)$. Suppose that there exists a positive integer j such that $\beta_n(t) \leq u(t) \leq \alpha_n(t)$. Then, setting $m = \beta_{n+1} - u$, we have

$$\begin{aligned} m'(t) &= \Delta\beta_{n+1}(t) - u'(t) \\ &= M(t)\beta_{n+1}(t) + (\mathcal{L}\beta_{n+1})(t) + (Q\beta_n)(t) - M(t)\beta_n(t) - (\mathcal{L}\beta_n)(t) - (Qy)(t) \\ &\geq M(t)m(t) + (\mathcal{L}m)(t), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} m(0) &= \beta_{n+1}(0) - y(0) \\ &= -\frac{1}{M_1} g\bigl(\beta_n(0), \beta_n(T)\bigr) + \frac{M_2}{M_1}\bigl(\beta_{n+1}(T) - \beta_n(T)\bigr) \end{aligned}$$

$$
\begin{aligned}
&\quad + \beta_n(0) - y(0) + \frac{1}{M_1} g\bigl(y(0), y(T)\bigr) \\
&\geq \frac{M_2}{M_1} m(T).
\end{aligned}
$$

By Lemma 2.1, $m \leq 0$, *i.e.*, $\beta_{n+1} \leq u$. Similarly, we can get $u \leq \alpha_{n+1}$ on J. Since $\beta_0(t) \leq y(t) \leq \alpha_0(t)$, by induction we have $\beta_n \leq u \leq \alpha_n$, which implies $r(t) \leq u(t) \leq \rho(t)$. The proof is complete. □

4 Coupled lower and upper solutions

Definition 4.1 We say that $\alpha, \beta \in \Omega$ are called coupled lower and upper solutions of (1) if

$$
\begin{cases}
\alpha'(t) \leq (Q\alpha)(t), & t \in J, \\
g(\alpha(0), \beta(T)) \leq 0,
\end{cases}
$$

and

$$
\begin{cases}
\beta'(t) \geq (Q\beta)(t), & t \in J, \\
g(\beta(0), \alpha(T)) \geq 0.
\end{cases}
$$

Definition 4.2 Relative to the causal differential equations (1), $u, v \in \Omega$ are said to be coupled quasi-solution solutions if

$$
\begin{cases}
u'(t) = (Qu)(t), & t \in J, \\
g(u(0), v(T)) = 0,
\end{cases}
$$

$$
\begin{cases}
v'(t) = (Qv)(t), & t \in J, \\
g(v(0), u(T)) = 0.
\end{cases}
$$

Definition 4.3 Coupled quasi-solution $\rho, r \in C^1(J, \mathbb{R})$ are called coupled minimal and maximal coupled quasi-solution of problem (1), if for any coupled quasi-solution u, v, we have $\rho(t) \leq u(t), v(t) \leq r(t)$ on J.

Theorem 4.1 *Assume that* (H_2), (3), (10) *hold and* $Q \in C[E, E]$. *In addition, suppose that*

(H_4) $\alpha, \beta \in \Omega$ *are coupled lower and upper solutions of problem* (1) *such that* $\beta \leq \alpha$;
(H_5) *the function* $g(u, v) \in C(\mathbb{R}^2, \mathbb{R})$ *is non-increasing in the second variable and*

$$
g(\bar{u}, v) - g(u, v) \leq M_1(\bar{u} - u), \quad \textit{for } \beta(0) \leq u(0) \leq \bar{u}(0) \leq \alpha(0),
$$

where $M_2 \geq M_1 > 0$ *and* $M_1 \neq M_2 e^{\int_0^T M(s)\,ds}$.

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$ *with* $\alpha_0 = \alpha, \beta_0 = \beta$, *such that* $\lim_{n\to\infty} \beta_n(t) = \rho(t), \lim_{n\to\infty} \alpha_n(t) = r(t)$, *uniformly and monotonically on* J *and such that* ρ, r *are coupled minimal and maximal quasi-solutions of* (1) *in the sector* $[\beta, \alpha]$.

