Peng et al. *Boundary Value Problems* (2016) 2016:142
DOI 10.1186/s13661-016-0651-2

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A viscous thin-film equation with a singular diffusion

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Abstract

The paper is devoted to studying a viscous thin-film equation with a singular diffusion term and the periodic boundary conditions in multidimensional space, which has a lot of applications in fluids theory such as draining of foams and the movement of contact lenses. In order to obtain the necessary uniform estimates and overcome the difficulty of a singular diffusion term, the entropy functional method is used. Finally, the existence of nonnegative weak solutions is obtained by some compactness arguments.

Keywords: fourth-order parabolic; thin-film equation; entropy functional; singular diffusion

1 Introduction

The research of the Cahn-Hilliard equation and the thin-film equation has become a hot topic recently. The Cahn-Hilliard equation (see [1]) can describe the evolution of a conserved concentration field during phase separation, which has the form $u_t + \nabla \cdot (m\nabla(\varepsilon^2 \Delta u + f'(u))) = 0$ where m, f, ε^2 denote the atomic mobility, the free energy, the parameter proportional to the interface energy, respectively. $-(\varepsilon^2 \Delta u + f'(u))$ can be taken as the chemical potential. For the linear or degenerate mobility, Elliott, Zheng, and Garcke [2, 3] have studied its existence and obtained some properties of solutions. Besides, Liang and Zheng [4] obtained the existence and stability results for this model with a gradient mobility by studying the corresponding semi-discrete problems.

The thin-film equation is usually used to describe the motion of a very thin layer of viscous incompressible fluids along an inclined plane such as the draining of foams and the movement of contact lenses. It can be taken as a class of fourth-order degenerate parabolic equations [5]:

$$u_t + \big(m(u)u_{xxx} + f(u, u_x, u_{xx})\big)_x = 0,$$

where the mobility $m(u)$ degenerates at $u = 0$. For example, thin-film flows driven by the surface tension can be modeled by the following fourth-order degenerate parabolic equations:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^3}{3}(Ch_{xxx} - \delta Bh_x \cos\alpha + B\sin\alpha) + A\frac{u_x}{u} + \frac{M}{2}\sigma_x u^2\right) = 0.$$

For the simplified thin-film equation $u_t + (u^n u_{xxx})_x = 0$, Bernis and Friedman [6] gave the first result to the existence and nonnegativity of weak solutions. Bertozzi and Pugh [7] have studied the existence in the distributional sense and the long time decay for the model of the thin-film equation with a second-order diffusion term. Boutat *et al.* [8] studied a generalized thin-film equation with period boundary in multidimensional space. Furthermore, Liang [9] has investigated the existence of the weak solutions and strong solutions with the initial function near a steady state solution. For other results, the reader may refer to [10–14] and [15].

In this paper, we study the following viscous thin-film equation with a singular diffusion:

$$\begin{cases} u_t - \nabla \cdot (u^n \nabla w) + A\nabla \cdot (\frac{\nabla u}{u^\alpha}) = 0 & \text{in } Q_T, \\ w = -\Delta u + \nu u_t & \text{in } Q_T, \\ u \text{ is } \Omega\text{-periodic}, \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1}$$

where $\Omega = (-1,1)^N$, $Q_T = \Omega \times (0,T)$. n, A, α, and ν are all constants with $n, \alpha, \nu > 0$.

For convenience, we introduce some notations:

- C is denoted as a positive constant and may change from line to line.
- $\Omega = (-1,1)^N$, $\Gamma_j = \partial\Omega \cap \{x_j = -1\}$, $\Gamma_{j+N} = \partial\Omega \cap \{x_j = 1\}$.
- $H^m_{\mathrm{per}}(\Omega)$ is the periodic Sobolev space *i.e.*

$$H^m_{\mathrm{per}}(\Omega) = \bigl\{u \in H^m(\Omega) \big| D^\xi u\big|_{\Gamma_j} = D^\xi u|_{\Gamma_{j+N}}, j = 1, \ldots, N, |\xi| \le m-1\bigr\}.$$

- The following norms on $H^m_{\mathrm{per}}(\Omega)$ $(m \ge 1)$ are equivalent:

$$\|u\|_{H^m(\Omega)}, \qquad \|u\|_{L^2(\Omega)} + \|D^m u\|_{L^2(\Omega)} \quad \text{and} \quad |\overline{u}| + \|D^m u\|_{L^2(\Omega)},$$

  where $\overline{u} = \frac{1}{2^N}\int_\Omega u(x)\,\mathrm{d}x$ (see [8]).
- $C^m_{\mathrm{per}}(\overline{\Omega}) = \{u \in C^m(\overline{\Omega}) | D^\xi u|_{\Gamma_j} = D^\xi u|_{\Gamma_{j+N}}, j = 1, \ldots, N, |\xi| \le m-1\}$.
- $a_+ = \max\{a, 0\}$, $a_- = \min\{a, 0\}$ for $a \in R$.

Our main result is the following theorem.

Theorem 1 *Let* $\alpha \in (0, \frac{1}{2}]$, $u_0 \in H^1(\Omega)$,

$$n \in \begin{cases} (\frac{6}{7}, 2), & N = 1; \\ (\frac{8}{9}, 2), & N = 2; \\ (\frac{16}{17}, 2), & N = 3. \end{cases}$$

Suppose $A \le 0$ *or* $\alpha \le 1 - n$. *Then there exist at least one pair of solutions* (u, w) *satisfying*

1. $u \in L^2(0,T; H^2_{\mathrm{per}}(\Omega)) \cap C([0,T]; H^1(\Omega))$, $u^{-\frac{1}{2}}|\nabla u| \in L^4(Q_T)$, $w, u_t \in L^2(Q_T)$;
2. *for any test function* $\phi \in C^1([0,T]; C^2_{\mathrm{per}}(\Omega))$, *one has*

$$\iint_{Q_T} u_t\phi\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} u^n w \Delta\phi\,\mathrm{d}x\,\mathrm{d}t + n\iint_{Q_T} u^{n-1}\nabla u w \nabla\phi\,\mathrm{d}x\,\mathrm{d}t - A\iint_{Q_T} \frac{\nabla u \nabla\phi}{u^\alpha}\,\mathrm{d}x\,\mathrm{d}t = 0,$$

$$\iint_{Q_T} w\phi\,\mathrm{d}x\,\mathrm{d}t = -\iint_{Q_T} \Delta u\phi\,\mathrm{d}x\,\mathrm{d}t + \nu\iint_{Q_T} u_t\phi\,\mathrm{d}x\,\mathrm{d}t.$$

The following lemmas are needed in the paper.

