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Pullback $\mathcal{D}$-attractors for three-dimensional Navier-Stokes equations with nonlinear damping

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Abstract
We investigate the asymptotic behavior of solutions of the non-autonomous Navier-Stokes equation with nonlinear damping in three-dimensional bounded domain. When $3 < \beta \leq 5$, the existence of pullback attractors is proved in V and $\mathbf{H}^2(\Omega)$, respectively.

MSC: 35Q30; 35Q35; 35B40

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1 Introduction

In this paper, we investigate the following non-autonomous Navier-Stokes equation with nonlinear damping:

$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \alpha|u|^{\beta-1}u + \nabla p = f(x,t), & x\in\Omega, t>\tau, \\ \operatorname{div} u = 0, & x\in\Omega, t>\tau, \\ u|_{t=\tau} = u_\tau, & x\in\Omega, \\ u|_{\partial\Omega} = 0, & t>\tau. \end{cases} \tag{1}$$

$\mu > 0$ is the kinematic viscosity of the fluid and $f = f(x,t)$ is the external body force. The unknown functions here are $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$, which stand for the velocity field and the pressure of the flow, respectively. In the damping term, $\beta \geq 1$ and $\alpha > 0$ are two constants. The given function $u_\tau = u_\tau(x)$ is the initial velocity.

When $\alpha = 0$, problem (1) becomes the classical 3D Navier-Stokes equation with external force, which has been studied by many authors (see [1–8]). The damping arises from the resistance to the motion of the flow and describes various physical phenomena, such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [9–11]). For the autonomous case, in [12], Cai and Jiu proved that Cauchy problem (1) possesses global strong solutions when $\beta \geq \frac{7}{2}$, and the global strong solution is unique when $\frac{7}{2} \leq \beta \leq 5$. In [13], Zhang, Wu and Lu also investigated the uniqueness of strong solution of problem (1). They established that the strong solution exists when $\beta > 3$, and the global strong solution is unique when $3 < \beta \leq 5$. This improved the earlier results in [12]. In [14, 15], some authors discussed the L^2-decay rate of solutions of problem (1). In [16, 17], we studied the existence of global attractors and uniform attractors of strong solution of problem (1) when $\frac{7}{2} \le \beta \le 5$.

In this paper, our aim is to study the long-time behavior of strong solution of problem (1) by the theory of pullback attractors. Pullback attractor theory is a natural generalization of the theory of global attractors developed to study autonomous dynamical systems, and it is well suited to study the non-autonomous dynamical systems. We shall prove the existence of pullback attractors in $(H_0^1(\Omega))^3$ and $(H^2(\Omega))^3$ under the assumption of an external force $f(x,t)$ satisfying a certain integrability condition. To attain our goal we use the methods introduced in [18–20], which will be explained in more detail in Section 2. Before formulating the main results of the paper, we shall introduce some function spaces and some notations.

We define the function spaces

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}, \qquad H = \mathrm{cl}_{(L^2(\Omega))^3}\,\mathcal{V}, \qquad V = \mathrm{cl}_{(H_0^1(\Omega))^3}\,\mathcal{V},$$

where cl_X denotes the closure in the space X. It is well known that H, V are separable Hilbert spaces and identifying H and its dual H', we have $V \hookrightarrow H \hookrightarrow V'$ with dense and continuous injections, and $V \hookrightarrow H$ is compact. H and V endowed, respectively, with the inner products

$$(u,v) = \int_\Omega u \cdot v \,\mathrm{d}x, \quad \forall u,v \in H,$$

$$((u,v)) = \sum_{i=1}^{3} \int_\Omega \nabla u_i \cdot \nabla v_i \,\mathrm{d}x, \quad \forall u,v \in V,$$

and norms $|\cdot|_2 = (\cdot,\cdot)^{1/2}$, $\|\cdot\| = ((\cdot,\cdot))^{1/2}$. In this paper, $\mathbf{H}^2(\Omega) = (H^2(\Omega))^3$, $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$, and we use $|\cdot|_p$ to denote the norm in $\mathbf{L}^p(\Omega)$.

If $u \in L^\infty(\tau,T;H) \cap L^2(\tau,T;V) \cap L^{\beta+1}(\tau,T;\mathbf{L}^{\beta+1}(\Omega))$ satisfies

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u,v) + \mu((u,v)) + b(u,u,v) + (\alpha|u|^{\beta-1}u, v) = (f,v), \quad \forall v \in V, \forall t > \tau, \\ u(\tau) = u_\tau, \end{cases} \tag{2}$$

then we say that u is a weak solution of (1) on $[\tau, T]$.

The weak formulation (2) is equivalent to the function equation

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + \mu A u + B(u) + G(u) = f(x,t), \quad \text{for } t > \tau, \\ u(\tau) = u_\tau, \end{cases} \tag{3}$$

where $Au = -\tilde{P}\Delta u$ is the Stokes operator defined by $\langle Au, v\rangle = ((u,v))$, and $\tilde{P}$ is the orthogonal projection of $(L^2(\Omega))^3$ onto H. $G(u) = \tilde{P}F(u)$ and $F(u) = \alpha|u|^{\beta-1}u$. $B : V \times V \to V'$ is a bilinear operator defined by $\langle B(u,v), w\rangle = b(u,v,w)$, $B(u) = B(u,u)$, where

$$b(u,v,w) = \sum_{i=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \,\mathrm{d}x,$$

and $\langle\cdot,\cdot\rangle$ is the duality product between V and V'.