Proof Let us consider the following equations:

$$
\begin{cases}
\alpha_n'(t) = M(t)\alpha_n(t) + (\mathcal{L}\alpha_n)(t) + (Q\alpha_{n-1})(t) - M(t)\alpha_{n-1}(t) - (\mathcal{L}\alpha_{n-1})(t), \\
g(\alpha_{n-1}(0), \beta_{n-1}(T)) + M_1(\alpha_n(0) - \alpha_{n-1}(0)) - M_2(\alpha_n(T) - \alpha_{n-1}(T)) = 0,
\end{cases}
\tag{14}
$$

$$\begin{cases} \beta_n'(t) = M(t)\beta_n(t) + (\mathcal{L}\beta_n)(t) + (Q\beta_{n-1})(t) - M(t)\beta_{n-1}(t) - (\mathcal{L}\beta_{n-1})(t), \\ g(\beta_{n-1}(0), \alpha_{n-1}(T)) + M_1(\beta_n(0) - \beta_{n-1}(0)) - M_2(\beta_n(T) - \beta_{n-1}(T)) = 0, \end{cases} \tag{15}$$

for $n = 1, 2, \ldots$, where $\alpha_0 = \alpha$, $\beta_0 = \beta$.

This is an adequate definition since by Theorem 3.2 the existence and uniqueness of the solution for (14) and (15) are guaranteed. First, we show that $\beta_0 \leq \beta_1 \leq \alpha_1 \leq \alpha_0$ and setting $m = \beta_1 - \beta_0$, employing (H_4), (H_5), we acquire

$$\begin{aligned} m'(t) &= \alpha_1'(t) - \alpha'(t) \\ &\geq M(t)\alpha_1(t) + (\mathcal{L}\alpha_1)(t) + (Q\alpha)(t) - M(t)\alpha(t) - (\mathcal{L}\alpha)(t) - (Q\alpha)(t) \\ &= M(t)m(k) + (\mathcal{L}m)(t), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} m(0) &= -\frac{1}{M_1} g\bigl(\alpha(0), \beta(T)\bigr) + \frac{M_2}{M_1}\bigl(\alpha_1(T) - \alpha(T)\bigr) + \alpha(0) - \alpha(0) \\ &\geq \frac{M_2}{M_1} m(T). \end{aligned}$$

It follows that $m(t) \leq 0$ on J, which implies $\alpha_1 \leq \alpha_0$ on J. Similarly, we may obtain $\beta_0 \leq \beta_1$ on J.

Next, take $m(t) = \beta_1(t) - \alpha_1(t)$, by ($H_2$), ($H_4$), and ($H_5$), we get

$$\begin{aligned} m'(t) &= \beta_1'(t) - \alpha_1'(k) \\ &= M(t)\beta_1(t) + (\mathcal{L}\beta_1)(t) + (Q\beta)(t) - M(t)\beta(t) - (\mathcal{L}\beta)(t) \\ &\quad - M(t)\alpha_1(t) - (\mathcal{L}\alpha_1)(t) - (Q\alpha)(t) + M(t)\alpha(t) + (\mathcal{L}\alpha)(t) \\ &\geq M(t)m(t) + (\mathcal{L}m)(t), \quad t \in J, \end{aligned}$$

and

$$\begin{aligned} m(0) &= \beta_1(0) - \alpha_1(0) \\ &= -\frac{1}{M_1} g\bigl(\beta(0), \alpha(T)\bigr) + \frac{M_2}{M_1}\bigl(\beta_1(T) - \beta(T)\bigr) + \beta(0) \\ &\quad - \left[-\frac{1}{M_1} g\bigl(\alpha(0), \beta(T)\bigr) + \frac{M_2}{M_1}\bigl(\alpha_1(T) - \alpha(T)\bigr) + \alpha(0)\right] \\ &\geq \frac{M_2}{M_1} m(T). \end{aligned}$$

This implies that $m(t) \leq 0$ on J, and $\beta_1 \leq \alpha_1$.