Lemma 1 (Bernis, see [8]) *Let $u \in H^2_{\mathrm{per}}(\Omega)$ be a nonnegative function. There exists a constant $\mu > 0$ such that the following inequality holds*:

$$\int_\Omega \frac{|\nabla u|^4}{u^2}\,\mathrm{d}x \le \mu \|u\|^2_{H^2(\Omega)}.$$

Lemma 2 (Aubin-Lions, see [16]) *Let X, B, and Y be Banach spaces and assume $X \hookrightarrow B \hookrightarrow Y$ with compact imbedding $X \hookrightarrow B$.*

(1) *Let $\mathfrak{F}$ be bounded in $L^p(0,T;X)$ where $1 \le p < \infty$, and $\frac{\partial \mathfrak{F}}{\partial t} = \{\frac{\partial f}{\partial t} : f \in \mathfrak{F}\}$ be bounded in $L^1(0,T;Y)$. Then $\mathfrak{F}$ is relatively compact in $L^p(0,T;B)$.*

(2) *Let $\mathfrak{F}$ be bounded in $L^\infty(0,T;X)$, and $\frac{\partial \mathfrak{F}}{\partial t} = \{\frac{\partial f}{\partial t} : f \in \mathfrak{F}\}$ be bounded in $L^r(0,T;Y)$ where $r > 1$. Then $\mathfrak{F}$ is relatively compact in $C([0,T];B)$.*

Lemma 3 (see [17] or [18]) *Let V be a real, separable, reflexive Banach space and H is a real, separable, Hilbert space. $V \hookrightarrow H$ is continuous and V is dense in H. Then $\{u \in L^2(0,T;V) | u_t \in L^2(0,T;V')\}$ is continuously imbedded in $C([0,T];H)$.*

The paper is arranged as follows. The existence of solutions to the approximate problem will be proved in Section 2. In Sections 3 and 4, we will take the limit for small parameters $\delta \to 0$ and $\varepsilon \to 0$, respectively.

2 Approximate problem

This section is devoted to studying the following approximate problem:

$$\begin{cases} u_t - \nabla\cdot((u_+ + \delta)^n \nabla w) + A\nabla\cdot(\frac{(u_+ + \delta)^n \nabla u}{(u_+ + \varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)}) = 0 \quad \text{in } Q_T, \\ w = -\Delta u + \nu u_t \quad \text{in } Q_T, \\ u \text{ is } \Omega\text{-periodic}, \\ u(x,0) = u_{0\delta\varepsilon}(x) \quad \text{on } \Omega \end{cases} \tag{2}$$

for $0 < \delta < \varepsilon < 1$ and $u_+ = \max\{u, 0\}$.

Lemma 4 *Let $u_{0\delta\varepsilon} \in H^1_{\mathrm{per}}(\Omega)$, $\alpha > 0$, and $0 < n < 2$. Then there exist at least a pair of solutions (u,w) to (2) satisfying*

1. *$u \in L^2(0,T;H^3_{\mathrm{per}}(\Omega)) \cap C([0,T];H^2_{\mathrm{per}}(\Omega))$, $u_t \in L^2(Q_T)$, $w \in L^2(0,T;H^1_{\mathrm{per}}(\Omega))$, and $u(x,0) = u_0$;*
2. *for any test function $\phi \in L^2(0,T;H^1_{\mathrm{per}}(\Omega))$, one has*

$$\begin{aligned} &\iint_{Q_T} u_t\phi\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} (u_+ + \delta)^n \nabla w \nabla w\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - A\iint_{Q_T}\left(\frac{(u_+ + \delta)^n \nabla u}{(u_+ + \varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)}\right)\nabla\phi\,\mathrm{d}x\,\mathrm{d}t = 0, \\ &\iint_{Q_T} w\phi\,\mathrm{d}x\,\mathrm{d}t = -\iint_{Q_T} \Delta u\phi\,\mathrm{d}x\,\mathrm{d}t + \nu\iint_{Q_T} u_t\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Proof We will apply the Galerkin method to obtain the existence of solutions. Let $\{\phi_i\}_{i=1,2,3,\dots}$ be the eigenfunctions of the Laplace operator $-\Delta\phi_i = \lambda_i\phi_i$ with periodic boundary value conditions. Moreover, those eigenfunctions are orthogonal in H^1 and L^2 spaces and we can normalize ϕ_i such that $(\phi_i, \phi_j) = \delta_{ij} = \left\{\begin{smallmatrix} 1, & i=j, \\ 0, & i\neq j, \end{smallmatrix}\right.$ where we define $\lambda_1 = 0$, $\phi_1 = 1$, and $(\cdot,\cdot)$ denotes the scalar product of the L^2 space.