In this paper, we assume the external force $f(x,t) \in L^2_{\mathrm{loc}}(\mathbb{R};H)$, and the derivative $\frac{\mathrm{d}f}{\mathrm{d}t} \in L^2_b(\mathbb{R};H)$. Recall that a function $g(t)$ is said to be translation bounded (tr.b.) in $L^2_{\mathrm{loc}}(\mathbb{R};H)$ if

$$\|g\|^2_{L^2_b} = \|g\|^2_{L^2_b(\mathbb{R};H)} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \bigl|g(s)\bigr|_2^2 \, \mathrm{d}t < \infty.$$

$L^2_b(\mathbb{R};H)$ denotes the collection of functions that are tr.b. in $L^2_{\mathrm{loc}}(\mathbb{R};H)$. Furthermore, we assume that $f(x,t)$ is uniformly bounded in H, *i.e.*, there exists a positive constant K, such that

$$\sup_{t\in\mathbb{R}} \bigl|f(x,t)\bigr|_2 \le K.$$

Throughout this paper, we assume that the external force $f(x,t)$ satisfies

$$\int_{-\infty}^{t} e^{\sigma s} \bigl|f(s)\bigr|_2^2 \, \mathrm{d}s < \infty, \quad \text{for all } t \in \mathbb{R}, \tag{4}$$

where $0 < \sigma < \frac{\mu\lambda_1}{2}$, and λ_1 is the first eigenvalue of the Stokes operator. Let $\mathcal{D}$ be the class of all families $\{D(t) : t \in \mathbb{R}\}$ of nonempty subsets of $(H_0^1(\Omega))^3$ such that

$$\lim_{t\to-\infty} e^{\sigma t}\bigl[D(t)\bigr] = 0, \tag{5}$$

where $[D(t)] = \sup\{\|u\|^2_{(H_0^1(\Omega))^3} : u \in D(t)\}$.

In this paper, the letter C is a generic positive constant, which may change its value from line to line, even in the same line.

In the next section, we provide basic definitions and results we shall use in this paper. In Section 3 we give some prior estimates of solutions. Based on these uniform estimation, in Section 4 we prove the existence of pullback attractors.

2 Preliminaries and abstract results

In this section, we will recall some basic definitions and abstract results about pullback attractor and state the theorems about the existence and uniqueness of global solutions of problem (1).

Let X be a complete metric space. A two-parameter family of mappings acting on X: $U(t,\tau): X \to X$, $t \ge \tau$, $\tau \in \mathbb{R}$, is said to be an evolutionary process if

(1) $U(t,\tau) = U(t,r)U(r,\tau)$, for all $\tau \le r \le t$,

(2) $U(\tau,\tau) = \mathrm{Id}$ is the identity operator, $\forall \tau \in \mathbb{R}$.

Let $\mathcal{D}$ be a nonempty class of families $\hat{D} = \{D(t) : t \in \mathbb{R}\}$ of nonempty subsets of X.

Definition 2.1 A family $\hat{A} = \{A(t) : t \in \mathbb{R}\}$ of nonempty subsets of X is said to be a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\ge\tau}$ in X, if

(1) $\hat{A}(t)$ is compact in X for any $t \in \mathbb{R}$,

(2) $\hat{A}$ is invariant, *i.e.*, $U(t,\tau)A(\tau) = A(t)$ for any $\tau \le t$,

(3) $\hat{A}$ is pullback $\mathcal{D}$-attracting, *i.e.*,

$$\lim_{\tau\to-\infty} \mathrm{dist}\bigl(U(t,\tau)D(\tau), A(t)\bigr) = 0,$$

for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$.

Such a family $\hat{\mathcal{A}}$ is called minimal if $A(t) \subset C(t)$ for any family $\hat{C} = \{C(t) : t \in \mathbb{R}\}$ of closed subsets of X such that $\lim_{\tau \to -\infty} \operatorname{dist}(U(t,\tau)B(\tau), C(t)) = 0$ for any $\hat{B} \in \mathcal{D}$.

Definition 2.2 It is said that $\hat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}_{t\geq\tau}$, if for any $\hat{D} \in \mathcal{D}$ and any $t \in \mathbb{R}$, there exists a $\tau_0(t,\hat{D}) \leq t$ such that $U(t,\tau)D(\tau) \subset B(t)$ for any $\tau \leq \tau_0(t,\hat{D})$.

Definition 2.3 Let X be a Banach space. A process $U(t,\tau)$ is said to be norm-to-weak continuous on X if for all $t,\tau \in \mathbb{R}$ with $t \geq \tau$ and for every sequence $x_n \in X$,

$$x_n \to x \quad \text{strongly in } X \quad \Rightarrow \quad U(t,\tau)x_n \rightharpoonup U(t,\tau)x \quad \text{weakly in } X.$$

Obviously, a continuous process is a norm-to-weak continuous process. The following result is very useful to check that the process is norm-to-weak continuous.

