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Existence and multiplicity of positive periodic solutions of ratio-dependent food chain model

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Abstract

By utilizing the coincidence degree theory and the related continuation theorem, as well as some prior estimates, we investigate the existence and multiplicity of positive periodic solutions of ratio-dependent food chain model with exploited terms. Some sufficient criteria are established for the existence and multiplicity of periodic solutions.

MSC: 34K13; 92D25

Keywords: coincidence degree; periodic solution; exploited term; ratio-dependent

1 Introduction

The last years have seen very important progress made on Michaelis-Menten type ratio-dependent predator-prey model in mathematical ecology literature, which can be roughly stated as that the per capita predator growth rate should be a function of the ratio of prey to predator abundance and usually takes the form

$$x'(t) = x(r - kx) - c\frac{mxy}{ay + x}, \qquad y'(t) = \left(\frac{mx}{ay + x} - d\right)y, \tag{1.1}$$

where x, y stand for prey and predator density, respectively, r, k, a, c, d, m are positive constants that stand for prey intrinsic growth rate, carrying capacity, half-saturation constant, conversion rate, predator's death rate, and maximal predator growth rate, respectively. System (1.1) is capable of producing far richer and biologically more realistic dynamics. Specifically, it will not produce the paradox of biological control and the paradox of enrichment. In view of these features it has been studied by many authors leading to great progress [1–7]. Moreover, the ratio-dependence form is applied successfully to some other models, for example, in [8], the authors investigated the following three trophic level food chain model with ratio dependence:

$$\begin{aligned} x'(t) &= x\left(r - kx - \frac{b_1 y}{m_1 y + x}\right), \\ y'(t) &= y\left(\frac{c_1 x}{m_1 y + x} - d_1 - \frac{b_2 z}{m_2 z + y}\right), \end{aligned} \tag{1.2}$$

$$z'(t) = z\left(\frac{c_2 y}{m_2 z + y} - d_2\right),$$

where x, y, z stand for the population densities of prey, predator and top predator, respectively. For $i = 1, 2$, m_i, d_i, b_i, c_i are half-saturation constants and the death rates of predator, capture rates, and maximal predator growth rates, respectively, r/k gives the carrying capacity of the prey. This model reflects the simple relation of these three species: z prey on y and only on y, and y prey on x and nutrient recycling is not accounted for. It was shown that this model is rich in boundary dynamics and is capable of generating extinction dynamics.

Recently, there has been a rich body of literature on ecological systems with exploited term(s) and numerous good results have been obtained, for example; see [6, 7, 9–13]. In these references, instead of studying the existence of a periodic solution, one investigated the existence of multiple periodic solutions for considering the inclusion of the effect of periodic changing environment. This is due to the fact that it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. In the present paper, we study the following ratio-dependent food chain model with exploited terms in a periodically varying environment because the variation of the environment plays an important role in many biological and ecological systems:

$$\begin{aligned}
x'(t) &= x(t)\left[r(t) - k(t)x(t) - \frac{b_1(t)y(t)}{m_1(t)y(t) + x(t)}\right] - h_1(t), \\
y'(t) &= y(t)\left[\frac{c_1(t)x(t)}{m_1(t)y(t) + x(t)} - d_1(t) - \frac{b_2(t)z(t)}{m_2(t)z(t) + y(t)}\right] - h_2(t), \\
z'(t) &= z(t)\left[\frac{c_2(t)y(t)}{m_2(t)z(t) + y(t)} - d_2(t)\right] - h_3(t),
\end{aligned} \tag{1.3}$$

where h_1, h_2, h_3 are nonnegative continuous ω-periodic functions representing exploited terms, the other variables and parameters have the same biological meanings as in system (1.2) except that these parameters are ω-periodic functions now.

The paper is organized as follows. In Section 2, the original contributions of this work are summarized. In Section 3, some conclusions are given. Finally, the proofs of our main results are reported in the Appendix to close this paper.

2 Main results

We are now ready to present the main contributions involving eight theorems. For simplicity, we will discuss in detail for Theorem 2.1, the remainder results are similar and their proofs are presented in the Appendix.

For the reader's convenience, we now recall Mawhin's coincidence degree [14], which our study is based upon.

Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N

will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Theorem A (Continuation theorem) *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*:
(i) *For each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(ii) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(iii) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.
Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

For a bounded continuous function $g(t)$ on $\mathbb{R}$, we use the following notations:

$$g^U = \max_{t\in[0,\omega]} g(t), \qquad g^\ell = \min_{t\in[0,\omega]} g(t),$$

where $g(t)$ is s continuous function.

Theorem 2.1 *If $h_1(t) \neq 0$, $h_2(t) \neq 0$, $h_3(t) \neq 0$, and the following conditions are satisfied*:

$$\text{(H1)} \quad r^\ell - \left(\frac{b_1}{m_1}\right)^U > 2\sqrt{k^U h_1^U},$$

$$\text{(H2)} \quad \left[c_1^\ell - d_1^U - \left(\frac{b_2}{m_2}\right)^U\right]\frac{h_1^\ell}{r^U} - m_1^U h_2^U > 2\sqrt{m_1^U\left[d_1^U + \left(\frac{b_2}{m_2}\right)^U\right]\frac{r^U h_2^U}{k^\ell}},$$

$$\text{(H3)} \quad \left(c_2^\ell - d_2^U\right)\frac{h_2^\ell}{c_1^U} - m_2^U h_3^U > 2\sqrt{\frac{r^U c_1^U m_2^U d_2^U h_3^U}{k^\ell m_1^\ell d_1^\ell}}.$$

Then system (1.3) *has at least eight positive periodic solutions.*

Proof We make the change of variables

$$x(t) = \exp\{u(t)\}, \qquad y(t) = \exp\{v(t)\}, \qquad z(t) = \exp\{w(t)\}.$$

Then system (1.3) can be written as

$$\begin{cases} u'(t) = r(t) - k(t)e^{u(t)} - \frac{b_1(t)e^{v(t)}}{m_1(t)e^{v(t)}+e^{u(t)}} - \frac{h_1(t)}{e^{u(t)}}, \\ v'(t) = \frac{c_1(t)e^{u(t)}}{m_1(t)e^{v(t)}+e^{u(t)}} - d_1(t) - \frac{b_2(t)e^{w(t)}}{m_2(t)e^{w(t)}+e^{v(t)}} - \frac{h_2(t)}{e^{v(t)}}, \\ w'(t) = \frac{c_2(t)e^{v(t)}}{m_2(t)e^{w(t)}+e^{v(t)}} - d_2(t) - \frac{h_3(t)}{e^{w(t)}}. \end{cases} \tag{2.1}$$

It is easy to see that if system (2.1) has an ω-periodic solution $(u^*, v^*, w^*)^T$, then $(x^*, y^*, z^*)^T = (e^{u^*}, e^{v^*}, e^{w^*})^T$ is a positive ω-periodic solution of system (1.3). To this end, it suffices to prove that system (2.1) has at least eight ω-periodic solutions.

For $\lambda \in (0,1)$, we consider the following system:

$$\begin{cases} u'(t) = \lambda[r(t) - k(t)e^{u(t)} - \frac{b_1(t)e^{v(t)}}{m_1(t)e^{v(t)}+e^{u(t)}} - \frac{h_1(t)}{e^{u(t)}}], \\ v'(t) = \lambda[\frac{c_1(t)e^{u(t)}}{m_1(t)e^{v(t)}+e^{u(t)}} - d_1(t) - \frac{b_2(t)e^{w(t)}}{m_2(t)e^{w(t)}+e^{v(t)}} - \frac{h_2(t)}{e^{v(t)}}], \\ w'(t) = \lambda[\frac{c_2(t)e^{v(t)}}{m_2(t)e^{w(t)}+e^{v(t)}} - d_2(t) - \frac{h_3(t)}{e^{w(t)}}]. \end{cases} \tag{2.2}$$

Suppose that $(u(t), v(t), w(t))^T$ is an arbitrary ω-periodic solution of system (2.2) for a certain $\lambda \in (0,1)$. Then we can choose ξ_i, η_i, $i = 1,2,3$ such that

$$u(\xi_1) = \max_{t\in[0,\omega]}\{u(t)\}, \qquad u(\eta_1) = \min_{t\in[0,\omega]}\{u(t)\}, \tag{2.3}$$

$$v(\xi_2) = \max_{t\in[0,\omega]}\{v(t)\}, \qquad v(\eta_2) = \min_{t\in[0,\omega]}\{v(t)\}, \tag{2.4}$$

$$w(\xi_3) = \max_{t\in[0,\omega]}\{w(t)\}, \qquad w(\eta_3) = \min_{t\in[0,\omega]}\{w(t)\}. \tag{2.5}$$