Let M denote a positive integer and define $w^M(x,t) = \sum_{i=1}^M d_i(t)\phi_i(x)$, $u^M(x,t) = \sum_{i=1}^M c_i(t)\phi_i(x)$, $u^M(x,0) = \sum_{i=1}^M (u_0,\phi_i)\phi_i$. For $j = 1,\dots,M$, we consider the following system of ordinary differential equations:

$$\begin{aligned} \frac{d}{dt}\bigl(u^M,\phi_j\bigr) &= -\bigl(\bigl(u_+^M+\delta\bigr)^n\nabla w^M,\nabla\phi_j\bigr) \\ &\quad + A\biggl(\biggl(\frac{(u_+^M+\delta)^n\nabla u^M}{(u_+ +\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u^M|^2)}\biggr),\nabla\phi_j\biggr), \end{aligned} \tag{3}$$

$$\bigl(w^M,\phi_j\bigr) = -\bigl(\Delta u^M,\phi_j\bigr) + \nu\frac{d}{dt}\bigl(u^M,\phi_j\bigr). \tag{4}$$

The ODE existence theorem yields the local unique existence of this initial value problem since the right hand side depends on c_i continuously. In order to show the global solvability, we take $-\Delta u^M$ as the test function and apply the Young inequality to get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl|\nabla u^M\bigr|^2\,\mathrm{d}x + \nu\int_\Omega\bigl|u_t^M\bigr|^2\,\mathrm{d}x + \int_\Omega\bigl(u_+^M+\delta\bigr)^n\bigl|\nabla w^M\bigr|^2\,\mathrm{d}x \\ &\quad = A\int_\Omega\frac{(u_+^M+\delta)^n\nabla u^M\nabla w^M}{(u_+^M+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u^M|^2)}\,\mathrm{d}x \\ &\quad \le \frac{1}{2}\int_\Omega\bigl(u_+^M+\delta\bigr)^n\bigl|\nabla w^M\bigr|^2\,\mathrm{d}x + \frac{C}{\varepsilon^\alpha}\int_\Omega\bigl|\nabla u^M\bigr|^2\,\mathrm{d}x. \end{aligned} \tag{5}$$

It gives

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigl|\nabla u^M\bigr|^2\,\mathrm{d}x + \nu\int_\Omega\bigl|u_t^M\bigr|^2\,\mathrm{d}x + \int_\Omega\bigl(u_+^M+\delta\bigr)^n\bigl|\nabla\Delta u^M\bigr|^2\,\mathrm{d}x \\ &\quad = \frac{C}{\varepsilon^\alpha}\int_\Omega\bigl|\nabla u^M\bigr|^2\,\mathrm{d}x. \end{aligned} \tag{6}$$

The mass conservation property $\int_\Omega u^M(x,t)\,\mathrm{d}x = \int_\Omega u_0^M(x)\,\mathrm{d}x$ (by letting $j=1$) ensures that Poincaré's inequality can be applied. On the other hand, the Gronwall inequality yields

$$\begin{aligned} &\sup_{t\in(0,T)}\int_\Omega\bigl(\bigl|u^M\bigr|^2+\bigl|\nabla u^M\bigr|^2\bigr)(x,t)\,\mathrm{d}x + \iint_{Q_T}\bigl|u_t^M\bigr|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T}\bigl|\nabla w^M\bigr|^2\,\mathrm{d}x \\ &\quad \le C. \end{aligned} \tag{7}$$

Therefore, we have obtained

$$u^M\in L^\infty\bigl(0,T;H^1_{\mathrm{per}}(\Omega)\bigr),\qquad w^M\in L^2\bigl(0,T;H^1_{\mathrm{per}}(\Omega)\bigr),\qquad u_t^M\in L^2(Q_T). \tag{8}$$

The classic L^p-estimate of the second-order elliptic equations implies

$$u^M\in L^2\bigl(0,T;H^3_{\mathrm{per}}(\Omega)\bigr). \tag{9}$$

By (8), (10), and Lemma 2, we conclude that there exist a pair of functions (u,w) and a subsequence of (u^M,w^M) such that as $M\to\infty$,

$$u^M \rightharpoonup u \quad \text{weakly}^* \text{ in } L^\infty\bigl(0,T;H^1_{\mathrm{per}}(\Omega)\bigr); \tag{10}$$

$$w^M \rightharpoonup w \quad \text{weakly in } L^2\bigl(0,T;H^1_{\mathrm{per}}(\Omega)\bigr); \tag{11}$$

$$u^M_t \rightharpoonup u_t \quad \text{weakly in } L^2(Q_T); \tag{12}$$

$$u^M \to u \quad \text{strongly in } L^2\bigl(0,T;H^2_{\mathrm{per}}(\Omega)\bigr); \tag{13}$$

$$\nabla u^M \to \nabla u \quad \text{a.e. in } (Q_T)^N; \tag{14}$$

$$u^M \to u \quad \text{a.e. in } Q_T \text{ and strongly in } C\bigl([0,T];L^2(\Omega)\bigr). \tag{15}$$

Moreover, by Lemma 3, (12), (13), and the embedding $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$, we have

$$u^M, u \in C\bigl([0,T];H^1_{\mathrm{per}}(\Omega)\bigr), \tag{16}$$

$$\nabla u^M, \nabla u \in L^4(Q_T). \tag{17}$$

From Vitali's theorem, we get

$$\bigl(u^M_+ + \delta\bigr)^n \to \bigl(u^M_+ + \delta\bigr)^n \quad \text{strongly in } L^4(Q_T); \tag{18}$$

$$\nabla u^M \to \nabla u \quad \text{strongly in } L^4(Q_T); \tag{19}$$

$$\frac{(u^M_+ + \delta)^n \nabla u^M}{(u^M_+ + \varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u^M|^2)} \to \frac{(u_+ + \delta)^n \nabla u}{(u_+ + \varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)} \quad \text{strongly in } L^2(Q_T). \tag{20}$$

Let T_M denote the projection from the space $L^2(\Omega)$ to $Span\{\phi_1,\dots,\phi_M\}$. By multiplying equation (3) by $T_M\phi$ for $\phi\in L^2(0,T;H^1_{\mathrm{per}}(\Omega))$, one has

$$\begin{aligned}
&\iint_{Q_T} u^M_t T_M\phi \,\mathrm{d}t + \iint_{Q_T} \bigl(u^M_+ + \delta\bigr)^n \nabla w^M \nabla T_M\phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = A\iint_{Q_T} \biggl(\frac{(u^M_+ + \delta)^n \nabla u^M}{(u^M_+ + \varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u^M|^2)}\biggr) \nabla T_M\phi \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{21}$$

$$\iint_{Q_T} w^M T_M\phi \,\mathrm{d}x\,\mathrm{d}t = -\iint_{Q_T} \Delta u^M T_M\phi \,\mathrm{d}x\,\mathrm{d}t + \nu \iint_{Q_T} u^M_t T_M\phi \,\mathrm{d}x\,\mathrm{d}t. \tag{22}$$

By (10)-(20), we can perform the limit $M\to\infty$ in each term of (21)-(22). □

3 The limit $\delta \to 0$

We shall perform the limit $\delta\to 0$ in the section to the solutions obtained by Lemma 4 and we suppose that the initial function $u_{0\delta\varepsilon}\to u_{0\varepsilon}\in H^1(\Omega)$ as $\delta\to 0$ and $u_{0\varepsilon}\geq 0$.