Theorem 2.1 (see [19, 21, 22]) *Let X, Y be two Banach spaces. X^*, Y^* be, respectively, their dual spaces. Assume that X is dense in Y, the injection $i : X \to Y$ is continuous, its adjoint $i^* : Y^* \to X^*$ is dense, and U is a norm-to-weak continuous process on Y. Then U is a norm-to-weak continuous process on X if and only if for any $\tau \in \mathbb{R}$, $t \geq \tau$, $U(t,\tau)$ maps compact sets of X into bounded sets of X.*

Definition 2.4 The process $\{U(t,\tau)\}_{t\geq\tau}$ is said to be pullback $\mathcal{D}$-asymptotically compact, if for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$ and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ is relatively compact in X.

Lemma 2.1 (see [18–20]) *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process in X satisfying the following conditions*:

(1) *$\{U(t,\tau)\}_{t\geq\tau}$ is norm-to-weak continuous in X*;
(2) *there exists a family $\hat{B}$ of pullback $\mathcal{D}$-absorbing sets $\{B(t) : t \in \mathbb{R}\}$ in X*;
(3) *$\{U(t,\tau)\}_{t\geq\tau}$ is pullback $\mathcal{D}$-asymptotically compact.*

Then there exists a minimal pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ in X given by

$$A(t) = \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} U(t,\tau)B(\tau)}.$$

Now, we recall the existence and uniqueness theorem for strong solution of problem (1).

Theorem 2.2 ([17]) *Suppose $f \in L^2_b(\mathbb{R};H)$, $u_\tau \in H$, and $\beta \geq 1$. Then for any given $T > \tau$, there exists at least one solution u that satisfies* (2). *Moreover,*

$$u \in L^\infty(\tau,T;H) \cap L^2(\tau,T;V) \cap L^{\beta+1}\big(\tau,T;\mathbf{L}^{\beta+1}(\Omega)\big).$$

We say that $u(x,t)$ is a strong solution of (1), if it is a weak solution of (1), and satisfies

$$u \in L^\infty(\tau,T;V) \cap L^2\big(\tau,T;\mathbf{H}^2(\Omega)\big) \cap L^\infty\big(\tau,T;\mathbf{L}^{\beta+1}(\Omega)\big).$$

Theorem 2.3 *Suppose $\beta > 3$, $u_\tau \in V \cap \mathbf{L}^{\beta+1}(\Omega)$, and $f \in L^2_b(\mathbb{R};H)$. Then there exists a strong solution $u(x,t)$ that satisfies* (2),

$$u \in L^\infty(\tau,T;V) \cap L^\infty\big(\tau,T;\mathbf{L}^{\beta+1}(\Omega)\big) \cap L^2\big(\tau,T;\mathbf{H}^2(\Omega)\big),$$
$$\nabla u|u|^{\frac{\beta-1}{2}} \in L^2(\tau,T;H), \quad u_t \in L^2(\tau,T;H).$$

Moreover, when $3 < \beta \le 5$, the strong solution is unique.

Proof In [13], Zhang, Wu and Lu have proved this theorem in the autonomous case. For the non-autonomous case, it is similar to the proof of Theorem 3.1 in [13], so we omit it here. □

Because $\Omega \subset R^3$ is sufficiently regular, so $V \hookrightarrow \mathbf{L}^6(\Omega)$, and because $\mathbf{L}^6(\Omega) \hookrightarrow \mathbf{L}^{\beta+1}(\Omega)$ $(3 < \beta \le 5)$, so $V \cap \mathbf{L}^{\beta+1}(\Omega) = V$. In the following, we use $u_\tau \in V$ to replace $u_\tau \in V \cap \mathbf{L}^{\beta+1}(\Omega)$.

In [16], we have proved that the strong solution $u(x,t)$ is continuous with respect to the initial value condition u_0 in the space V when $\frac{7}{2} \le \beta \le 5$ (Proposition 7). From the proof, we can easily deduce that, when $3 < \beta \le 5$, for the non-autonomous case, the strong solution $u(x,t)$ is also continuous with respect to the initial data u_τ in V.

In order to construct a process $\{U(t,\tau)\}_{t\ge\tau}$ for problem (1), we define $U(t,\tau): V \to V$ by $U(t,\tau)u_\tau = u(t)$, $t \ge \tau$. Obviously, the process $\{U(t,\tau)\}_{t\ge\tau}$ is a continuous process in V, so it is also a norm-to-weak continuous process in V.

3 Uniform estimates of solutions

In this section, we derive uniform estimates on solutions of problem (1) when $\tau \to -\infty$. These estimates are necessary to prove the existence of pullback $\mathcal{D}$-absorbing sets and the pullback asymptotic compactness of process $\{U(t,\tau)\}_{t\ge\tau}$ associated with the system.