By the first equation of (2.2) and (2.3), we have

$$r^U \ge r(\xi_1) > k(\xi_1)e^{u(\xi_1)} \ge k^\ell e^{u(\xi_1)},$$

which reduces to

$$u(\xi_1) < \ln\left\{\frac{r^U}{k^\ell}\right\}. \tag{2.6}$$

Again from the first equation of (2.2) and (2.3), it follows that

$$h_1^\ell e^{-u(\eta_1)} \le h_1(\eta_1)e^{-u(\eta_1)} < r(\eta_1) \le r^U,$$

which implies

$$u(\eta_1) > \ln\left\{\frac{h_1^\ell}{r^U}\right\}. \tag{2.7}$$

From the second equation of (2.2) and (2.4), (2.6), we obtain

$$d_1^\ell \le d_1(\xi_2) < \frac{c_1(\xi_2)e^{u(\xi_2)}}{m_1(\xi_2)e^{v(\xi_2)}} < \frac{r^U c_1^U}{k^\ell m_1^\ell e^{v(\xi_2)}},$$

which reduces to

$$v(\xi_2) < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}. \tag{2.8}$$

Moreover, from the second equation of (2.2) and (2.4), we get

$$h_2^\ell e^{-v(\eta_2)} \le h_2(\eta_2)e^{-v(\eta_2)} < c_1(\eta_2) \le c_1^U,$$

that is,

$$v(\eta_2) > \ln\left\{\frac{h_2^\ell}{c_1^U}\right\}. \tag{2.9}$$

From the third equation of (2.2), (2.5), and (2.8), we have

$$d_2^\ell \le d_2(\xi_3) < \frac{c_2(\xi_3)e^{v(\xi_3)}}{m_2(\xi_3)e^{w(\xi_3)}} < \frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell e^{w(\xi_3)}},$$

which implies

$$w(\xi_3) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \tag{2.10}$$

It follows from the third equation of (2.2) and (2.5) that

$$h_3^\ell e^{-w(\eta_3)} \le h_3(\eta_3)e^{-w(\eta_3)} < c_2(\eta_3) \le c_2^U,$$

which reduces to

$$w(\eta_3) > \ln\left\{\frac{h_3^\ell}{c_2^U}\right\}. \tag{2.11}$$

Furthermore, by the definition of ξ_1 and the first equation of (2.2), we know

$$r(\xi_1) - k(\xi_1)e^{u(\xi_1)} - \frac{b_1(\xi_1)e^{v(\xi_1)}}{m_1(\xi_1)e^{v(\xi_1)} + e^{u(\xi_1)}} - \frac{h_1(\xi_1)}{e^{u(\xi_1)}} = 0.$$

Then

$$k^U e^{2u(\xi_1)} + \left[\left(\frac{b_1}{m_1}\right)^U - r^\ell\right]e^{u(\xi_1)} + h_1^U > 0,$$

which produces

$$u(\xi_1) > \ln A_0^+ \quad \text{or} \quad u(\xi_1) < \ln A_0^-, \tag{2.12}$$

where

$$A_0^\pm = \frac{1}{2k^U}\left\{\left[r^\ell - \left(\frac{b_1}{m_1}\right)^U\right] \pm \sqrt{\left[r^\ell - \left(\frac{b_1}{m_1}\right)^U\right]^2 - 4k^U h_1^U}\right\}.$$

By the definition of η_1 and the parallel argument to (2.12), it is easy to prove that

$$u(\eta_1) > \ln A_0^+ \quad \text{or} \quad u(\eta_1) < \ln A_0^-. \tag{2.13}$$

Similarly, by the definition of ξ_2 and the second equation of (2.2), we have

$$d_1(\xi_2) + \frac{b_2(\xi_2)e^{w(\xi_2)}}{m_2(\xi_2)e^{w(\xi_2)} + e^{v(\xi_2)}} + h_2(\xi_2)e^{-v(\xi_2)} - \frac{c_1(\xi_2)e^{u(\xi_2)}}{m_1(\xi_2)e^{v(\xi_2)} + e^{u(\xi_2)}} = 0.$$

Then

$$m_1^U\left[d_1^U + \left(\frac{b_2}{m_2}\right)^U\right]e^{2v(\xi_2)} - \left\{\left[c_1^\ell - d_1^U - \left(\frac{b_2}{m_2}\right)^U\right]\frac{h_1^\ell}{r^U} - h_2^U m_1^U\right\}e^{v(\xi_2)} + \frac{r^U h_2^U}{k^\ell} > 0.$$

Solving the inequality, we get

$$v(\xi_2) > \ln B_0^+ \quad \text{or} \quad v(\xi_2) < \ln B_0^-, \tag{2.14}$$

where

$$B_0^{\pm} = \frac{1}{2m_1^U[d_1^U + (\frac{b_2}{m_2})^U]}\left\{\left[c_1^\ell - d_1^U - \left(\frac{b_2}{m_2}\right)^U\right]\frac{h_1^\ell}{r^U} - m_1^U h_2^U \right.$$
$$\left. \pm\sqrt{\left\{\left[c_1^\ell - d_1^U - \left(\frac{b_2}{m_2}\right)^U\right]\frac{h_1^\ell}{r^U} - m_1^U h_2^U\right\}^2 - 4m_1^U\left[d_1^U + \left(\frac{b_2}{m_2}\right)^U\right]\frac{h_2^U r^U}{k^\ell}}\right\}.$$

In the same way, we can obtain

$$v(\eta_2) > \ln B_0^+ \quad \text{or} \quad v(\eta_2) < \ln B_0^-. \tag{2.15}$$

Using the definition of ξ_3 and the third equation of (2.2), we get

$$d_2(\xi_3)m_2(\xi_2)e^{2w(\xi_3)} + \left[d_2(\xi_3) - c_2(\xi_3)\right]e^{v(\xi_3)+w(\xi_3)} + h_3(\xi_3)e^{v(\xi_3)} + m_2(\xi_3)h_3(\xi_3)e^{w(\xi_3)} = 0,$$

which, combined with (2.8) and (2.9), yields

$$d_2^U m_2^U e^{2w(\xi_3)} + h_3^U\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell} - \left[\left(c_2^\ell - d_2^U\right)\frac{h_2^\ell}{c_1^U} - m_2^U h_3^U\right]e^{w(\xi_3)} > 0.$$

Solving the inequality, we have

$$w(\xi_3) > \ln C_0^+ \quad \text{or} \quad w(\xi_3) < \ln C_0^-, \tag{2.16}$$

where

$$C_0^{\pm} = \frac{1}{2d_2^U m_2^U}\left\{\left(c_2^\ell - d_2^U\right)\frac{h_2^\ell}{c_1^U} - m_2^U h_3^U \right.$$
$$\left. \pm\sqrt{\left[\left(c_2^\ell - d_2^U\right)\frac{h_2^\ell}{c_1^U} - m_2^U h_3^U\right]^2 - 4\frac{r^U c_1^U m_2^U d_2^U h_3^U}{k^\ell m_1^\ell d_1^\ell}}\right\}.$$

Likewise, it follows that

$$w(\eta_3) > \ln C_0^+ \quad \text{or} \quad w(\eta_3) < \ln C_0^-. \tag{2.17}$$

From (2.6), (2.7), (2.12), and (2.13), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{h_1^\ell}{r^U}\right\} < u(t) < \ln A_0^- \tag{2.18}$$

or

$$\ln A_0^+ < u(t) < \ln\left\{\frac{r^U}{k^\ell}\right\}. \tag{2.19}$$

From (2.8), (2.9), (2.14), and (2.15), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{h_2^\ell}{c_1^U}\right\} < v(t) < \ln B_0^- \tag{2.20}$$

or

$$\ln B_0^+ < v(t) < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}. \tag{2.21}$$

From (2.10), (2.11), (2.16), and (2.17), we obtain, for any $t \in [0, \omega]$,

$$\ln\left\{\frac{h_3^U}{c_2^U}\right\} < w(t) < \ln C_0^- \tag{2.22}$$

or

$$\ln C_0^+ < w(t) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \tag{2.23}$$

It is easily seen that $\ln A_0^\pm$, $\ln B_0^\pm$, $\ln C_0^\pm$, $\ln\{\frac{h_1^\ell}{r^U}\}$, $\ln\{\frac{r^U}{k^\ell}\}$, $\ln\{\frac{h_2^\ell}{c_1^U}\}$, $\ln\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\}$, $\ln\{\frac{h_3^U}{c_2^U}\}$, $\ln\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\}$ are independent of λ.

In the following, we will show that (i)-(iii) in Theorem A are satisfied.

First, let us take

$$X = Z = \left\{\big(u(t), v(t), w(t)\big)^T \in C\big(\mathbb{R}, \mathbb{R}^3\big) | u(t+\omega) = u(t), v(t+\omega) = v(t), w(t+\omega) = w(t)\right\}$$

and

$$\left\| \big(u(t), v(t), w(t)\big)^T \right\| = \max_{t\in[0,\omega]} \left|u(t)\right| + \max_{t\in[0,\omega]} \left|v(t)\right| + \max_{t\in[0,\omega]} \left|w(t)\right|.$$

Then X and Z are Banach spaces equipped with the norm $\|\cdot\|$.

Let

$$L\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} u'(t) \\ v'(t) \\ w'(t) \end{pmatrix},$$

where $\operatorname{Dom} L = \{(u, v, w)^T \in X : (u, v, w)^T \in C^1(\mathbb{R}, \mathbb{R}^3)\}$.