In the following, we show that β_1, α_1 are coupled lower and upper solutions of (1). Using (H_1), (H_5), and (14), we have

$$\begin{aligned} \alpha_1'(t) &= M(t)\alpha_1(t) + (\mathcal{L}\alpha_1)(t) + (Q\alpha)(t) - M(t)\alpha(t) - (\mathcal{L}\alpha)(t) \\ &\leq M(t)\alpha_1(t) + (\mathcal{L}\alpha_1)(t) + (Q\alpha_1)(t) + M(t)\bigl(\alpha(t) - \alpha_1(t)\bigr) \end{aligned}$$

$$\begin{aligned}
&+ \big(\mathcal{L}(\alpha - \alpha_1)\big)(t) - M(t)\alpha(t) - (\mathcal{L}\alpha)(t) \\
&\leq (Q\alpha_1)(t),
\end{aligned} \tag{16}$$

and by means of the fact $\beta_0 \leq \beta_1 \leq \alpha_1 \leq \alpha_0$, ($H_4$), and ($H_5$), we obtain

$$\begin{aligned}
g\big(\alpha_1(0), \beta_1(T)\big) &\leq g\big(\alpha_1(0), \beta_1(T)\big) - g\big(\alpha(0), \beta(T)\big) \\
&\leq g\big(\alpha_1(0), \beta(T)\big) - g\big(\alpha(0), \beta(T)\big) \\
&\leq M_1\big(\alpha_1(0) - \alpha_0(0)\big) \\
&\leq 0.
\end{aligned} \tag{17}$$

Similarly, we can get

$$\beta_1'(t) \geq (Q\beta_1)(t), \qquad g\big(\beta_1(0), \alpha_1(T)\big) \geq 0, \tag{18}$$

from (17)-(18), we see that α_1, β_1 are coupled lower and upper solutions of problem (1).

Now employing the mathematical induction, assume that, for some integer $k > 1$,

$$\beta_{k-1} \leq \beta_k \leq \alpha_k \leq \alpha_{k-1} \quad \text{on } J.$$

We need to prove that

$$\beta_k \leq \beta_{k+1} \leq \alpha_{k+1} \leq \alpha_k \quad \text{on } J.$$

For this purpose, let $m(t) = \beta_k(t) - \beta_{k+1}(t)$ and using (H_2), (H_5), we note that

$$\begin{aligned}
m'(t) &= \beta_k'(t) - \beta_{k+1}'(t) \\
&\geq (Q\beta_k)(t) - M(t)\beta_{k+1}(t) - (\mathcal{L}\beta_{k+1})(t) - (Q\beta_k)(t) + M(t)\beta_k(t) + (\mathcal{L}\beta_k)(t) \\
&= M(t)m(k) + (\mathcal{L}m)(t), \quad t \in J,
\end{aligned}$$

and

$$\begin{aligned}
m(0) &= \beta_k(0) - \beta_{k+1}(0) \\
&= \beta_k(0) + \frac{1}{M_1} g\big(\beta_k(0), \alpha_k(T)\big) - \frac{M_2}{M_1}\big(\beta_{k+1}(T) - \beta_k(T)\big) - \beta_k(0) \\
&\geq \frac{M_2}{M_1} m(T).
\end{aligned}$$

By Lemma 2.1, $\beta_k \leq \beta_{k+1}$ on J. Similarly, we can prove that $\alpha_{k+1} \leq \alpha_k$ on J.

Next, we prove $\beta_{k+1} \leq \alpha_{k+1}$, set $m(t) = \beta_{k+1}(t) - \alpha_{k+1}(t)$, from ($H_4$), ($H_5$), and the fact $\beta_k \leq \alpha_k$, we can obtain

$$\begin{aligned}
m'(t) &= \beta_{k+1}'(t) - \alpha_{k+1}'(t) \\
&= M(t)\beta_{k+1}(t) + (\mathcal{L}\beta_{k+1})(t) + (Q\beta_k)(t) - M(t)\beta_k(t) - (\mathcal{L}\beta_k)(t)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \bigl[M(t)\alpha_{k+1}(t) + (\mathcal{L}\alpha_{k+1})(t) + (Q\alpha_k)(t) - M(t)\alpha_k(t) - (\mathcal{L}\alpha_k)(t)\bigr] \\
&\geq M(t)m(k) + (\mathcal{L}m)(t), \quad t \in J, \\
m(0) &= \beta_{k+1}(0) - \alpha_{k+1}(0) \\
&= -\frac{1}{M_1} g\bigl(\beta_k(0), \alpha_k(T)\bigr) + \frac{M_2}{M_1}\bigl(\beta_{k+1}(T) - \beta_k(T)\bigr) + \beta_k(0) \\
&\quad - \biggl[-\frac{1}{M_1} g\bigl(\alpha_k(0), \alpha_k(T)\bigr) + \frac{M_2}{M_1}\bigl(\alpha_{k+1}(T) - \alpha_k(T)\bigr) + \alpha_k(0)\biggr] \\
&\geq \frac{M_2}{M_1} m(T).
\end{aligned}
$$