Lemma 3.1 *Under the assumptions* (4)-(5) *and $f \in L^2_{\mathrm{loc}}(\mathbb{R};H)$. Let $3 < \beta \le 5$, $\tau \in \mathbb{R}$, and $u(t)$ be the solution of problem* (1). *Then for any $\hat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$, there exists $\tau_0 = \tau_0(t,\hat{D}) < t$, such that*

$$\big|u(t)\big|_2^2 \le Ce^{-\sigma t} \int_{-\infty}^t e^{\sigma\xi} \big|f(\xi)\big|_2^2 \,\mathrm{d}\xi;$$
$$\int_\tau^t e^{\sigma\xi} \big\|u(\xi)\big\|^2 \,\mathrm{d}\xi \le C \int_{-\infty}^t e^{\sigma\xi} \big|f(\xi)\big|_2^2 \,\mathrm{d}\xi;$$
$$\int_\tau^t e^{\sigma\xi} \big|u(\xi)\big|_{\beta+1}^{\beta+1} \,\mathrm{d}\xi \le C \int_{-\infty}^t e^{\sigma\xi} \big|f(\xi)\big|_2^2 \,\mathrm{d}\xi,$$

for any $u_\tau \in D(\tau)$ and $\tau \le \tau_0(t,\hat{D})$.

Proof Taking the inner product of $(1)_1$ with u, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + 2\mu\|u\|^2 + 2\alpha|u|_{\beta+1}^{\beta+1} = 2(f,u) &\le \mu\lambda_1|u|_2^2 + \frac{1}{\mu\lambda_1}|f|_2^2 \\ &\le \mu\|u\|^2 + \frac{1}{\mu\lambda_1}|f|_2^2,\end{aligned}$$

where λ_1 is the first eigenvalue of the Stokes operator. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + \frac{\mu\lambda_1}{2}|u|_2^2 + \frac{\mu}{2}\|u\|^2 + 2\alpha|u|_{\beta+1}^{\beta+1} \leq \frac{1}{\mu\lambda_1}|f|_2^2. \tag{6}$$

Multiplying (6) by $e^{\sigma t}$ and then integrating over (τ, t), we derive that

$$\begin{aligned} &\left|u(t)\right|_2^2 + \frac{\mu}{2}e^{-\sigma t}\int_\tau^t e^{\sigma\xi}\left\|u(\xi)\right\|^2 \mathrm{d}\xi + 2\alpha e^{-\sigma t}\int_\tau^t e^{\sigma\xi}\left|u(\xi)\right|_{\beta+1}^{\beta+1}\mathrm{d}\xi \\ &\quad\leq \left(\sigma - \frac{\mu}{2}\lambda_1\right)e^{-\sigma t}\int_\tau^t e^{\sigma\xi}\left|u(\xi)\right|_2^2 \mathrm{d}\xi + \frac{1}{\mu\lambda_1}e^{-\sigma t}\int_\tau^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \mathrm{d}\xi + e^{-\sigma t}e^{\sigma\tau}\left|u(\tau)\right|_2^2. \end{aligned}$$

Since $0 < \sigma < \frac{\mu\lambda_1}{2}$, we have

$$\begin{aligned} &\left|u(t)\right|_2^2 + \frac{\mu}{2}e^{-\sigma t}\int_\tau^t e^{\sigma\xi}\left\|u(\xi)\right\|^2 \mathrm{d}\xi + 2\alpha e^{-\sigma t}\int_\tau^t e^{\sigma\xi}\left|u(\xi)\right|_{\beta+1}^{\beta+1}\mathrm{d}\xi \\ &\quad\leq \frac{1}{\mu\lambda_1}e^{-\sigma t}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \mathrm{d}\xi + e^{-\sigma t}e^{\sigma\tau}\left|u(\tau)\right|_2^2. \end{aligned} \tag{7}$$

Since $u(\tau) \in D(\tau)$, for every $t \in \mathbb{R}$, there exists $\tau_0 = \tau_0(t, \hat{D}) < t$ such that, for all $\tau \leq \tau_0$,

$$e^{\sigma\tau}\left|u(\tau)\right|_2^2 \leq \frac{1}{\mu\lambda_1}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \mathrm{d}\xi. \tag{8}$$

By (7) and (8), we find that

$$\begin{aligned} &\left|u(t)\right|_2^2 + \frac{\mu}{2}e^{-\sigma t}\int_\tau^t e^{\sigma\xi}\left\|u(\xi)\right\|^2 \mathrm{d}\xi + 2\alpha e^{-\sigma t}\int_\tau^t e^{\sigma\xi}\left|u(\xi)\right|_{\beta+1}^{\beta+1}\mathrm{d}\xi \\ &\quad\leq \frac{2}{\mu\lambda_1}e^{-\sigma t}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \mathrm{d}\xi. \end{aligned}$$

□

Lemma 3.2 *Under the assumptions* (4)-(5) *and* $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$. *Let* $3 < \beta \leq 5$, $\tau \in \mathbb{R}$, *and* $u(t)$ *be the solution of problem* (1). *Then for any* $\hat{D} \in \mathcal{D}$ *and* $t \in \mathbb{R}$, *there exists* $\tau_1 = \tau_1(t, \hat{D}) < t - 2$, *such that*

$$\begin{aligned} &\int_{t-2}^t e^{\sigma\xi}\left|u(\xi)\right|_2^2 \mathrm{d}\xi \leq C\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \mathrm{d}\xi; \\ &\int_{t-2}^t e^{\sigma\xi}\left\|u(\xi)\right\|^2 \mathrm{d}\xi \leq C\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \mathrm{d}\xi; \\ &\int_{t-2}^t e^{\sigma\xi}\left|u(\xi)\right|_{\beta+1}^{\beta+1} \mathrm{d}\xi \leq C\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \mathrm{d}\xi, \end{aligned}$$

for any $u_\tau \in D(\tau)$ *and* $\tau \leq \tau_1(t, \hat{D})$.