The main result of this section is the following.

Proposition 1 *Let*

$$n \in \begin{cases} (\frac{6}{7},2), & N=1; \\ (\frac{8}{9},2), & N=2; \\ (\frac{16}{17},2), & N=3. \end{cases}$$

Then there exist at least a pair of functions $(\overline{u},\overline{w})$ *satisfying*

1. $\overline{w}\in L^2(Q_T)$, $\overline{u}\in L^2(0,T;H^2_{\mathrm{per}}(\Omega))\cap C([0,T];H^1_{\mathrm{per}}(\Omega))$, $\overline{u}_t\in L^2(Q_T)$, *and* $\overline{u}(x,0)=u_{0\varepsilon}$;
2. *for any test function* $\phi\in L^2(0,T;C^\infty_{\mathrm{per}}(\overline{\Omega}))$, *one has*

$$\begin{aligned}&\iint_{Q_T}\overline{u}_t\phi\,\mathrm{d}x\,\mathrm{d}t-\iint_{Q_T}\overline{u}^n\overline{w}\Delta\phi\,\mathrm{d}x\,\mathrm{d}t-n\iint_{Q_T}\overline{u}^{n-1}\overline{w}\nabla\overline{u}\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\&\quad-A\iint_{Q_T}\frac{\overline{u}^n\nabla\overline{u}\nabla\phi}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)}\,\mathrm{d}x\,\mathrm{d}t=0,\\&\iint_{Q_T}\overline{w}\phi\,\mathrm{d}x\,\mathrm{d}t=-\iint_{Q_T}\Delta\overline{u}\phi\,\mathrm{d}x\,\mathrm{d}t+\nu\iint_{Q_T}\overline{u}_t\phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

In order to prove this proposition, we have to establish some uniform energy estimates independent of δ and thus we introduce a nonnegative convex functional $\Phi_\delta(\cdot)$ (see[8]):

If $0\le n<2$, $n\neq 1$,

$$\Phi_\delta(\sigma)=\begin{cases}\frac{1}{(1-n)(2-n)}(\sigma+\delta)^{2-n}-\frac{1}{1-n}(\sigma+\delta)+\frac{1}{2-n}, & \sigma\ge 0;\\ \frac{(\sigma)^2}{2\delta^n}+\frac{1}{1-n}(\delta^{1-n}-1)\sigma+\frac{1}{2-n}, & \sigma<0.\end{cases}$$

If $n=1$,

$$\Phi_\delta(\sigma)=\begin{cases}(\sigma+\delta)Ln(\sigma+\delta)-(\sigma+\delta)+1, & \sigma\ge 0;\\ \frac{(\sigma)^2}{2\delta}+\sigma(Ln\delta)+\delta(Ln\delta)-\delta+1, & \sigma<0.\end{cases}$$

It is easy to check that $\Phi_\delta\in W^{2,+\infty}_{\mathrm{loc}}(R)$, $\Phi''_\delta(\sigma)=\frac{1}{(\sigma_++\delta)^n}$.

By applying this functional, we can get the following estimates.

Lemma 5 *There exist some constants* C *independent of* δ (*may depend on* ε) *such that*

1. $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Phi(u(x,t))\,\mathrm{d}x+\int_\Omega|w|^2\,\mathrm{d}x+\nu\int_\Omega|u_t|^2\,\mathrm{d}x\le C$;
2. $\|w\|_{L^2(Q_T)}\le C$, $\|u\|_{L^2(0,T;H^2_{\mathrm{per}}(\Omega))}\le C$;
3. $\|u\|_{L^\infty(0,T;H^1_{\mathrm{per}}(\Omega))}\le C$;
4. $\iint_{Q_T}(u_++\delta)^n|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t\le C$;
5. $\|u_t\|_{L^2(Q_T)}\le C$.

Proof By choosing $\Phi'(u)$ as the test function in (2), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Phi\big(u(x,t)\big)\,\mathrm{d}x+\int_\Omega|w|^2\,\mathrm{d}x+\nu\int_\Omega|u_t|^2\,\mathrm{d}x\\&\quad=A\int_\Omega\frac{|\nabla u|^2}{(u_++\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)}\,\mathrm{d}x\le\frac{|A|}{\varepsilon^{n+\alpha+1}}.\end{aligned}\tag{23}$$

This implies

$$\|u\|_{L^2(0,T;H^2_{\mathrm{per}}(\Omega))}\le C\|w\|_{L^2(Q_T)}\le C,\tag{24}$$

which yields the results 1-2. Similar to (5), we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla u|^2\,\mathrm{d}x+\nu\int_\Omega|u_t|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega(u_++\delta)^n|\nabla w|^2\,\mathrm{d}x\le C(\varepsilon)\int_\Omega|\nabla u|^2\,\mathrm{d}x,$$

which gives 3 and 4. □

Lemma 6 *There exist a pair of functions* $(\overline{u},\overline{w})$ *such that, as* $\delta\to 0$,

1. $u\rightharpoonup\overline{u}$ *weakly in* $L^2(0,T;H^2_{\mathrm{per}}(\Omega))$;
2. $w\rightharpoonup\overline{w}$ *weakly in* $L^2(Q_T)$;
3. $u_t\to\overline{u}_t$ *weakly in* $L^2(Q_T)$;
4. $u\to\overline{u}$ *strongly in* $L^2(0,T;H^1_{\mathrm{per}}(\Omega))$ *and a.e. in* Q_T;
5. $u\to\overline{u}$ *strongly in* $C([0,T];L^2(\Omega))$;
6. *if* $\sup_{\delta\in(0,1)}\int_\Omega\Phi(u_0)\,\mathrm{d}x<\infty$, *then* $\overline{u}\geq 0$ *in* $\overline{Q}_T$ *and* $\sup_{t\leq T}\|u_-(t)\|_{L^2(\Omega)}\leq C\delta^{\frac{n}{2}}$ *when* $n\leq 1$;
7. $\overline{u}\in L^2(0,T;H^2_{\mathrm{per}}(\Omega))\cap C([0,T];L^2(\Omega))$, $\overline{u}_t\in L^2(Q_T)$, $\overline{w}\in L^2(Q_T)$.