Define $N : X \to X$ as follows:

$$N\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} r(t) - k(t)e^{u(t)} - \frac{b_1(t)e^{v(t)}}{m_1(t)e^{v(t)} + e^{u(t)}} - \frac{h(t)}{e^{u(t)}} \\ \frac{c_1(t)e^{u(t)}}{m_1(t)e^{v(t)} + e^{u(t)}} - d_1(t) - \frac{b_2(t)e^{w(t)}}{m_2(t)e^{w(t)} + e^{v(t)}} - \frac{h_2(t)}{e^{v(t)}} \\ \frac{c_2(t)e^{v(t)}}{m_2(t)e^{w(t)} + e^{v(t)}} - d_2(t) - \frac{h_3(t)}{e^{w(t)}} \end{pmatrix} := \begin{pmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \end{pmatrix}.$$

Define projectors P and Q by

$$P\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = Q\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega u(t)\,dt \\ \frac{1}{\omega}\int_0^\omega v(t)\,dt \\ \frac{1}{\omega}\int_0^\omega w(t)\,dt \end{pmatrix}, \qquad \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \in X.$$

Then it follows that $\operatorname{Ker} L = \mathbb{R}^3$, $\operatorname{Im} L = \operatorname{Ker} Q = \{(u(t), v(t))^T \in X : \bar{u} = \bar{v} = \bar{w} = 0\}$ is closed in X, and $\dim \operatorname{Ker} L = 3 = \operatorname{codim} \operatorname{Im} L$, and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Hence, L is a Fredholm operator of index zero. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ is given by

$$K_P \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \int_0^t u(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t u(s)\,ds\,dt \\ \int_0^t v(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t v(s)\,ds\,dt \\ \int_0^t w(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t w(s)\,ds\,dt \end{pmatrix}.$$

Then

$$QN \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega N_1(s)\,ds \\ \frac{1}{\omega}\int_0^\omega N_2(s)\,ds \\ \frac{1}{\omega}\int_o^\omega N_3(s)\,ds \end{pmatrix}$$

and

$$K_p(I-Q)N \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \int_0^\omega N_1(t)\,dt - \frac{1}{\omega}\int_0^\omega \int_0^t N_1(s)\,ds\,dt + (\frac{1}{2} - \frac{t}{\omega})\int_0^\omega N_1(s)\,ds \\ \int_0^\omega N_2(t)\,dt - \frac{1}{\omega}\int_0^\omega \int_0^t N_2(s)\,ds\,dt + (\frac{1}{2} - \frac{t}{\omega})\int_0^\omega N_2(s)\,ds \\ \int_0^\omega N_3(t)\,dt - \frac{1}{\omega}\int_0^\omega \int_0^t N_3(s)\,ds\,dt + (\frac{1}{2} - \frac{t}{\omega})\int_0^\omega N_3(s)\,ds \end{pmatrix}.$$

Now, we reach the point where we search for appropriate open bounded subsets Ω_i, $i = 1, 2, \ldots, 8$, for the application of the continuation theorem. To this end, we take

$$\Omega_1 = \left\{ (u,v,w)^T \in X \;\middle|\; \begin{array}{l} u(t) \in (\ln\{\frac{h_1^\ell}{r^U}\}, \ln A_0^-) \\ v(t) \in (\ln\{\frac{h_2^\ell}{c_1^U}\}, \ln B_0^-) \\ w(t) \in (\ln\{\frac{h_3^U}{c_2^U}\}, \ln C_0^-) \end{array} \right\},$$

$$\Omega_2 = \left\{ (u,v,w)^T \in X \;\middle|\; \begin{array}{l} u(t) \in (\ln\{\frac{h_1^\ell}{r^U}\}, \ln A_0^-) \\ v(t) \in (\ln\{\frac{h_2^\ell}{c_1^U}\}, \ln B_0^-) \\ w(t) \in (\ln C_0^+, \ln\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\}) \end{array} \right\},$$

$$\Omega_3 = \left\{ (u,v,w)^T \in X \;\middle|\; \begin{array}{l} u(t) \in (\ln\{\frac{h_1^\ell}{r^U}\}, \ln A_0^-) \\ v(t) \in (\ln B_0^+, \ln\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\}) \\ w(t) \in (\ln\{\frac{h_3^U}{c_2^U}\}, \ln C_0^-) \end{array} \right\},$$

$$\Omega_4 = \left\{ (u,v,w)^T \in X \;\middle|\; \begin{array}{l} u(t) \in (\ln\{\frac{h_1^\ell}{r^U}\}, \ln A_0^-) \\ v(t) \in (\ln B_0^+, \ln\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\}) \\ w(t) \in (\ln C_0^+, \ln\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\}) \end{array} \right\},$$

$$\Omega_5 = \left\{ (u,v,w)^T \in X \;\middle|\; \begin{array}{l} u(t) \in (\ln A_0^+, \ln\{\frac{r^U}{k^\ell}\}) \\ v(t) \in (\ln\{\frac{h_2^\ell}{c_1^U}\}, \ln B_0^-) \\ w(t) \in (\ln\{\frac{h_3^U}{c_2^U}\}, \ln C_0^-) \end{array} \right\},$$

$$\Omega_6 = \left\{ (u,v,w)^T \in X \;\middle|\; \begin{array}{l} u(t) \in (\ln A_0^+, \ln\{\frac{r^U}{k^\ell}\}) \\ v(t) \in (\ln\{\frac{h_2^\ell}{c_1^U}\}, \ln B_0^-) \\ w(t) \in (\ln C_0^+, \ln\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\}) \end{array} \right\},$$

$$\Omega_7 = \left\{ (u,v,w)^T \in X \,\middle|\, \begin{array}{l} u(t) \in (\ln A_0^+, \ln\{\frac{r^U}{k^\ell}\}) \\ v(t) \in (\ln B_0^+, \ln\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\}) \\ w(t) \in (\ln\{\frac{h_3^U}{c_2^U}\}, \ln C_0^-) \end{array} \right\},$$

$$\Omega_8 = \left\{ (u,v,w)^T \in X \,\middle|\, \begin{array}{l} u(t) \in (\ln A_0^+, \ln\{\frac{r^U}{k^\ell}\}) \\ v(t) \in (\ln B_0^+, \ln\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\}) \\ w(t) \in (\ln C_0^+, \ln\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\}) \end{array} \right\}.$$

Then Ω_i $(i = 1,\dots,8)$ are bounded open subset of X, $\Omega_i \cap \Omega_j = \phi$, $i \neq j$, $i,j = 1,\dots,8$. Hence Ω_i $(i = 1,\dots,8)$ satisfies the requirement (i) in Theorem A.

Second, we will prove that (ii) holds. If it is not true, then when $(u,v,w)^T \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap \mathbb{R}^3$, $i = 1,\dots,8$, $QNx \neq 0$. There exist three points $t_1, t_2, t_3 \in [0,\omega]$ such that

$$\begin{cases} r(t_1) - k(t_1)e^u - \frac{b_1(t_1)e^v}{m_1(t_1)e^v + e^u} - \frac{h_1(t_1)}{e^u} = 0, \\ \frac{c_1(t_2)e^u}{m_1(t_2)e^v + e^u} - d_1(t_2) - \frac{b_2(t_2)e^w}{m_2(t_2)e^w + e^v} - \frac{h_2(t_2)}{e^v} = 0, \\ \frac{c_2(t_3)e^v}{m_2(t_3)e^w + e^v} - d_2(t_3) - \frac{h_3(t_3)}{e^w} = 0. \end{cases}$$

From the above arguments, we have

$$\begin{aligned} &\ln\left\{\frac{h_1^\ell}{r^U}\right\} < u(t) < \ln A_0^- \quad \text{or} \quad \ln A_0^+ < u(t) < \ln\left\{\frac{r^U}{k^\ell}\right\}, \\ &\ln\left\{\frac{h_2^\ell}{c_1^U}\right\} < v(t) < \ln B_0^- \quad \text{or} \quad \ln B_0^+ < v(t) < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}, \\ &\ln\left\{\frac{h_3^U}{c_2^U}\right\} < w(t) < \ln C_0^- \quad \text{or} \quad \ln C_0^+ < w(t) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \end{aligned}$$

Then we know $(u,v,w)^T$ belongs to one of $\Omega_i \cap \mathbb{R}^3$, $i = 1,\dots,8$. This leads to a contradiction.

Finally, we show that (iii) in Theorem A is satisfied. We proceed in our proofs by two steps.

On one hand, we show that, for $i = 1,\dots,8$,

$$\begin{aligned} \deg\{JQNx, \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\} &= \deg\{(N_1(t_1), N_2(t_2), N_3(t_3))^T, \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\} \\ &= \deg\{(\widehat{N}_1, \widehat{N}_2, \widehat{N}_3)^T, \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\}. \end{aligned} \tag{2.24}$$

Here

$$\begin{bmatrix} \widehat{N}_1 \\ \widehat{N}_2 \\ \widehat{N}_3 \end{bmatrix} = \begin{bmatrix} \hat{r} - \hat{k}e^u - \hat{h}_1 e^{-u} \\ (\hat{c}_1 - \hat{d}_1)e^{u+v} - \widehat{m}_1\hat{d}_1 e^{2v} - \hat{h}_2 e^u - \widehat{m}_1\hat{h}_2 e^v \\ (\hat{c}_2 - \hat{d}_2)e^{v+w} - \hat{h}_3(\widehat{m}_2 e^w + e^v) - \widehat{m}_2\hat{d}_2 e^{2w} \end{bmatrix},$$

and $\hat{r}$, $\hat{k}$, $\hat{b}_i$, $\hat{c}_i$, $\widehat{m}_i$ $(i = 1,2)$, $\hat{h}_j$, $j = 1,2,3$ are some chosen positive constants satisfying the following conditions:

$$\begin{aligned} &\hat{r}k^\ell < r^U\hat{k}, \qquad \hat{r}h_1^\ell < r^U\hat{h}_1, \qquad \hat{c}_1 m_1^\ell d_1^\ell < c_1^U\widehat{m}_1\hat{d}_1, \\ &\hat{c}_1 h_2^\ell < c_1^U\hat{h}_2, \qquad \hat{c}_2 m_2^\ell d_2^\ell < c_2^U\widehat{m}_2\hat{d}_2, \qquad \hat{c}_2 h_3^\ell < c_2^U\hat{h}_3, \end{aligned}$$