Proof It follows from (6) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|u(t)\right|_2^2 + \frac{\mu\lambda_1}{2}\left|u(t)\right|_2^2 \leq \frac{1}{\mu\lambda_1}\left|f(t)\right|_2^2. \tag{9}$$

Let $s \in [t-2, t]$. Multiplying (9) by $e^{\sigma t}$, then relabeling t as ξ and integrating with respect to ξ over (τ, s), we get

$$\begin{aligned} e^{\sigma s}\left|u(s)\right|_2^2 &\leq e^{\sigma\tau}\left|u(\tau)\right|_2^2 + \left(\sigma - \frac{\mu}{2}\lambda_1\right)\int_\tau^s e^{\sigma\xi}\left|u(\xi)\right|_2^2 \,\mathrm{d}\xi + \frac{1}{\mu\lambda_1}\int_\tau^s e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi \\ &\leq e^{\sigma\tau}\left|u(\tau)\right|_2^2 + \frac{1}{\mu\lambda_1}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi. \end{aligned} \tag{10}$$

Since $u(\tau) \in D(\tau)$, for every $t \in \mathbb{R}$, there exists $\tau_1 = \tau_1(t, \hat{D}) < t-2$, such that, for all $\tau \leq \tau_1$,

$$e^{\sigma\tau}\left|u(\tau)\right|_2^2 \leq \frac{1}{\mu\lambda_1}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi. \tag{11}$$

By (10) and (11), we have, for $s \in [t-2, t]$,

$$e^{\sigma s}\left|u(s)\right|_2^2 \leq \frac{2}{\mu\lambda_1}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi. \tag{12}$$

Integrating (12) with respect to s over the interval $(t-2, t)$ produces

$$\int_{t-2}^t e^{\sigma s}\left|u(s)\right|_2^2 \,\mathrm{d}s \leq \frac{4}{\mu\lambda_1}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi. \tag{13}$$

Multiplying (6) by $e^{\sigma t}$, then relabeling t as ξ and integrating with respect to ξ over $(t-2, t)$, by (12) we obtain, for all $\tau \leq \tau_1$,

$$\begin{aligned} &e^{\sigma t}\left|u(t)\right|_2^2 + \frac{\mu}{2}\int_{t-2}^t e^{\sigma\xi}\left\|u(\xi)\right\|^2 \,\mathrm{d}\xi + 2\alpha\int_{t-2}^t e^{\sigma\xi}\left|u(\xi)\right|_{\beta+1}^{\beta+1} \,\mathrm{d}\xi \\ &\quad\leq e^{\sigma(t-2)}\left|u(t-2)\right|_2^2 + \left(\sigma - \frac{\mu}{2}\lambda_1\right)\int_{t-2}^t e^{\sigma\xi}\left|u(\xi)\right|_2^2 \,\mathrm{d}\xi + \frac{1}{\mu\lambda_1}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi \\ &\quad\leq \frac{3}{\mu\lambda_1}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi, \end{aligned}$$

which along with (13) completes the proof. □

Corollary 3.1 *Under the assumptions* (4)-(5) *and* $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$. *Let* $3 < \beta \leq 5$, $\tau \in \mathbb{R}$, *and* $u(t)$ *be the solution of problem* (1). *Then for any* $\hat{D} \in \mathcal{D}$ *and* $t \in \mathbb{R}$,

$$\begin{aligned} &\int_{t-2}^t \left|u(\xi)\right|_2^2 \,\mathrm{d}\xi \leq Ce^{-\sigma t}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi; \\ &\int_{t-2}^t \left\|u(\xi)\right\|^2 \,\mathrm{d}\xi \leq Ce^{-\sigma t}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi; \\ &\int_{t-2}^t \left|u(\xi)\right|_{\beta+1}^{\beta+1} \,\mathrm{d}\xi \leq Ce^{-\sigma t}\int_{-\infty}^t e^{\sigma\xi}\left|f(\xi)\right|_2^2 \,\mathrm{d}\xi, \end{aligned}$$

for any $u_\tau \in D(\tau)$ *and* $\tau \leq \tau_1(t, \hat{D})$.