Proof The results 1-3 can be obtained from Lemma 5, and Lemma 2 can give 4 and 5. In order to prove 6-7, we integrate (23) over $(0,T)$ to get

$$0\leq\int_\Omega\Phi\bigl(u(x,t)\bigr)\,\mathrm{d}x\leq\frac{n|A|T}{\varepsilon^{n+\alpha+1}}+\sup_{\delta\in(0,1)}\int_\Omega\Phi\bigl(u_0(x)\bigr)\,\mathrm{d}x\leq C(\varepsilon).$$

If $n<1$, we have

$$0\leq\frac{1}{2}\int_\Omega u_-^2(x,t)\,\mathrm{d}x\leq\frac{\delta^n}{n-1}\bigl(\delta^{1-n}-1\bigr)\int_\Omega u_-(x,t)\,\mathrm{d}x+C(\varepsilon)\delta^n.$$

If $n=1$, we have

$$0\leq\frac{1}{2}\int_\Omega u_-^2(x,t)\,\mathrm{d}x\leq-\delta\ln\delta\int_\Omega u_-(x,t)\,\mathrm{d}x+C(\varepsilon)\delta.$$

By performing the limit $\delta\to 0$, we get $\int_\Omega\overline{u}_-^2(x,t)\,\mathrm{d}x=0$, which implies 6. Besides, the result 7 can be obtained from 1-3 and Lemma 3. □

Proof of Proposition 1 For any function $\phi\in L^2(0,T;C^\infty_{\mathrm{per}}(\Omega))$, Lemma 4 gives

$$\begin{aligned}&\iint_{Q_T}u_t\phi\,\mathrm{d}x\,\mathrm{d}t-\iint_{Q_T}(u_++\delta)^n w\Delta\phi\,\mathrm{d}x\,\mathrm{d}t-n\iint_{Q_T}(u_++\delta)^{n-1}w\nabla u\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\&\quad+A\iint_{Q_T}\left(\frac{(u_++\delta)^n\nabla u\nabla\phi}{(u_++\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)}\right)\mathrm{d}x\,\mathrm{d}t=0,\end{aligned}\tag{25}$$

$$\iint_{Q_T}w\phi\,\mathrm{d}x\,\mathrm{d}t=-\iint_{Q_T}\Delta u\phi\,\mathrm{d}x\,\mathrm{d}t+\nu\iint_{Q_T}u_t\phi\,\mathrm{d}x\,\mathrm{d}t.\tag{26}$$

Similar to the proof of (18)-(20) and applying Lemma 5, Lemma 6, and Vitali's theorem, we can get

$$(u_++\delta)^n\to\overline{u}^n\quad\text{strongly in }L^4(Q_T);\tag{27}$$

$$\nabla u\to\nabla\overline{u}\quad\text{strongly in }L^4(Q_T);\tag{28}$$

$$(u_++\delta)^{n-1}\nabla u\to\overline{u}^{n-1}\nabla\overline{u}\quad\text{strongly in }L^2(Q_T)\text{ if }n\geq 1;\tag{29}$$

$$\frac{(u_++\delta)^n\nabla u}{(u_++\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)}\to\frac{(\overline{u}_++\delta)^n\nabla\overline{u}}{(\overline{u}_++\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)}\quad\text{strongly in }L^2(Q_T).\tag{30}$$

If (29) holds for $n < 1$, (25)-(30) ensure that the limit $\delta \to 0$ can be performed in (25)-(26) and then we can complete the proof of Proposition 1.

Therefore, we only need to prove

$$(u_+ + \delta)^{n-1}\nabla u \to \overline{u}^{n-\frac{1}{2}}\frac{\nabla\overline{u}}{\overline{u}^{\frac{1}{2}}} = \overline{u}^{n-1}\nabla\overline{u} \quad \text{strongly in } L^2(Q_T) \tag{31}$$

if $n < 1$.

From the following three steps, we can prove (29).

Step 1. Define $m(\delta) = \delta + \|u_-\|_{C(\overline{Q_T})}$ and we have $u + m(\delta) \geq \delta > 0$. By applying the Bernis inequality, we get

$$\iint_{Q_T}\frac{|\nabla u|^4}{(u+m(\delta))^2}\,\mathrm{d}x\,\mathrm{d}t \leq \iint_{Q_T}|\Delta u|^2\,\mathrm{d}x\,\mathrm{d}t \leq C, \tag{32}$$

where C is independent of δ.

Step 2. In this step, we define $U_\delta = (u_+ + \delta)^{n-1}(u + m(\delta))^{\frac{1}{2}}$ and we want to prove that the limit $\lim_{\delta\to 0}\|U_\delta - \overline{u}^{n-\frac{1}{2}}\|_{L^4(Q_T)} = 0$ holds.