$$
\begin{aligned}
& A_0^+ < u^+ \triangleq \frac{\hat{r} + \sqrt{\hat{r}^2 - 4\hat{k}\hat{h}_1}}{2\hat{k}}, \qquad A_0^- > u^- \triangleq \frac{\hat{r} - \sqrt{\hat{r}^2 - 4\hat{k}\hat{h}_1}}{2\hat{k}}, \\
& B_0^+ < v^+ \\
& \quad \triangleq \frac{1}{2\widehat{m}_1\hat{d}_1}\left[(\hat{c}_1 - \hat{d}_1)\frac{h_1^\ell}{r^U} - \widehat{m}_1\hat{h}_2 + \sqrt{\left[(\hat{c}_1 - \hat{d}_1)\frac{h_1^\ell}{r^U} - \widehat{m}_1\hat{h}_2\right]^2 - \frac{4r^U\widehat{m}_1\hat{d}_1\hat{h}_2}{k^\ell}}\right], \\
& B_0^- > v^- \qquad\qquad (2.25) \\
& \quad \triangleq \frac{1}{2\widehat{m}_1\hat{d}_1}\left[(\hat{c}_1 - \hat{d}_1)\frac{h_1^\ell}{r^U} - \widehat{m}_1\hat{h}_2 - \sqrt{\left[(\hat{c}_1 - \hat{d}_1)\frac{h_1^\ell}{r^U} - \widehat{m}_1\hat{h}_2\right]^2 - \frac{4r^U\widehat{m}_1\hat{d}_1\hat{h}_2}{k^\ell}}\right], \\
& C_0^+ < w^+ \\
& \quad \triangleq \frac{1}{2\widehat{m}_2\hat{d}_2}\left[(\hat{c}_2 - \hat{d}_2)\frac{h_2^\ell}{c_1^U} - \widehat{m}_2\hat{h}_3 + \sqrt{\left[(\hat{c}_2 - \hat{d}_2)\frac{h_2^\ell}{c_1^U} - \widehat{m}_2\hat{h}_3\right]^2 - \frac{4r^U c_1^U\widehat{m}_2\hat{d}_2\hat{h}_3}{k^\ell m_1^\ell d_1^\ell}}\right], \\
& C_0^- > w^- \\
& \quad \triangleq \frac{1}{2\widehat{m}_2\hat{d}_2}\left[(\hat{c}_2 - \hat{d}_2)\frac{h_2^\ell}{c_1^U} - \widehat{m}_2\hat{h}_3 - \sqrt{\left[(\hat{c}_2 - \hat{d}_2)\frac{h_2^\ell}{c_1^U} - \widehat{m}_2\hat{h}_3\right]^2 - \frac{4r^U c_1^U\widehat{m}_2\hat{d}_2\hat{h}_3}{k^\ell m_1^\ell d_1^\ell}}\right].
\end{aligned}
$$

To this end, define a mapping $\phi_1 : \operatorname{Dom} L \times [0,1] \to X$ by

$$
\phi_1(u,v,w,\mu_1) = \mu_1 \begin{bmatrix} N_1(t_1) \\ N_2(t_2) \\ N_3(t_3) \end{bmatrix} + (1-\mu_1) \begin{bmatrix} \widehat{N}_1 \\ \widehat{N}_2 \\ \widehat{N}_3 \end{bmatrix},
$$

where $\mu_1 \in [0,1]$ is a parameter.

Now we show that $\phi_1(u,v,w,\mu_1) \neq 0$, $(u,v,w)^T \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap \mathbb{R}^3$, $i = 1,\ldots,8$. If it is not the case, then when $(u,v,w)^T \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap \mathbb{R}^3$, $i = 1,\ldots,8$, $\phi_1(u,v,w,\mu_1) = 0$. Therefore, the constant vector $(u,v,w)^T \in \mathbb{R}^3$ satisfies

$$
\mu_1\left[r(t_1) - k(t_1)e^u - \frac{b_1(t_1)e^v}{m_1(t_1)e^v + e^u} - \frac{h_1(t_1)}{e^u}\right] + (1-\mu_1)\left(\hat{r} - \hat{k}e^u - \hat{h}_1 e^{-u}\right) = 0, \qquad (2.26)
$$

$$
\begin{aligned}
& \mu_1\left[\frac{c_1(t_2)e^u}{m_1(t_2)e^v + e^u} - d_1(t_2) - \frac{b_2(t_2)e^w}{m_2(t_2)e^w + e^v} - \frac{h_2(t_2)}{e^v}\right] + (1-\mu_1) \\
& \quad \times \left[(\hat{c}_1 - \hat{d}_1)e^{u+v} - \widehat{m}_1\hat{d}_1 e^{2v} - \hat{h}_2 e^u - \widehat{m}_1\hat{h}_2 e^v\right] = 0, \qquad (2.27) \\
& \mu_1\left[\frac{c_2(t_3)e^v}{m_2(t_3)e^w + e^v} - d_2(t_3) - \frac{h_3(t_3)}{e^w}\right] + (1-\mu_1) \\
& \quad \times \left[(\hat{c}_2 - \hat{d}_2)e^{v+w} - \hat{h}_3\left(\widehat{m}_2 e^w + e^v\right) - \widehat{m}_2\hat{d}_2 e^{2w}\right] = 0. \qquad (2.28)
\end{aligned}
$$

From (2.26)-(2.28), we make the following nine claims.

(1) $u < \ln\{\frac{r^U}{k^\ell}\}$. Otherwise, $u \geq \ln\{\frac{r^U}{k^\ell}\}$. Then

$$
\begin{aligned}
& \mu_1\left[r(t_1) - k(t_1)e^u - \frac{b_1(t_1)e^v}{m_1(t_1)e^v + e^u} - \frac{h_1(t_1)}{e^u}\right] + (1-\mu_1)\left(\hat{r} - \hat{k}e^u - \hat{h}_1 e^{-u}\right) \\
& \quad < \mu_1\left(r^U - k^\ell e^u\right) + (1-\mu_1)\left(\hat{r} - \hat{k}e^u\right)
\end{aligned}
$$

$$\begin{aligned}
&< \mu_1\left(r^U - k^\ell \frac{r^U}{k^\ell}\right) + (1-\mu_1)\left(\hat{r} - \hat{k}\frac{r^U}{k^\ell}\right) \\
&< 0.
\end{aligned}$$

(2) $u > \ln\{\frac{h_1^\ell}{r^U}\}$. Otherwise, $u \le \ln\{\frac{h_1^\ell}{r^U}\}$. Then

$$\begin{aligned}
&\mu_1\left[r(t_1) - k(t_1)e^u - \frac{b_1(t_1)e^v}{m_1(t_1)e^v + e^u} - \frac{h_1(t_1)}{e^u}\right] + (1-\mu_1)\left(\hat{r} - \hat{k}e^u - \hat{h}_1e^{-u}\right) \\
&< \mu_1\left(r^U - h_1^\ell \frac{r^U}{h_1^\ell}\right) + (1-\mu_1)\left(\hat{r} - \hat{h}_1\frac{r^U}{h_1^\ell}\right) \\
&< 0.
\end{aligned}$$

(3) $u > \ln A_0^+$ or $u < \ln A_0^-$. Otherwise, $\ln A_0^- \le u \le \ln A_0^+$. Then

$$\begin{aligned}
&\mu_1\left[r(t_1) - k(t_1)e^u - \frac{b_1(t_1)e^v}{m_1(t_1)e^v + e^u} - \frac{h_1(t_1)}{e^u}\right] + (1-\mu_1)\left(\hat{r} - \hat{k}e^u - \hat{h}_1e^{-u}\right) \\
&= \frac{-\mu_1}{e^u}\left[k(t_1)e^{2u} + \frac{b_1(t_1)e^{v+u}}{m_1(t_1)e^v + e^u} - r(t_1)e^u + h_1(t_1)\right] \\
&\quad - \frac{1-\mu_1}{e^u}\left(\hat{k}e^{2u} - \hat{r}e^u + \hat{h}_1\right) \\
&> \frac{-\mu_1}{e^u}\left[k^U e^{2u} + \left(\frac{b_1}{m_1}\right)^U e^u - r^\ell e^u + h_1^U\right] - \frac{1-\mu_1}{e^u}\left(\hat{k}e^{2u} - \hat{r}e^u + \hat{h}_1\right) \\
&> -\frac{1-\mu_1}{e^u}\left(\hat{k}e^{2u} - \hat{r}e^u + \hat{h}_1\right) \\
&> 0.
\end{aligned}$$

Clearly, the above three inequalities contradict (2.26). Hence Claims 1-3 hold.

(4) $v < \{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\}$. Otherwise, $v \ge \{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\}$. Then

$$\begin{aligned}
&\mu_1\left[\frac{c_1(t_2)e^u}{m_1(t_2)e^v + e^u} - d_1(t_2) - \frac{b_2(t_2)e^w}{m_2(t_2)e^w + e^v} - \frac{h_2(t_2)}{e^v}\right] \\
&\quad + (1-\mu_1) \times \left[(\hat{c}_1 - \hat{d}_1)e^{u+v} - \widehat{m}_1\hat{d}_1e^{2v} - \hat{h}_2e^u - \widehat{m}_1\hat{h}_2e^v\right] \\
&< \mu_1\left[-d_1^\ell + \frac{c_1^U e^u}{m_1^\ell e^v}\right] + (1-\mu_1)e^{2v}\left(\frac{\hat{c}_1e^u}{e^v} - \widehat{m}_1\hat{d}_1\right) \\
&< (1-\mu_1)e^{2v}\left(\frac{\hat{c}_1 r^U k^\ell m_1^\ell d_1^\ell}{k^\ell r^U c_1^U} - \widehat{m}_1\hat{d}_1\right) \\
&< 0.
\end{aligned}$$