Proof It is straightforward from Lemma 3.2. □

Lemma 3.3 *Under the assumptions* (4)-(5) *and* $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$. *Let* $3 < \beta \leq 5$, $\tau \in \mathbb{R}$, *and* $u(t)$ *be the solution of problem* (1). *Then, for any* $\hat{D} \in \mathcal{D}$ *and* $t \in \mathbb{R}$,

$$\|u(t)\|^2 + |u(t)|_{\beta+1}^{\beta+1} \leq Ce^{-\sigma t} \int_{-\infty}^{t} e^{\sigma \xi} |f(\xi)|_2^2 \, \mathrm{d}\xi, \tag{14}$$

for any $u_\tau \in D(\tau)$ *and* $\tau \leq \tau_1(t, \hat{D})$.

Proof Multiplying the first equation of (1) by u_t, $-\Delta u$, respectively, and then integrating the resulting equation on Ω, we obtain

$$\begin{aligned} &|u_t|_2^2 + \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{\beta+1}^{\beta+1} \\ &\quad = -\int_\Omega (u \cdot \nabla) u u_t \, \mathrm{d}x + (f, u_t), \end{aligned} \tag{15}$$

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \mu|\Delta u|_2^2 + \alpha \int_\Omega |u|^{\beta-1}|\nabla u|^2 \, \mathrm{d}x + \frac{\alpha(\beta-1)}{4}\int_\Omega |u|^{\beta-3}\big|\nabla|u|^2\big|^2 \, \mathrm{d}x \\ &\quad = \int_\Omega (u \cdot \nabla) u \Delta u \, \mathrm{d}x - (f, \Delta u). \end{aligned} \tag{16}$$

From (15) we have

$$\mu \frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{2\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{\beta+1}^{\beta+1} \leq \big|(u \cdot \nabla)u\big|_2^2 + \big|f(t)\big|_2^2. \tag{17}$$

From (16) we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \mu|\Delta u|_2^2 + 2\alpha \int_\Omega |u|^{\beta-1}|\nabla u|^2 \, \mathrm{d}x + \frac{\alpha(\beta-1)}{2}\int_\Omega |u|^{\beta-3}\big|\nabla|u|^2\big|^2 \, \mathrm{d}x \\ &\quad \leq \frac{2|(u \cdot \nabla)u|_2^2}{\mu} + \frac{2|f(t)|_2^2}{\mu}. \end{aligned} \tag{18}$$

Taking (17), (18) together, it follows that

$$\begin{aligned} &(\mu+1)\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{2\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{\beta+1}^{\beta+1} + \mu|\Delta u|_2^2 + 2\alpha \int_\Omega |u|^{\beta-1}|\nabla u|^2 \, \mathrm{d}x \\ &\qquad + \frac{\alpha(\beta-1)}{2}\int_\Omega |u|^{\beta-3}\big|\nabla|u|^2\big|^2 \, \mathrm{d}x \\ &\quad \leq \left(\frac{2}{\mu}+1\right)\big|(u \cdot \nabla)u\big|_2^2 + \left(\frac{2}{\mu}+1\right)\big|f(t)\big|_2^2. \end{aligned} \tag{19}$$

From the proof of Theorem 3.1 in [13], we can find that, when $3 < \beta \leq 5$,

$$\begin{aligned} J &= C\big|(u \cdot \nabla)u\big|_2^2 \\ &\leq C\int_\Omega |u|^2|\nabla u|^2 \, \mathrm{d}x \\ &\leq \alpha \int_\Omega |u|^{\beta-1}|\nabla u|^2 \, \mathrm{d}x + \frac{\mu}{4}\int_\Omega |\Delta u|^2 \, \mathrm{d}x + C\int_\Omega |u|^{\beta+1} \, \mathrm{d}x. \end{aligned} \tag{20}$$

Substituting (20) into (19), we find that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[(\mu+1)\|u(s)\|^2+\frac{2\alpha}{\beta+1}|u(s)|_{\beta+1}^{\beta+1}\right]\\
&\quad\le C|u(s)|_{\beta+1}^{\beta+1}+\left(\frac{2}{\mu}+1\right)|f(s)|_2^2\\
&\quad\le C\left[(\mu+1)\|u(s)\|^2+\frac{2\alpha}{\beta+1}|u(s)|_{\beta+1}^{\beta+1}\right]+\left(\frac{2}{\mu}+1\right)|f(s)|_2^2.
\end{aligned}\tag{21}$$

Applying the uniform Gronwall lemma to (21) on interval $[t-1,t]$, we have

$$\begin{aligned}
&(\mu+1)\|u(t)\|^2+\frac{2\alpha}{\beta+1}|u(t)|_{\beta+1}^{\beta+1}\\
&\quad\le C\left(\int_{t-1}^{t}\left[(\mu+1)\|u(\xi)\|^2+\frac{2\alpha}{\beta+1}|u(\xi)|_{\beta+1}^{\beta+1}\right]\mathrm{d}\xi+\left(\frac{2}{\mu}+1\right)\int_{t-1}^{t}|f(\xi)|_2^2\,\mathrm{d}\xi\right).
\end{aligned}$$