At first, it is obvious that we have

$$U_\delta \geq \bigl(u_+ + m(\delta)\bigr)^{n-\frac{1}{2}} \quad \text{in } Q_T. \tag{33}$$

Now we choose

$$r\begin{cases} = +\infty, & N = 1; \\ < +\infty, & N = 2; \\ < 6, & N = 3, \end{cases}$$

such that $H^s(\Omega) \hookrightarrow W^{1,r}(\Omega)$ with $\frac{7}{4} < s < 2$. By using the Gagliardo-Nirenberg interpolation inequality and Lemma 6, we get

$$\begin{aligned} \bigl\|u_-(t)\bigr\|_{L^\infty(\Omega)} &\leq C\bigl\|u_-(t)\bigr\|^{\gamma}_{W^{1,r}(\Omega)}\bigl\|u_-(t)\bigr\|^{1-\gamma}_{L^2(\Omega)} \\ &\leq C\bigl\|u_-(t)\bigr\|^{\gamma}_{H^s_{\mathrm{per}}(\Omega)}\delta^{\frac{n}{2}(1-\gamma)} \\ &\leq C(\varepsilon,s)\delta^{\frac{n}{2}(1-\gamma)} \end{aligned} \tag{34}$$

with $\gamma = \frac{\frac{1}{2}}{\frac{N+2}{2N}-\frac{1}{r}}$. It implies

$$\begin{aligned} U_\delta(x,t) &\leq (u_+ + \delta)^{n-1}\bigl(u_+ + \delta + 2\bigl\|u_-(t)\bigr\|_{L^\infty(\Omega)}\bigr)^{\frac{1}{2}} \\ &\leq (u_+ + \delta)^{n-\frac{1}{2}} + \delta^{n-1}\bigl(2\bigl\|u_-(t)\bigr\|_{L^\infty(\Omega)}\bigr)^{\frac{1}{2}} \\ &\leq (u_+ + \delta)^{n-\frac{1}{2}} + C(\varepsilon)\delta^{n-1+\frac{n}{4}(1-\gamma)} \end{aligned} \tag{35}$$

with

$$n \in \begin{cases} (\frac{6}{7}, 2), & N = 1; \\ (\frac{8}{9}, 2), & N = 2; \\ (\frac{16}{17}, 2), & N = 3. \end{cases}$$

Equations (33) and (35) yield

$$\lim_{\delta\to 0} U_\delta(x,t) = \overline{u}^{n-\frac{1}{2}} \quad \text{a.e. in } Q_T. \tag{36}$$

The Lebesgue-dominated theorem yields

$$\lim_{\delta\to 0} \iint_{Q_T} \big|U_\delta - \overline{u}^{n-\frac{1}{2}}\big|^4 \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{37}$$

Step 3. This step is devoted to the proof of (31). For any positive constant η, one has

$$\begin{aligned}
&\iint_Q \left|(u_+ + \delta)^{n-1}\nabla u - \overline{u}^{n-\frac{1}{2}} \frac{\nabla\overline{u}}{\sqrt{\overline{u}}}\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad\le \iint_Q \big|U_\delta - \overline{u}^{n-\frac{1}{2}}\big|^2 \left|\frac{\nabla u}{\sqrt{u+m(\delta)}}\right|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q \overline{u}^{2n-1}\left|\frac{\nabla u}{u+m(\delta)} - \frac{\nabla\overline{u}}{\sqrt{\overline{u}}}\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad\le \left(\iint_Q \left|\frac{\nabla u}{\sqrt{u+m(\delta)}}\right|^4 \mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\iint_Q \big|U_\delta - \overline{u}^{n-\frac{1}{2}}\big|^4 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad + \iint_{\{\overline{u}\ge\eta\}} \overline{u}^{2n-1}\left|\frac{\nabla u}{u+m(\delta)} - \frac{\nabla\overline{u}}{\sqrt{\overline{u}}}\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\qquad + \iint_{\{\overline{u}<\eta\}} \overline{u}^{2n-1}\left|\frac{\nabla u}{u+m(\delta)} - \frac{\nabla\overline{u}}{\sqrt{\overline{u}}}\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad= \mathtt{I}_1 + \mathtt{I}_2 + \mathtt{I}_3.
\end{aligned} \tag{38}$$

From Step 1 and Step 2, we know $\mathtt{I}_1 \to 0$ as $\delta \to 0$ and by applying Lemma 6, we have $\mathtt{I}_2 \to 0$ as $\delta \to 0$. For the last term, we have

$$\begin{aligned}
\mathtt{I}_3 &= \iint_{\{\overline{u}<\eta\}} \overline{u}^{2n-1}\left|\frac{\nabla u}{u+m(\delta)} - \frac{\nabla\overline{u}}{\sqrt{\overline{u}}}\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\le \eta^{2n-1}\left[\iint_{\{\overline{u}<\eta\}} \frac{|\nabla u|^2}{|u+m(\delta)|}\,\mathrm{d}x\,\mathrm{d}t + \iint_{\{\overline{u}<\eta\}} \frac{|\nabla\overline{u}|^2}{|\overline{u}|}\,\mathrm{d}x\,\mathrm{d}t\right] \\
&\le C\eta^{2n-1}\left[\left(\iint_{\{\overline{u}<\eta\}} \frac{|\nabla u|^4}{|u+m(\delta)|^2}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \left(\iint_{\{\overline{u}<\eta\}} \frac{|\nabla\overline{u}|^4}{|\overline{u}|^2}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\right] \\
&\le C\eta^{2n-1}.
\end{aligned} \tag{39}$$

Therefore, by performing the limit $\eta \to 0$, we get $\mathtt{I}_3 \to 0$ and then the estimate (31) holds. □

4 The limit $\varepsilon \to 0$

We will perform the last limit $\varepsilon \to 0$ in this section and assume that the initial function $u_{0\varepsilon}$ converges to u_0 strongly in $L^2(\Omega)$.