(5) $v > \ln\{\frac{h_2^\ell}{c_1^U}\}$. Otherwise, $v \le \ln\{\frac{h_2^\ell}{c_1^U}\}$. Then

$$\begin{aligned}
&\mu_1\left[\frac{c_1(t_2)e^u}{m_1(t_2)e^v + e^u} - d_1(t_2) - \frac{b_2(t_2)e^w}{m_2(t_2)e^w + e^v} - \frac{h_2(t_2)}{e^v}\right] \\
&\quad + (1-\mu_1) \times \left[(\hat{c}_1 - \hat{d}_1)e^{u+v} - \widehat{m}_1\hat{d}_1e^{2v} - \hat{h}_2e^u - \widehat{m}_1\hat{h}_2e^v\right]
\end{aligned}$$

$$
\begin{aligned}
&< \mu_1\left[c_1^U - h_2^\ell \frac{c_1^U}{h_2^\ell}\right] + (1-\mu_1)\left(\widehat{m}_1 e^v + e^u\right)e^v\left[\frac{\hat{c}_1 e^u}{\widehat{m}_1 e^v + e^u} - \frac{\hat{h}_2}{e^v}\right] \\
&< (1-\mu_1)\left(\widehat{m}_1 e^v + e^u\right)e^v\left[\hat{c}_1 - \hat{h}_2\frac{c_1^U}{h_2^\ell}\right] \\
&< 0.
\end{aligned}
$$

(6) $v > \ln B_0^+$ or $v < \ln B_0^-$. Otherwise, $\ln B_0^- \leq v \leq \ln B_0^+$. Then

$$
\begin{aligned}
&\mu_1\left[\frac{c_1(t_2)e^u}{m_1(t_2)e^v + e^u} - d_1(t_2) - \frac{b_2(t_2)e^w}{m_2(t_2)e^w + e^v} - \frac{h_2(t_2)}{e^v}\right] \\
&\quad + (1-\mu_1)\times\left[(\hat{c}_1 - \hat{d}_1)e^{u+v} - \widehat{m}_1\hat{d}_1 e^{2v} - \hat{h}_2 e^u - \widehat{m}_1\hat{h}_2 e^v\right] \\
&> \frac{-\mu_1}{e^v(m_1^U e^v + e^u)}\left\{m_1^U\left[d_1^U + \left(\frac{b_2}{m_2}\right)^U\right]e^{2v}\right. \\
&\quad \left. - \left[\left(c_1^\ell - d_1^U - \left(\frac{b_2}{m_2}\right)^U\right)\frac{h_1^\ell}{r^U} - h_2^U m_1^U\right]e^v + \frac{h_2^U r^U}{k^\ell}\right\} \\
&\quad - (1-\mu_1)\left\{\widehat{m}_1\hat{d}_1 e^{2v} - \left[(\hat{c}_1 - \hat{d}_1)\frac{h_1^\ell}{r^U} - \widehat{m}_1\hat{h}_2\right]e^v + \hat{h}_2\frac{r^U}{k^\ell}\right\} \\
&> 0.
\end{aligned}
$$

It is easy to see the above three inequalities contradict (2.27). Therefore, Claims 4-6 hold.

(7) $w < \ln\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\}$. Otherwise, $w \geq \ln\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\}$. Then

$$
\begin{aligned}
&\mu_1\left[\frac{c_2(t_3)e^v}{m_2(t_3)e^w + e^v} - d_2(t_3) - \frac{h_3(t_3)}{e^w}\right] + (1-\mu_1) \\
&\quad \times\left[(\hat{c}_2 - \hat{d}_2)e^{v+w} - \hat{h}_3\left(\widehat{m}_2 e^w + e^v\right) - \widehat{m}_2\hat{d}_2 e^{2w}\right] \\
&< \mu_1\left[\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell e^w} - d_2^\ell\right] + (1-\mu_1)e^{2w}\left[\frac{(\hat{c}_2 - \hat{d}_2)e^v}{e^w} - \widehat{m}_2\hat{d}_2\right] \\
&< (1-\mu_1)e^{2w}\left[\frac{r^U c_1^U \hat{c}_2}{k^\ell m_1^\ell d_1^\ell e^w} - \widehat{m}_2\hat{d}_2\right] \\
&< 0.
\end{aligned}
$$

(8) $w > \ln\{\frac{h_3^\ell}{c_2^U}\}$. Otherwise, $w \leq \ln\{\frac{h_3^\ell}{c_2^U}\}$. Then

$$
\begin{aligned}
&\mu_1\left[\frac{c_2(t_3)e^v}{m_2(t_3)e^w + e^v} - d_2(t_3) - \frac{h_3(t_3)}{e^w}\right] + (1-\mu_1) \\
&\quad \times\left[(\hat{c}_2 - \hat{d}_2)e^{v+w} - \hat{h}_3\left(\widehat{m}_2 e^w + e^v\right) - \widehat{m}_2\hat{d}_2 e^{2w}\right] \\
&< \mu_1\left[c_2^U - h_3^\ell\frac{c_2^U}{h_3^\ell}\right] + (1-\mu_1)\left(\widehat{m}_2 e^w + e^v\right)e^w\left[\frac{\hat{c}_2 e^v}{\widehat{m}_2 e^w + e^v} - \frac{\hat{h}_3}{e^w}\right] \\
&< (1-\mu_1)\left(\widehat{m}_2 e^w + e^v\right)e^w\left[\hat{c}_2 - \hat{h}_3\frac{c_2^U}{h_3^\ell}\right] \\
&< 0.
\end{aligned}
$$

(9) $w > \ln C_0^+$ or $w < \ln C_0^-$. Otherwise, $\ln C_0^- \le w \le \ln C_0^+$. Then

$$\begin{aligned}
&\mu_1\left[\frac{c_2(t_3)e^v}{m_2(t_3)e^w + e^v} - d_2(t_3) - \frac{h_3(t_3)}{e^w}\right] + (1-\mu_1) \\
&\quad\times\left[(\hat{c}_2 - \hat{d}_2)e^{v+w} - \hat{h}_3\left(\widehat{m}_2 e^w + e^v\right) - \widehat{m}_2\hat{d}_2 e^{2w}\right] \\
&> \frac{-\mu_1}{(m_2(t_3)e^w + e^v)e^w}\left[d_2^U m_2^U e^{2w} + h_3^U \frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell} - \left((c_2 - d_2)^\ell \frac{h_2^\ell}{c_1^U} - m_2^U h_3^U\right)e^w\right] \\
&\quad - (1-\mu_1)\left\{\widehat{m}_2\hat{d}_2 e^{2w} - \left[(\hat{c}_2 - \hat{d}_2)\frac{h_2^\ell}{c_1^U} - \widehat{m}_2\hat{h}_3\right]e^w + \hat{h}_3\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\} \\
&> -(1-\mu_1)\left\{\widehat{m}_2\hat{d}_2 e^{2w} - \left[(\hat{c}_2 - \hat{d}_2)\frac{h_2^\ell}{c_1^U} - \widehat{m}_2\hat{h}_3\right]e^w + \hat{h}_3\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\} \\
&> 0.
\end{aligned}$$

Obviously, the above three inequalities contradict (2.28). Hence Claims 7-9 hold.

From the above arguments (1)-(9), we have

$$\begin{aligned}
&\ln\left\{\frac{h_1^\ell}{r^U}\right\} < u < \ln A_0^- \quad\text{or}\quad \ln A_0^+ < u < \ln\left\{\frac{r^U}{k^\ell}\right\}, \\
&\ln\left\{\frac{h_2^\ell}{c_1^U}\right\} < v < \ln B_0^- \quad\text{or}\quad \ln B_0^+ < v < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}, \\
&\ln\left\{\frac{h_3^U}{c_2^U}\right\} < w < \ln C_0^- \quad\text{or}\quad \ln C_0^+ < w < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}.
\end{aligned}$$

These indicate that $(u,v,w)^T$ belongs to one of $\Omega_i \cap \mathbb{R}^3$, $i = 1,\ldots,8$. This is a contradiction.

On the other hand, we prove that, for $i = 1,\ldots,8$,

$$\begin{aligned}
&\deg\left\{(\widehat{N}_1, \widehat{N}_2, \widehat{N}_3)^T, \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\right\} \\
&\quad = \deg\left\{\left[\hat{r} - \hat{k}e^u - \hat{h}_1 e^{-u}, \hat{c}_1 e^{u+v} - \widehat{m}_1\hat{d}_1 e^{2v} - \hat{h}_2 e^u, \hat{c}_2 e^{v+w} - \widehat{m}_2\hat{d}_2 e^{2w} - \hat{h}_3 e^v\right]^T,\right. \\
&\quad\quad \left.\Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\right\}. \qquad (2.29)
\end{aligned}$$

To this end, we define a mapping $\psi_2 : \operatorname{Dom} L \times [0,1] \to X$ by

$$\psi_2(u,v,w,\mu_2) = \begin{bmatrix} \hat{r} - \hat{k}e^u - \hat{h}_1 r^{-u} \\ \hat{c}_1 e^{u+v} - \widehat{m}_1\hat{d}_1 e^{2v} - \hat{h}_2 e^u - \mu_2(\widehat{m}_1\hat{h}_2 e^v + \hat{d}_1 e^{u+v}) \\ \hat{c}_2 e^{v+w} - \widehat{m}_2\hat{d}_2 e^{2w} - \hat{h}_3 e^v - \mu_2(\widehat{m}_2\hat{h}_3 e^w + \hat{d}_2 e^{v+w}) \end{bmatrix},$$

where $\mu_2 \in [0,1]$ is a parameter. We prove that when $(u,v,w)^T \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap \mathbb{R}^3$, $i = 1,\ldots,8$, $\psi_2(u,v,w,\mu_2) \neq (0,0,0)^T$. If it is not true, then the constant vector $(u,v,w)^T \in \partial\Omega_i \cap \mathbb{R}^3$, $i = 1,\ldots,8$ satisfies the following equalities:

$$\begin{cases} \hat{r} - \hat{k}e^u - \hat{h}_1 e^{-u} = 0, \\ \hat{c}_1 e^{u+v} - \widehat{m}_1\hat{d}_1 e^{2v} - \hat{h}_2 e^u - \mu_2(\widehat{m}_1\hat{h}_2 e^v + \hat{d}_1 e^{u+v}) = 0, \\ \hat{c}_2 e^{v+w} - \widehat{m}_2\hat{d}_2 e^{2w} - \hat{h}_3 e^v - \mu_2(\widehat{m}_2\hat{h}_3 e^w + \hat{d}_2 e^{v+w}) = 0. \end{cases}$$