Then $m(t) \leq 0$, *i.e.*, $\beta_{k+1} \leq \alpha_{k+1}$. From the above discussion, we have

$$
\beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \leq \cdots \leq \alpha_n \leq \cdots \leq \alpha_2 \leq \alpha_1 \leq \alpha_0.
$$

Obviously, the constructed sequences $\{\alpha_n\}$, $\{\beta_n\}$ are equicontinuous and uniform bounded. By the Ascoli-Arzela theorem, the sequences $\{\alpha_n\}, \{\beta_n\}$ converge uniformly to limit functions r, ρ on J, respectively. Using the definition of (14), (15), and passing to the limit when $n \to \infty$, we see that ρ, r are coupled quasi-solutions of problem (1).

It remains to show that ρ, r are coupled minimal and maximal solutions of problem (1). Let $u_1, u_2 \in [\beta, \alpha]$ be any coupled quasi-solutions of (1). Assume that there exists a positive integer k such that $\beta_k \leq u_1, u_2 \leq \alpha_k$ on J. Then, putting $m(t) = \beta_{k+1} - u_1$, and employing (H_2) and (H_5), we have

$$
\begin{aligned}
m'(t) &= \Delta\beta_{n+1}(t) - u'(t) \\
&= M(t)\beta_{n+1}(t) + (\mathcal{L}\beta_{n+1})(t) + (Q\beta_n)(t) - M(t)\beta_n(t) - (\mathcal{L}\beta_n)(t) - (Qu)(t) \\
&\geq M(t)m(t) + (\mathcal{L}m)(t), \quad t \in J,
\end{aligned}
$$

and

$$
\begin{aligned}
m(0) &= \beta_{n+1}(0) - u(0) \\
&= -\frac{1}{M_1} g\bigl(\beta_n(0), \beta_n(T)\bigr) + \frac{M_2}{M_1}\bigl(\beta_{n+1}(T) - \beta_n(T)\bigr) \\
&\quad + \beta_n(0) - u(0) + \frac{1}{M_1} g\bigl(u(0), u(T)\bigr) \\
&\geq \frac{M_2}{M_1} m(T).
\end{aligned}
$$

By Lemma 2.1, $m(t) \leq 0$, which proves $\beta_{k+1} \leq u$. Using similar arguments we can conclude $\beta_{k+1} \leq u_1, u_2 \leq \alpha_{k+1}$ on J. Since $\beta \leq u_1, u_2 \leq \alpha$, by the principle of induction, $\beta_n \leq u_1, u_2 \leq \alpha_n$ holds for all n. Taking the limit as $n \to \infty$, we have $\rho \leq u_1, u_2 \leq r$ on J proving ρ, r are coupled minimal and maximal quasi-solutions of (1). Since any natural solution u of (1) can be considered as (u, u) coupled quasi-solutions, we also have $\rho \leq u \leq r$ on J. This completes the proof. □

5 Example

In this section, we give two simple but illustrative examples, thereby validating the proposed theorems.

Example 5.1 Consider the following problem:

$$\begin{cases} u'(t) = \frac{1}{2}t^3u^2(t) + t^2u(t) + \frac{2}{5}\int_0^t su^2(s)\,ds, & t \in J = [0,1], \\ \sin(u(0)) + 2u(0) - 3u(1) - c = 0, & 0 \le c \le 0.15. \end{cases} \tag{19}$$

Let $\alpha_0(t) = 0, \beta_0(t) = -1$, we can easily verify that $\alpha_0(t)$ is a lower solution and $\beta_0(t)$ is an upper solution with $\beta_0(t) \le \alpha_0(t)$.