Noticing

$$\begin{aligned}
\int_{t-1}^{t}|f(\xi)|_2^2\,\mathrm{d}\xi&=e^{-\sigma(t-1)}\int_{t-1}^{t}e^{\sigma(t-1)}|f(\xi)|_2^2\,\mathrm{d}\xi\\
&\le e^{-\sigma(t-1)}\int_{t-1}^{t}e^{\sigma\xi}|f(\xi)|_2^2\,\mathrm{d}\xi\\
&\le Ce^{-\sigma t}\int_{-\infty}^{t}e^{\sigma\xi}|f(\xi)|_2^2\,\mathrm{d}\xi,
\end{aligned}$$

according to Corollary 3.1, we have

$$(\mu+1)\|u(t)\|^2+\frac{2\alpha}{\beta+1}|u(t)|_{\beta+1}^{\beta+1}\le Ce^{-\sigma t}\int_{-\infty}^{t}e^{\sigma\xi}|f(\xi)|_2^2\,\mathrm{d}\xi. \qquad \square$$

Lemma 3.4 *Under the assumptions* (4)-(5) *and* $f\in L^2_{\mathrm{loc}}(\mathbb{R};H)$. *Let* $3<\beta\le 5$, $\tau\in\mathbb{R}$, *and* $u(t)$ *be the solution of problem* (1). *Then for any* $\hat{D}\in\mathcal{D}$ *and* $t\in\mathbb{R}$,

$$\int_{t-1}^{t}|\Delta u(\xi)|_2^2\,\mathrm{d}\xi\le Ce^{-\sigma t}\int_{-\infty}^{t}e^{\sigma\xi}|f(\xi)|_2^2\,\mathrm{d}\xi,\quad \forall t\in\mathbb{R},$$

for any $u_\tau\in D(\tau)$ *and* $\tau\le\tau_1(t,\hat{D})$.

Proof Similar to the proof of Lemma 3.3, applying the uniform Gronwall lemma to (21) on interval $[t-2,t-1]$, we can also prove

$$(\mu+1)\|u(t-1)\|^2+\frac{2\alpha}{\beta+1}|u(t-1)|_{\beta+1}^{\beta+1}\le Ce^{-\sigma t}\int_{-\infty}^{t}e^{\sigma\xi}|f(\xi)|_2^2\,\mathrm{d}\xi.\tag{22}$$

Thanks to (18), (20), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\frac{3}{4}\mu|\Delta u|_2^2+\alpha\int_\Omega|u|^{\beta-1}|\nabla u|^2\,\mathrm{d}x\le C|u|_{\beta+1}^{\beta+1}+\frac{2}{\mu}|f(t)|_2^2.\tag{23}$$

Integrating (23) from $t-1$ to t, according to Corollary 3.1 and (22), we can obtain

$$\begin{aligned}
&\|u(t)\|^2 + \frac{3}{4}\mu \int_{t-1}^{t} \left|\Delta u(\xi)\right|_2^2 \,\mathrm{d}\xi + \alpha \int_{t-1}^{t}\int_\Omega \left|u(\xi)\right|^{\beta-1}\left|\nabla u(\xi)\right|^2 \,\mathrm{d}x\,\mathrm{d}\xi \\
&\quad\le \|u(t-1)\|^2 + C\int_{t-1}^{t} \left|u(\xi)\right|_{\beta+1}^{\beta+1}\,\mathrm{d}\xi + \frac{2}{\mu}\int_{t-1}^{t}\left|f(\xi)\right|_2^2\,\mathrm{d}\xi \\
&\quad\le Ce^{-\sigma t}\int_{-\infty}^{t} e^{\sigma\xi}\left|f(\xi)\right|_2^2\,\mathrm{d}\xi,
\end{aligned} \tag{24}$$

which completes the proof. □

Lemma 3.5 *Under the assumptions* (4)-(5) *and* $f \in L^2_{\mathrm{loc}}(\mathbb{R};H)$, $\frac{\mathrm{d}f}{\mathrm{d}t} \in L^2_b(\mathbb{R};H)$. *Let* $3 < \beta \le 5$, $\tau \in \mathbb{R}$, *and* $u(t)$ *be the solution of problem* (1). *Then for any* $\hat{D} \in \mathcal{D}$ *and* $t \in \mathbb{R}$, *there exists a family of positive constants* $\{r_1(t) : t \in \mathbb{R}\}$ *such that*

$$\left|u_t(t)\right|_2^2 \le r_1(t),$$

for any $u_\tau \in D(\tau)$ *and* $\tau \le \tau_1(t,\hat{D})$, *where* $r_1(t)$ *is a positive constant which is independent of the initial data.*

Proof From (15) and (20) we deduce that

$$\begin{aligned}
&|u_t|_2^2 + \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{\beta+1}^{\beta+1} \\
&\quad\le \frac{|u_t|_2^2}{2} + \left|(u\cdot\nabla)u\right|_2^2 + \left|f(t)\right|_2^2 \\
&\quad\le \frac{|u_t|_2^2}{2} + \alpha\int_\Omega |u|^{\beta-1}|\nabla u|^2\,\mathrm{d}x + \frac{\mu}{4}|\Delta u|_2^2 + C|u|_{\beta+1}^{\beta+1} + \left|f(t)\right|_2^2.
\end{aligned} \tag{25}$$