By similar arguments to the above estimation of $(u,v,w)^T$, we can obtain

$$\begin{aligned}
&\ln\left\{\frac{\hat{h}_1}{\hat{r}}\right\} < u < \ln u^- \quad \text{or} \quad \ln u^+ < u < \ln\left\{\frac{\hat{r}}{\hat{k}}\right\},\\
&\ln\left\{\frac{\hat{h}_2}{\hat{c}_1}\right\} < v < \ln v^- \quad \text{or} \quad \ln v^+ < v < \ln\left\{\frac{\hat{r}\hat{c}_1}{\hat{k}\widehat{m}_1\hat{d}_1}\right\},\\
&\ln\left\{\frac{\hat{h}_3}{\hat{c}_2}\right\} < w < \ln w^- \quad \text{or} \quad \ln w^+ < w < \ln\left\{\frac{\hat{r}\hat{c}_1\hat{c}_2}{\hat{k}\widehat{m}_1\widehat{m}_2\hat{d}_1\hat{d}_2}\right\}.
\end{aligned}$$

Therefore, combined with the conditions in (2.25), it follows that

$$\begin{aligned}
&\ln\left\{\frac{h_1^\ell}{r^U}\right\} < u < \ln A_0^- \quad \text{or} \quad \ln A_0^+ < u < \ln\left\{\frac{r^U}{k^\ell}\right\},\\
&\ln\left\{\frac{h_2^\ell}{c_1^U}\right\} < v < \ln B_0^- \quad \text{or} \quad \ln B_0^+ < v < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\},\\
&\ln\left\{\frac{h_3^U}{c_2^U}\right\} < w < \ln C_0^- \quad \text{or} \quad \ln C_0^+ < w < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\},
\end{aligned}$$

which implies $(u,v,w)^T$ belongs to one of Ω_i, $i = 1,\dots,8$. This is a contradiction. Hence $\psi_2(u,v,w,\mu_2) \neq (0,0,0)^T$, $(u,v,w)^T \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap \mathbb{R}^3$, $i = 1,\dots,8$.

By using homotopy invariance of topological degree and (2.24), (2.29), we have, for $i = 1,\dots,8$,

$$\begin{aligned}
&\deg\bigl\{JQNx, \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\bigr\}\\
&\quad = \deg\bigl\{\psi_1(u,v,w,1), \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\bigr\}\\
&\quad = \deg\bigl\{\psi_1(u,v,w,0), \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\bigr\}\\
&\quad = \deg\bigl\{\psi_2(u,v,w,1), \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\bigr\}\\
&\quad = \deg\bigl\{\psi_2(u,v,w,0), \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\bigr\}\\
&\quad = \deg\bigl\{\bigl[\hat{r} - \hat{k}e^u - \hat{h}_1e^{-u}, \hat{c}_1e^{u+v} - \widehat{m}_1\hat{d}_1e^{2v} - \hat{h}_2e^u, \hat{c}_2e^{v+w} - \widehat{m}_2\hat{d}_2e^{2w} - \hat{h}_3e^v\bigr]^T,\\
&\qquad \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\bigr\}.
\end{aligned}$$

Now, we consider the following algebraic equations:

$$\begin{cases}
\hat{r} - \hat{k}e^u - \hat{h}_1e^{-u} = 0,\\
\hat{c}_1e^{u+v} - \widehat{m}_1\hat{d}_1e^{2v} - \hat{h}_2e^u = 0,\\
\hat{c}_2e^{v+w} - \widehat{m}_2\hat{d}_2e^{2w} - \hat{h}_3e^v = 0.
\end{cases}$$

It is not difficult to find the equations has eight distinct solutions,

$$\begin{aligned}
&(u_1^*, v_1^*, w_1^*) = (\ln u^+, \ln v_+^+, \ln w_{++}^+), \qquad (u_2^*, v_2^*, w_2^*) = (\ln u^+, \ln v_+^+, \ln w_{++}^-),\\
&(u_3^*, v_3^*, w_3^*) = (\ln u^+, \ln v_-^+, \ln w_{+-}^+), \qquad (u_4^*, v_4^*, w_4^*) = (\ln u^+, \ln v_-^+, \ln w_{+-}^-),\\
&(u_5^*, v_5^*, w_5^*) = (\ln u^-, \ln v_+^-, \ln w_{-+}^+), \qquad (u_6^*, v_6^*, w_6^*) = (\ln u^-, \ln v_+^-, \ln w_{-+}^-),
\end{aligned}$$

$$\left(u_7^*, v_7^*, w_7^*\right) = \left(\ln u^-, \ln v_-^-, \ln w_{--}^+\right), \qquad \left(u_8^*, v_8^*, w_8^*\right) = \left(\ln u^-, \ln v_-^-, \ln w_{--}^-\right),$$

where

$$v_+^{\pm} = \ln\left\{\frac{\hat{c}_1 u^+ \pm \sqrt{(\hat{c}_1 u^+)^2 - 4\widehat{m}_1 \hat{d}_1 \hat{h}_2 u^+}}{2\widehat{m}_1 \hat{d}_1}\right\},$$

$$v_-^{\pm} = \ln\left\{\frac{\hat{c}_1 u^- \pm \sqrt{(\hat{c}_1 u^-)^2 - 4\widehat{m}_1 \hat{d}_1 \hat{h}_2 u^-}}{2\widehat{m}_1 \hat{d}_1}\right\},$$

$$w_{++}^{\pm} = \ln\left\{\frac{\hat{c}_2 v_+^+ \pm \sqrt{(\hat{c}_2 v_+^+)^2 - 4\widehat{m}_2 \hat{d}_2 \hat{h}_3 v_+^+}}{2\widehat{m}_2 \hat{d}_2}\right\},$$

$$w_{-+}^{\pm} = \ln\left\{\frac{\hat{c}_2 v_+^- \pm \sqrt{(\hat{c}_2 v_+^-)^2 - 4\widehat{m}_2 \hat{d}_2 \hat{h}_3 v_+^-}}{2\widehat{m}_2 \hat{d}_2}\right\},$$

$$w_{+-}^{\pm} = \ln\left\{\frac{\hat{c}_2 v_-^+ \pm \sqrt{(\hat{c}_2 v_-^+)^2 - 4\widehat{m}_2 \hat{d}_2 \hat{h}_3 v_-^+}}{2\widehat{m}_2 \hat{d}_2}\right\},$$

$$w_{--}^{\pm} = \ln\left\{\frac{\hat{c}_2 v_-^- \pm \sqrt{(\hat{c}_2 v_-^-)^2 - 4\widehat{m}_2 \hat{d}_2 \hat{h}_3 v_-^-}}{2\widehat{m}_2 \hat{d}_2}\right\}.$$

It is easy to verify that (u_i^*, v_i^*, w_i^*) belongs to one of Ω_j, $i, j = 1, \ldots, 8$.

It follows from the definition of the topological degree that

$$\begin{aligned}
&\deg\left\{JQNx, \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\right\} \\
&\quad = \operatorname{sign}\begin{vmatrix} -\hat{k}e^{u_i^*} + \hat{h}_1 e^{-u_i^*} & 0 & 0 \\ \hat{c}_1 e^{u_i^* + v_i^*} - \hat{h}_2 e^{u_i^*} & \hat{c}_1 e^{u_i^* + v_i^*} - 2\widehat{m}_1 \hat{d}_1 e^{2v} & 0 \\ 0 & \hat{c}_2 e^{v_i^* + w_i^*} - \hat{h}_3 e^{v_i^*} & \hat{c}_2 e^{v_i^* + w_i^*} - 2\widehat{m}_2 \hat{d}_2 e^{2w_i^*} \end{vmatrix} \\
&\quad = \operatorname{sign}\left[\left(-\hat{k}e^{u_i^*} + \hat{h}_1 e^{-u_i^*}\right)\left(\hat{c}_1 e^{u_i^* + v_i^*} - 2\widehat{m}_1 \hat{d}_1 e^{2v_i^*}\right)\left(\hat{c}_2 e^{v_i^* + w_i^*} - 2\widehat{m}_2 \hat{d}_2 e^{2w_i^*}\right)\right] \\
&\quad = -\operatorname{sign}\left[\left(2\hat{k}e^{u_i^*} - \hat{r}\right)\left(\hat{c}_1 e^{u_i^*} - 2\widehat{m}_1 \hat{d}_1 e^{v_i^*}\right)\left(\hat{c}_2 e^{v_i^*} - 2\widehat{m}_2 \hat{d}_2 e^{w_i^*}\right)\right].
\end{aligned}$$

Then, by direct calculation, we obtain

$$\deg\left\{JQNx, \Omega_i \cap \operatorname{Ker} L, (0,0,0)^T\right\} \neq 0, \quad i = 1, \ldots, 8.$$

By now, we have proved that each Ω_i $(i = 1, \ldots, 8)$ satisfies all the requirements of Theorem A. Hence, system (2.1) has at least one ω-periodic solution in each of $\Omega_1, \ldots, \Omega_8$. The proof is completed. □

Theorem 2.2 *If* $h_1(t) \neq 0$, $h_2(t) \neq 0$, $h_3(t) = 0$, *and* (H1), (H2) *are satisfied. Moreover,*

(H4) $c_2^\ell > d_2^u$.