Set

$$(Qu)(t) = \frac{1}{2}t^3u^2(t) + t^2u(t) + \frac{2}{5}\int_0^t su^2(s)\,ds,$$

$$(\mathcal{L}u)(t) = \frac{2}{5}\int_0^t su^2(s)\,ds,$$

$$g(u,v) = \sin u + 2u - 3v - c,$$

by computing, we have

$$(Qu)(t) - (Qv)(t) \le t^2\big(u(t) - v(t)\big) + \big(\mathcal{L}(u-v)\big)(t),$$

where $\beta_0(t) \le v(t) \le u(t) \le \alpha_0(t)$ on $t \in J$, $M(t) = t^2$.

$$g(\bar{u},\bar{v}) - g(u,v) \ge 2(\bar{u} - u) - 3(\bar{v} - v),$$

where $\beta(0) \le u \le \bar{u} \le \alpha(0), \beta(1) \le v \le \bar{v} \le \alpha(1), M_1 = 2, M_2 = 3, \lambda = \frac{M_1}{M_2} = \frac{2}{3}$.

It is easy to prove that $\xi = \max_{t\in J}\{|\frac{2e^{\int_0^1 t^2\,dt}}{2-3e^{\int_0^1 t^2\,dt}}|, |\frac{3e^{\int_0^1 t^2\,dt}}{2-3e^{\int_0^1 t^2\,dt}}|\} < 2$ and

$$\int_0^1 \big(M(t) + (\mathcal{L}1)(t)\big)\,dt = \int_0^1 \left(t^2 + \frac{1}{5}t^2\right)dt = \frac{2}{5} \le \frac{\lambda}{\lambda+1},$$

$$\xi\|\mathcal{L}\|T = \xi\|\mathcal{L}\| \le \frac{2}{5} < 1.$$

Then all conditions of Theorem 3.3 are satisfied. Therefore, via Theorem 3.3, there exist monotone iterative sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ which converge uniformly on J to the extremal solutions of (14) in $[\beta_0, \alpha_0]$.

Example 5.2 Consider the following problem:

$$\begin{cases} u'(t) = \frac{1}{8}tu^2(t) + \frac{1}{4}tu(t) + \frac{1}{10t^3}\int_0^t s^2u(s)\,ds \equiv (Qu)(t), & t \in J = [0,1], \\ g(u(0),u(1)) = -3u^2(0) + u(0) - u(1) + c = 0, & 1 \le c \le 2. \end{cases} \tag{20}$$

Let $\alpha_0(t) = 1, \beta_0(t) = 0$, we can easily verify that $\alpha_0(t)$ is a lower solution and $\beta_0(t)$ is an upper solution with $\beta_0(t) \le \alpha_0(t)$.

Take $(\mathcal{L}u)(t) = \frac{1}{10t^3}\int_0^t s^2 u(s)\,ds$, by computing, we get

$$(Qu)(t) - (Qv)(t) \leq \frac{t}{2}\big(u(t) - v(t)\big) + \big(\mathcal{L}(u - v)\big)(t),$$

where $\beta_0(t) \leq v(t) \leq u(t) \leq \alpha_0(t)$ on $t \in J$, $M(t) = \frac{t}{2}$.

Set $g(u, v) = -3u^2 + u - v + c$, we see that the function $g(u, v)$ is non-increasing in the second variable and

$$g(\bar{u}, v) - g(u, v) \leq (\bar{u} - u),$$

where $\beta(0) \leq u \leq \bar{u} \leq \alpha(0)$, $M_1 = 1, M_2 = 2$, $\lambda = \frac{M_1}{M_2} = \frac{1}{2}$.

It is easy to prove that $\xi = \max_{t\in J}\{|\frac{e^{\int_0^1 \frac{t}{2}\,dt}}{1-2e^{\int_0^1 \frac{t}{2}\,dt}}|, |\frac{2e^{\int_0^1 \frac{t}{2}\,dt}}{1-2e^{\int_0^1 \frac{t}{2}\,dt}}|\} < 2$ and

$$\int_0^1 \big(M(t) + (\mathcal{L}1)(t)\big)\,dt = \int_0^1 \left(\frac{t}{2} + \frac{1}{30}\right)dt = \frac{17}{60} \leq \frac{\lambda}{\lambda + 1},$$

$$\xi\,\|\mathcal{L}\|T = \xi\,\|\mathcal{L}\| < \frac{1}{15} < 1.$$

Then all conditions of Theorem 4.1 are satisfied. So problem (20) has coupled minimal and maximal quasi-solutions of (20) in the segment $[\beta_0, \alpha_0]$.

Competing interests
The author declares that they have no competing interests.

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