By letting $\delta = 0$ in the definition of $\Phi_\delta(\cdot)$, we can define $\Phi_0(\cdot)$ as

$$\Phi_0(x) = \begin{cases} \frac{1}{(1-n)(2-n)}x^{2-n} - \frac{1}{1-n}x + \frac{1}{2-n} & \text{if } n \in [0,2), n \ne 1; \\ x\ln x - x + 1 & \text{if } n = 1. \end{cases}$$

Lemma 7 *In the sense of $\mathcal{D}'(0,T)$, there exists a constant $C_0 > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi_0(\overline{u})\,\mathrm{d}x + C_0\int_\Omega |\Delta\overline{u}|^2\,\mathrm{d}x + \nu\int_\Omega |\overline{u}_t|^2\,\mathrm{d}x \le A\int_\Omega \frac{|\nabla\overline{u}|^2}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)}\,\mathrm{d}x.$$

Proof From the idea of (23) and the L^p-estimate, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi\big(u(x,t)\big)\,\mathrm{d}x + C_0\int_\Omega |\Delta u|^2\,\mathrm{d}x + \nu\int_\Omega |u_t|^2\,\mathrm{d}x \\
&\quad\le \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi\big(u(x,t)\big)\,\mathrm{d}x + \int_\Omega |w|^2\,\mathrm{d}x + \nu\int_\Omega |u_t|^2\,\mathrm{d}x \\
&\quad= A\int_\Omega \frac{|\nabla u|^2}{(u_+ +\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)}\,\mathrm{d}x.
\end{aligned} \tag{40}$$

Since $u \to \overline{u}$ in $C(\overline{Q}_T)$ as $\delta \to 0$, we have

$$-\int_0^T \phi'(t)\int_\Omega \Phi(u)\,\mathrm{d}x\,\mathrm{d}t \to -\int_0^T \phi'(t)\int_\Omega \Phi_0(\overline{u})\,\mathrm{d}x\,\mathrm{d}t \tag{41}$$

for any nonnegative function $\phi \in \mathcal{D}'(0,T)$. By applying the limit $\Delta u \rightharpoonup \Delta\overline{u}$ in $L^2(Q_T)$ as $\delta \to 0$, one has

$$\liminf_{\delta\to 0}\int_0^T \phi(t)\int_\Omega |\Delta u|^2\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T \phi(t)\int_\Omega |\Delta\overline{u}|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{42}$$

Finally, it is easy to check that

$$\begin{aligned}
&A\iint_{Q_T} \frac{|\nabla u|^2\phi(t)}{(u_+ +\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla u|^2)}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\to A\iint_{Q_T} \frac{|\nabla\overline{u}|^2\phi(t)}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla\overline{u}|^2)}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{43}$$

Equations (40)-(43) give the result of this lemma. □

Lemma 8 *If one of the following conditions holds*:

(I) $\int_\Omega \Phi_0(w_0)\,\mathrm{d}x < \infty$, $A \le 0$, *and*

(II) $\int_\Omega \Phi_0(w_0)\,\mathrm{d}x < \infty$, $\alpha \le 1-n$, $n < 1$, *one has* $\overline{u} \in L^2(0,T;H^2_{\mathrm{per}}(\Omega))$, $\overline{w}, \overline{u}_t \in L^2(Q_T)$ *independent of* ε.

Proof By Lemma 7 and the condition (I), we can prove the result easily.

If the condition (II) holds, Lemma 1 and Lemma 7 give

$$\begin{aligned}
&\int_\Omega \Phi_0(\overline{u})\,\mathrm{d}x + C_0\int_\Omega |\Delta\overline{u}|^2\,\mathrm{d}x + \nu\iint_{Q_T} |\overline{u}_t|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_\Omega \Phi_0(\overline{u}_0)\,\mathrm{d}x + |A|\iint_{Q_T} \frac{|\nabla\overline{u}|^2}{(\overline{u}+\varepsilon)(1+\varepsilon|\nabla\overline{u}|^2)}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_\Omega \Phi_0(\overline{u}_0)\,\mathrm{d}x + |A|\iint_{Q_T} \frac{|\nabla\overline{u}|^2}{(\overline{u}+\varepsilon)^{\alpha+n}}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\le \int_\Omega \Phi_0(\overline{u}_0)\,\mathrm{d}x + |A|\left(\iint_{Q_T} \frac{|\nabla\overline{u}|^4}{(\overline{u}+\varepsilon)^2}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\iint_{Q_T}(\overline{u}+\varepsilon)^{2(1-(\alpha+n))}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \int_\Omega \Phi_0(\overline{u}_0)\,\mathrm{d}x + C\left(\int_0^T \|\overline{u}\|_{H^2(\Omega)}\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\iint_{Q_T}(\overline{u}+\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\right)^{1-(\alpha+n)} \\
&\le \frac{C_0}{2}\iint_{Q_T}|\Delta\overline{u}|^2\,\mathrm{d}x\,\mathrm{d}t + C,
\end{aligned} \tag{44}$$

which yields $\Delta\overline{u} \in L^2(Q_T)$. Applying the second equation of Proposition 1, we get $\overline{w} \in L^2(Q_T)$. □

Now we are in the position to prove Theorem 1.

Proof of Theorem 1 By Lemma 8, we can show the existence of two functions $u \ge 0$ and w such that, as $\varepsilon \to 0$,

$$\overline{u} \rightharpoonup u \quad \text{in } L^2\big(0,T;H^2_{\mathrm{per}}(\Omega)\big); \tag{45}$$

$$\overline{u}_t \rightharpoonup u_t \quad \text{in } L^2(Q_T); \tag{46}$$

$$\overline{w} \rightharpoonup w \quad \text{in } L^2(Q_T); \tag{47}$$

$$\overline{u} \to u \quad \text{in } C\big([0,T];H^1_{\mathrm{per}}(\Omega)\big); \tag{48}$$

$$\overline{u} \to u \quad \text{in } L^2\big(0,T;H^1_{\mathrm{per}}(\Omega)\big); \tag{49}$$

$$\overline{u} \to u, \qquad \nabla\overline{u} \to \nabla u \quad \text{a.e. in } Q_T. \tag{50}$$

Furthermore, Lemma 3 yields

$$\|\overline{u}\|_{C([0,T];H^s_{\mathrm{per}}(\Omega))} \le C; \tag{51}$$

$$\|u\|_{C([0,T];H^s_{\mathrm{per}}(\Omega))} \le C \tag{52}$$

for $\frac{3}{2} < s < 2$. By the Sobolev embedding theorem with the case $N \le 3$, we have $\|\overline{u}\|_{L^\infty(Q_T)} \le C$ and $\|u\|_{L^\infty(Q_T)} \le C$.