Then system (1.3) *has at least four positive periodic solutions.*

Theorem 2.3 *If* $h_1(t) \neq 0$, $h_2(t) = 0$, $h_3(t) \neq 0$, *and* (H1) *is satisfied. Moreover,*

$$\text{(H5)}\quad c_1^\ell > d_1^U + \left(\frac{b_2}{m_2}\right)^U,$$

$$\text{(H6)}\quad \left(c_2^\ell - d_2^U\right)\frac{[c_1^\ell - d_1^U - (b_2/m_2)^U]h_1^\ell}{r^U m_1^U[d_1^U + (b_2/m_2)^U]} - m_2^U h_3^U > 2\sqrt{\frac{r^U c_1^U m_2^U d_2^U h_3^U}{k^\ell m_1^\ell d_1^\ell}}.$$

Then system (1.3) *has at least four positive periodic solutions.*

Theorem 2.4 *If* $h_1(t) = 0$, $h_2(t) \neq 0$, $h_3(t) \neq 0$, *and* (H3) *is satisfied. Moreover,*

$$\text{(H7)}\quad r^\ell > \left(\frac{b_1}{m_1}\right)^U,$$

$$\text{(H8)}\quad \left[c_1^\ell - d_1^U - \left(\frac{b_2}{m_2}\right)^U\right]\frac{r^\ell - (b_1/m_1)^U}{k^U} - m_1^U h_2^U > 2\sqrt{m_1^U\left[d_1^U + \left(\frac{b_2}{m_2}\right)^U\right]\frac{r^U h_2^U}{k^\ell}}.$$

Then system (1.3) *has at least four positive periodic solutions.*

Theorem 2.5 *If* $h_1(t) \neq 0$, $h_2(t) = 0$, $h_3(t) = 0$, *and* (H1), (H4), (H5) *are satisfied, then system* (1.3) *has at least two positive periodic solutions.*

Theorem 2.6 *If* $h_1(t) = 0$, $h_2(t) \neq 0$, $h_3(t) = 0$, *and* (H4), (H7), (H8) *are satisfied, then system* (1.3) *has at least two positive periodic solutions.*

Theorem 2.7 *If* $h_1(t) = 0$, $h_2(t) = 0$, $h_3(t) \neq 0$, *and* (H5), (H7) *are satisfied. Moreover,*

$$\text{(H9)}\quad \left(c_2^\ell - d_2^U\right)\frac{[c_1^\ell - d_1^U - (b_2/m_2)^U][r^\ell - (b_1/m_1)^U]}{k^U m_1^U[d_1^U + (b_2/m_2)^U]} - m_2^U h_3^U > 2\sqrt{\frac{r^U c_1^U m_2^U d_2^U h_3^U}{k^\ell m_1^\ell d_1^\ell}}.$$

Then system (1.3) *has at least two positive periodic solutions.*

Theorem 2.8 *If* $h_1(t) = 0$, $h_2(t) = 0$, $h_3(t) = 0$, *and* (H4), (H5), (H7) *are satisfied, then system* (1.3) *has at least one positive periodic solution.*

3 Conclusions

In this paper, with the help of a continuation theorem based on Gaines and Mawhin's coincidence degree theory, we study the existence and multiplicity of periodic solutions of a ratio-dependent food chain model with exploited term(s). Under some appropriate conditions, some sufficient criteria are established for the existence and multiplicity of periodic solutions. It worth mentioning that the results reported here are rather interesting. To make this point clear, we take i = number of exploited terms, then by our main results, there are at least 2^i periodic solutions. In fact, by our observation, the same result is valid for the models with one prey and one predator in the literature; for example, see [6, 7, 11, 13]. So, a natural question that one may ask is whether the assertion is fit for higher-dimensional biological and ecological systems (≥ 4).

Appendix: Proofs of Theorems 2.2-2.8

Clearly, all the arguments used in the proof of Theorem 2.1 can be applied here. Therefore, in this part, we only make the estimation of $(u(t), v(t), w(t))^T$ and omit the detailed proofs for space reasons.

Proof of Theorem 2.2 From (2.18)-(2.21), we know, for any $t \in [0, \omega]$,

$$\ln\left\{\frac{h_1^\ell}{r^U}\right\} < u(t) < \ln A_0^- \quad \text{or} \quad \ln A_0^+ < u(t) < \ln\left\{\frac{r^U}{k^\ell}\right\}$$

and

$$\ln\left\{\frac{h_2^\ell}{c_1^U}\right\} < v(t) < \ln B_0^- \quad \text{or} \quad \ln B_0^+ < v(t) < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}.$$

From the third equation of (2.2) and (2.5), (2.9), we have

$$\frac{c_2^\ell h_2^\ell / c_1^U}{m_2^U e^{w(\eta_3)} + h_2^\ell / c_1^U} < \frac{c_2(\eta_3) e^{v(\eta_3)}}{m_2(\eta_3) e^{w(\eta_3)} + e^{v(\eta_3)}} = d_2(\eta_3) \leq d_2^U,$$

which reduces to

$$w(\eta_3) > \ln\left\{\frac{(c_2^\ell - d_2^U) h_2^\ell}{c_1^U m_2^U d_2^U}\right\}. \tag{A.1}$$

Then, from (2.10) and (A.1), we obtain, for any $t \in [0, \omega]$,

$$\ln\left\{\frac{(c_2^\ell - d_2^U) h_2^\ell}{c_1^U m_2^U d_2^U}\right\} < w(t) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}.$$

□

Proof of Theorem 2.3 From (2.18) and (2.19), we know, for any $t \in [0, \omega]$,

$$\ln\left\{\frac{h_1^\ell}{r^U}\right\} < u(t) < \ln A_0^- \quad \text{or} \quad \ln A_0^+ < u(t) < \ln\left\{\frac{r^U}{k^\ell}\right\}.$$

From the second equation of (2.2) and (2.4), (2.7), we get

$$\begin{aligned}\frac{c_1^\ell h_1^\ell / r^U}{m_1^U e^{v(\eta_2)} + h_1^\ell / r^U} &\leq \frac{c_1(\eta_2) e^{u(\eta_2)}}{m_1(\eta_2) e^{v(\eta_2)} + e^{u(\eta_2)}} \\ &= d_1(\eta_2) + \frac{b_2(\eta_2) e^{w(\eta_2)}}{m_2(\eta_2) e^{w(\eta_2)} + e^{v(\eta_2)}} < d_1^U + \left(\frac{b_2}{m_2}\right)^U,\end{aligned}$$

which implies

$$v(\eta_2) > \ln\left\{\frac{[c_1^\ell - d_1^U - (b_2/m_2)^U] h_1^\ell}{r^U m_1^U (d_1^U + (b_2/m_2)^U)}\right\} \triangleq \ln M. \tag{A.2}$$

From (2.8) and (A.2), we have, for any $t \in [0, \omega]$,

$$\ln M < v(t) < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}.$$

Using the definition of ξ_3 and the third equation of (2.2), we get

$$d_2(\xi_3)m_2(\xi_3)e^{2w(\xi_3)} + \left[d_2(\xi_3) - c_2(\xi_3)\right]e^{v(\xi_3)+w(\xi_3)} + h_3(\xi_3)e^{v(\xi_3)} + m_2(\xi_3)h_3(\xi_3)e^{w(\xi_3)} = 0,$$

which combined with (2.8) and (A.2) produces

$$d_2^U m_2^U e^{2w(\xi_3)} + h_3^U \frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell} - \left[(c_2 - d_2)^\ell M - m_2^U h_3^U\right]e^{w(\xi_3)} > 0.$$

Solving the inequality, we have

$$w(\xi_3) > \ln C_1^+ \quad \text{or} \quad w(\xi_3) < \ln C_1^-, \tag{A.3}$$

where

$$C_1^\pm = \frac{1}{2d_2^U m_2^U}\Bigg\{\left(c_2^\ell - d_2^U\right)M - m_2^U h_3^U \\ \pm \sqrt{\left[\left(c_2^\ell - d_2^U\right)M - m_2^U h_3^U\right]^2 - 4\frac{r^U c_1^U m_2^U d_2^U h_3^U}{k^\ell m_1^\ell d_1^\ell}}\Bigg\}.$$

In the same way, we derive

$$w(\eta_3) > \ln C_1^+ \quad \text{or} \quad w(\eta_3) < \ln C_1^-. \tag{A.4}$$

From (2.10), (2.11), (A.3), and (A.4), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{h_3^U}{c_2^U}\right\} < w(t) < \ln C_1^- \quad \text{or} \quad \ln C_1^+ < w(t) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \qquad \square$$

Proof of Theorem 2.4 By the first equation of (2.2) and (2.3), we get

$$r^\ell \le r(\eta_1) = k(\eta_1)e^{u(\eta_1)} + \frac{b_1(\eta_1)e^{v(\eta_1)}}{m_1(\eta_1)e^{v(\eta_1)} + e^{u(\eta_1)}} < k^U e^{u(\eta_1)} + \left(\frac{b_1}{m_1}\right)^U,$$

which produces

$$u(\eta_1) > \ln\left\{\frac{r^\ell - (b_1/m_1)^U}{k^U}\right\}. \tag{A.5}$$

From (2.6) and (A.5), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{r^\ell - (b_1/m_1)^U}{k^U}\right\} < u(t) < \ln\left\{\frac{r^U}{k^\ell}\right\}.$$