Step 1. By using (51)-(52) and Vitali's theorem, we get $\overline{u}^n \to u^n$ in $L^q(Q_T)$ for any $q > 0$ and thus one has

$$\iint_{Q_T} \overline{u}^n\overline{w}\Delta\phi\,\mathrm{d}x\,\mathrm{d}t \to \iint_{Q_T} u^n w\Delta\phi\,\mathrm{d}x\,\mathrm{d}t \tag{53}$$

as $\varepsilon \to 0$ for any test function $\phi \in C^\infty([0,T];C^2_{\mathrm{per}}(\overline{\Omega}))$.

Step 2. In this step, we will prove the limit $\overline{u}^{n-1}\nabla\overline{u} \to u^{n-1}\nabla u$ in $L^2(Q_T)$.

First of all, the Bernis inequality yields $\iint_{Q_T}|\frac{\nabla\overline{u}}{\sqrt{\overline{u}}}|^4\,\mathrm{d}x\,\mathrm{d}t \le C$ and then we have

$$\begin{aligned}
\iint_{\Delta_0} \overline{u}^{n-1}|\nabla\overline{u}|^2\,\mathrm{d}x\,\mathrm{d}t &= \iint_{\Delta_0} \overline{u}^{2n-1}\frac{|\nabla\overline{u}|^2}{\sqrt{\overline{u}}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\left(\iint_{\Delta_0}\overline{u}^{4n-2}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \to 0
\end{aligned} \tag{54}$$

as $\varepsilon \to 0$ with $\Delta_0 = \{(x,t) \in Q_T | u(x,t) = 0\}$. On the other hand, it is easy to get

$$\frac{\nabla \overline{u}}{\sqrt{\overline{u}}} \to \frac{\nabla u}{\sqrt{u}} \quad \text{a.e. in } Q_T \backslash \Delta_0$$

as $\varepsilon \to 0$. By Vitali's theorem, we have

$$\overline{u}^{n-1} \nabla \overline{u} \to u^{n-1} \nabla u \quad \text{in } L^2(Q_T \backslash \Delta_0). \tag{55}$$

Hence, we have

$$\overline{u}^{n-1} \nabla \overline{u} \to u^{n-1} \nabla u \quad \text{in } L^2(Q_T), \tag{56}$$

where we define $u^{n-1} \nabla u = 0$ on Δ_0.

Step 3. In this step, we prove the limit $F_\varepsilon = \frac{\overline{u}^n \nabla \overline{u}}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla \overline{u}|^2)} \to u^{-\alpha} \nabla u$ in $L^2(Q_T)$.

If $\alpha \le \frac{1}{2}$, we have

$$\begin{aligned} \iint_{\Delta_0} |F_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t &\le \iint_{\Delta_0} \overline{u}^{1-2\alpha} \frac{|\nabla \overline{u}|^2}{\overline{u}} \,\mathrm{d}x\,\mathrm{d}t \\ &\le C \biggl(\iint_{\Delta_0} \overline{u}^{2-4\alpha} \,\mathrm{d}x\,\mathrm{d}t \biggr)^{\frac{1}{2}} \to 0 \end{aligned} \tag{57}$$

as $\varepsilon \to 0$. Beside, it is easy to show $F_\varepsilon \to u^{-\alpha} \nabla u$ a.e. in $Q_T \backslash \Delta_0$ and Vitali's theorem yields

$$\iint_{\Delta_0} \bigl| F_\varepsilon - u^{-\alpha} \nabla u \bigr|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0 \tag{58}$$

as $\varepsilon \to 0$. By (57)-(58), we have

$$F_\varepsilon \to u^{-\alpha} \nabla u \quad \text{in } L^2(Q_T), \tag{59}$$

where we define $u^{-\alpha} \nabla u = 0$ on Δ_0.

As $\varepsilon \to 0$, the convergence (56) and (46)-(47) give $\iint_{Q_T} \overline{u}_t \phi \,\mathrm{d}x\,\mathrm{d}t \to \iint_{Q_T} u_t \phi \,\mathrm{d}x\,\mathrm{d}t$ and $\iint_{Q_T} \overline{u}^{n-1} \nabla \overline{u} \overline{w} \nabla \phi \,\mathrm{d}x\,\mathrm{d}t \to \iint_{Q_T} u^{n-1} \nabla u w \nabla \phi \,\mathrm{d}x\,\mathrm{d}t$. Step 3 yields

$$\iint_{Q_T} \frac{\overline{u}^n \nabla \overline{u} \nabla \phi}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla \overline{u}|^2)} \,\mathrm{d}x\,\mathrm{d}t \to \iint_{Q_T} u^{-\alpha} \nabla u \nabla \phi \,\mathrm{d}x\,\mathrm{d}t.$$

Now we can take the limit $\varepsilon \to 0$ in the equality

$$\begin{aligned} &\iint_{Q_T} \overline{u}_t \phi \,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} \overline{u}^n \overline{w} \Delta \phi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + n \iint_{Q_T} \overline{u}^{n-1} \nabla \overline{u} \overline{w} \nabla \phi \,\mathrm{d}x\,\mathrm{d}t - A \iint_{Q_T} \frac{\overline{u}^n \nabla \overline{u} \nabla \phi}{(\overline{u}+\varepsilon)^{n+\alpha}(1+\varepsilon|\nabla \overline{u}|^2)} \,\mathrm{d}x\,\mathrm{d}t = 0, \\ &\iint_{Q_T} \overline{w} \phi \,\mathrm{d}x\,\mathrm{d}t = -\iint_{Q_T} \Delta \overline{u} \phi \,\mathrm{d}x\,\mathrm{d}t + \nu \iint_{Q_T} \overline{u}_t \phi \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for any test function $\phi \in C([0,T]; C^2_{\mathrm{per}}(\overline{\Omega}))$. For the initial value, this holds in the sense of $u \in C([0,T]; H^1_{\mathrm{per}}(\Omega))$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XP and BL completed the main study. MP and YW verified the calculation. All authors read and approved the final manuscript.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (No. 11201045) and the Education Department Science Foundation of Liaoning Province of China.

Received: 11 July 2016 Accepted: 28 July 2016 Published online: 05 August 2016

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