By the definition of ξ_2 and the second equation of (2.2), we have

$$d_1(\xi_2) + \frac{b_2(\xi_2)e^{w(\xi_2)}}{m_2(\xi_2)e^{w(\xi_2)} + e^{v(\xi_2)}} + h_2(\xi_2)e^{-v(\xi_2)} - \frac{c_1(\xi_2)e^{u(\xi_2)}}{m_1(\xi_2)e^{v(\xi_2)} + e^{u(\xi_2)}} = 0,$$

which together with (A.5) means

$$
\begin{aligned}
& m_1^U\left[d_1^U+\left(\frac{b_2}{m_2}\right)^U\right]e^{2\nu(\xi_2)} \\
& \quad -\left\{\left[c_1^\ell-d_1^U-\left(\frac{b_2}{m_2}\right)^U\right]\frac{r^\ell-(b_1/m_1)^U}{k^U}-h_2^U m_1^U\right\}e^{\nu(\xi_2)}+\frac{r^U h_2^U}{k^\ell}>0.
\end{aligned}
$$

Solving the inequality, we get

$$
\nu(\xi_2)>\ln B_1^+ \quad \text{or} \quad \nu(\xi_2)<\ln B_1^-, \tag{A.6}
$$

where

$$
\begin{aligned}
B_1^\pm = {} & \frac{1}{2m_1^U[d_1^U+(\frac{b_2}{m_2})^U]}\left\{\left[c_1^\ell-d_1^U-\left(\frac{b_2}{m_2}\right)^U\right]\frac{r^\ell-(b_1/m_1)^U}{k^U}-m_1^U h_2^U\right. \\
& \left.\pm\sqrt{\left\{\left[c_1^\ell-d_1^U-\left(\frac{b_2}{m_2}\right)^U\right]\frac{r^\ell-(b_1/m_1)^U}{k^U}-m_1^U h_2^U\right\}^2-4m_1^U\left[d_1^U+\left(\frac{b_2}{m_2}\right)^U\right]\frac{h_2^U r^U}{k^\ell}}\right\}.
\end{aligned}
$$

In the same way, we can obtain

$$
\nu(\eta_2)>\ln B_1^+ \quad \text{or} \quad \nu(\eta_2)<\ln B_1^-. \tag{A.7}
$$

From (2.8), (2.9), (A.6), and (A.7), we obtain, for any $t\in[0,\omega]$,

$$
\ln\left\{\frac{h_2^\ell}{c_1^U}\right\}<\nu(t)<\ln B_1^- \quad \text{or} \quad \ln B_1^+<\nu(t)<\ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}. \tag{A.8}
$$

From (2.22) and (2.23), we obtain, for any $t\in[0,\omega]$,

$$
\ln\left\{\frac{h_3^U}{c_2^U}\right\}<w(t)<\ln C_0^- \quad \text{or} \quad \ln C_0^+<w(t)<\ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \qquad \square
$$

Proof of Theorem 2.5 From (2.18) and (2.19), (2.8) and (A.2), we obtain, for any $t\in[0,\omega]$,

$$
\begin{aligned}
& \ln\left\{\frac{h_1^\ell}{r^U}\right\}<u(t)<\ln A_0^- \quad \text{or} \quad \ln A_0^+<u(t)<\ln\left\{\frac{r^U}{k^\ell}\right\}, \\
& \ln\left\{\frac{[c_1^\ell-d_1^U-(b_2/m_2)^U]h_1^\ell}{r^U m_1^U(d_1^U+(b_2/m_2)^U)}\right\}<\nu(t)<\ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}.
\end{aligned}
$$

From the third equation of (2.2) and (2.5), (A.2), we get

$$
\frac{c_2^\ell M}{m_2^U e^{w(\eta_3)}+M}<\frac{c_2(\eta_3)e^{\nu(\eta_3)}}{m_2(\eta_3)e^{w(\eta_3)}+e^{\nu(\eta_3)}}=d_2(\eta_3)\leq d_2^U,
$$

which produces

$$
w(\eta_3)>\ln\left\{\frac{(c_2^\ell-d_2^U)M}{m_2^U d_2^U}\right\}. \tag{A.9}
$$

Then, from (2.10) and (A.9), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{[c_1^\ell - d_1^U - (b_2/m_2)^U](c_2^\ell - d_2^U)h_1^\ell}{r^U m_1^U m_2^U d_2^U (d_1^U + (b_2/m_2)^U)}\right\} < w(t) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \qquad \square$$

Proof of Theorem 2.6 From (2.6) and (A.5), (A.8), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{r^\ell - (b_1/m_1)^U}{k^U}\right\} < u(t) < \ln\left\{\frac{r^U}{k^\ell}\right\},$$
$$\ln\left\{\frac{h_2^\ell}{c_1^U}\right\} < v(t) < \ln B_1^- \quad \text{or} \quad \ln B_1^+ < v(t) < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}.$$

From (2.10) and (A.1), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{(c_2^\ell - d_2^U)h_2^\ell}{c_1^U m_2^U d_2^U}\right\} < w(t) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \qquad \square$$

Proof of Theorem 2.7 From (2.6) and (A.5), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{r^\ell - (b_1/m_1)^U}{k^U}\right\} < u(t) < \ln\left\{\frac{r^U}{k^\ell}\right\}.$$

From the second equation of (2.2) and (2.4), (A.5), we get

$$\begin{aligned}\frac{c_1^\ell[r^\ell - (b_1/m_1)^U]/k^U}{m_1^U e^{v(\eta_2)} + [r^\ell - (b_1/m_1)^U]/k^U} &\le \frac{c_1(\eta_2)e^{u(\eta_2)}}{m_1(\eta_2)e^{v(\eta_2)} + e^{u(\eta_2)}} \\ &= d_1(\eta_2) + \frac{b_2(\eta_2)e^{w(\eta_2)}}{m_2(\eta_2)e^{w(\eta_2)} + e^{v(\eta_2)}} < d_1^U + \left(\frac{b_2}{m_2}\right)^U,\end{aligned}$$

which implies

$$v(\eta_2) > \ln\left\{\frac{[c_1^\ell - d_1^U - (b_2/m_2)^U][r^\ell - (b_1/m_1)^U]}{k^U m_1^U [d_1^U + (b_2/m_2)^U]}\right\} \triangleq \ln N. \tag{A.10}$$

It follows from (2.8) and (A.10) that, for any $t \in [0,\omega]$,

$$\ln N < v(t) < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}.$$

Using the definition of ξ_3 and the third equation of (2.2), we get

$$d_2(\xi_3)m_2(\xi_2)e^{2w(\xi_3)} + \left[d_2(\xi_3) - c_2(\xi_3)\right]e^{v(\xi_3)+w(\xi_3)} + h_3(\xi_3)e^{v(\xi_3)} + m_2(\xi_3)h_3(\xi_3)e^{w(\xi_3)} = 0,$$

which combined with (2.8) and (A.10) produces

$$d_2^U m_2^U e^{2w(\xi_3)} + h_3^U \frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell} - \left[(c_2 - d_2)^\ell N - m_2^U h_3^U\right]e^{w(\xi_3)} > 0.$$

Solving the inequality, we have

$$w(\xi_3) > \ln C_2^+ \quad \text{or} \quad w(\xi_3) < \ln C_2^-, \tag{A.11}$$

where

$$C_2^{\pm} = \frac{1}{2d_2^U m_2^U}\left\{(c_2^\ell - d_2^U)N - m_2^U h_3^U \pm \sqrt{\left[(c_2-d_2)^\ell N - m_2^U h_3^U\right]^2 - 4\frac{r^U c_1^U m_2^U d_2^U h_3^U}{k^\ell m_1^\ell d_1^\ell}}\right\}.$$

From (2.10), (2.11) and (A.11), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{h_3^U}{c_2^U}\right\} < w(t) < \ln C_2^- \quad \text{or} \quad \ln C_2^+ < w(t) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \qquad \square$$

Proof of Theorem 2.8 From (2.6) and (A.5), (2.8) and (A.10), (2.10) and (A.1), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{r^\ell - (b_1/m_1)^U}{k^U}\right\} < u(t) < \ln\left\{\frac{r^U}{k^\ell}\right\},$$
$$\ln\left\{\frac{[c_1^\ell - d_1^U - (b_2/m_2)^U][r^\ell - (b_1/m_1)^U]}{k^U m_1^U [d_1^U + (b_2/m_2)^U]}\right\} < v(t) < \ln\left\{\frac{r^U c_1^U}{k^\ell m_1^\ell d_1^\ell}\right\}.$$

From the third equation of (2.2) and (2.5), (A.10), we get

$$\frac{c_2^\ell N}{m_2^U e^{w(\eta_3)} + N} < \frac{c_2(\eta_3) e^{v(\eta_3)}}{m_2(\eta_3) e^{w(\eta_3)} + e^{v(\eta_3)}} = d_2(\eta_3) \leq d_2^U,$$

which produces

$$w(\eta_3) > \ln\left\{\frac{(c_2^\ell - d_2^U)N}{m_2^U d_2^U}\right\}.$$

From this expression and (2.10), we obtain, for any $t \in [0,\omega]$,

$$\ln\left\{\frac{[c_1^\ell - d_1^U - (b_2/m_2)^U][r^\ell - (b_1/m_1)^U](c_2^\ell - d_2^U)}{k^U m_1^U m_2^U d_2^U [d_1^U + (b_2/m_2)^U]}\right\} < w(t) < \ln\left\{\frac{r^U c_1^U c_2^U}{k^\ell m_1^\ell m_2^\ell d_1^\ell d_2^\ell}\right\}. \qquad \square$$

Competing interests
The author declares that they have no competing interests.

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