Thus

$$\begin{aligned}
&|u_t|_2^2 + \mu\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{2\alpha}{\beta+1}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{\beta+1}^{\beta+1} \\
&\quad\le \frac{\mu}{2}|\Delta u|_2^2 + 2\alpha\int_\Omega |u|^{\beta-1}|\nabla u|^2\,\mathrm{d}x + C|u|_{\beta+1}^{\beta+1} + 2\left|f(t)\right|_2^2.
\end{aligned} \tag{26}$$

Integrating (26) from $t-1$ to t, according to Corollary 3.1, Lemma 3.4, (22), and (24), we can obtain

$$\begin{aligned}
\int_{t-1}^{t}\left|u_\xi(\xi)\right|_2^2\,\mathrm{d}\xi &\le \mu\left\|u(t-1)\right\|^2 + \frac{2\alpha}{\beta+1}\left|u(t-1)\right|_{\beta+1}^{\beta+1} + \frac{\mu}{2}\int_{t-1}^{t}\left|\Delta u(\xi)\right|_2^2\,\mathrm{d}\xi \\
&\quad + 2\alpha\int_{t-1}^{t}\int_\Omega\left|u(\xi)\right|^{\beta-1}\left|\nabla u(\xi)\right|^2\,\mathrm{d}x\,\mathrm{d}\xi \\
&\quad + C\int_{t-1}^{t}\left|u(\xi)\right|_{\beta+1}^{\beta+1}\,\mathrm{d}\xi + 2\int_{t-1}^{t}\left|f(\xi)\right|_2^2\,\mathrm{d}\xi \\
&\le Ce^{-\sigma t}\int_{-\infty}^{t} e^{\sigma\xi}\left|f(\xi)\right|_2^2\,\mathrm{d}\xi.
\end{aligned} \tag{27}$$

We now differentiate $(3)_1$ with respect to t and then take the inner product with u_t in H to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_t|_2^2 + \mu\|u_t\|^2 \le \bigl|b(u_t,u,u_t)\bigr| - \int_\Omega \bigl(F'(u)u_t\bigr)\cdot u_t\,\mathrm{d}x + (f_t,u_t).$$

According to Lemma 2.4 in [16], $(F'(u)u_t)\cdot u_t$ is positive definite, hence we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_t|_2^2 + \mu\|u_t\|^2 &\le \bigl|b(u_t,u,u_t)\bigr| + |f_t|_2\cdot|u_t|_2\\ &\le C|u_t|_2^{1/2}\|u_t\|^{3/2}\|u\| + \frac{1}{2}|u_t|_2^2 + \frac{1}{2}|f_t|_2^2\\ &\le \frac{\mu}{2}\|u_t\|^2 + C\bigl(1+\|u\|^4\bigr)|u_t|_2^2 + \frac{1}{2}|f_t|_2^2.\end{aligned} \tag{28}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_t|_2^2 \le C\bigl(1+\|u\|^4\bigr)|u_t|_2^2 + |f_t|_2^2. \tag{29}$$

Thanks to (14),we have

$$\begin{aligned}C\int_{t-1}^t \bigl(1+\bigl\|u(\xi)\bigr\|^4\bigr)\,\mathrm{d}\xi &\le C\left(1+\int_{t-1}^t\left(Ce^{-\sigma s}\int_{-\infty}^s e^{\sigma\xi}\bigl|f(\xi)\bigr|_2^2\,\mathrm{d}\xi\right)^2\mathrm{d}s\right)\\ &= C + C\int_{t-1}^t r_0^2(s)\,\mathrm{d}s,\end{aligned}$$

where $r_0(s) = Ce^{-\sigma s}\int_{-\infty}^s e^{\sigma\xi}|f(\xi)|_2^2\,\mathrm{d}\xi$.

Applying the uniform Gronwall lemma to (29) on interval $[t-1,t]$, we can get

$$\begin{aligned}\bigl|u_t(t)\bigr|_2^2 &\le \left\{Ce^{-\sigma t}\int_{-\infty}^t e^{\sigma\xi}\bigl|f(\xi)\bigr|_2^2\,\mathrm{d}\xi + \int_{t-1}^t \bigl|f_\xi(\xi)\bigr|_2^2\,\mathrm{d}\xi\right\}\cdot\exp\left\{C + C\int_{t-1}^t r_0^2(s)\,\mathrm{d}s\right\}\\ &\equiv r_1(t).\end{aligned}$$

□

Lemma 3.6 *Under the assumptions* (4)-(5) *and* $f\in L^2_{\mathrm{loc}}(\mathbb{R};H)$, $\frac{\mathrm{d}f}{\mathrm{d}t}\in L^2_b(\mathbb{R};H)$. *Let* $3<\beta\le 5$, $\tau\in\mathbb{R}$, *and* $u(t)$ *be the solution of problem* (1). *Then for any* $\hat{D}\in\mathcal{D}$ *and* $t\in\mathbb{R}$, *there exists a family of positive constants* $\{r_2(t): t\in\mathbb{R}\}$ *such that*

$$\bigl|Au(t)\bigr|_2 \le r_2(t), \tag{30}$$

for any $u_\tau\in D(\tau)$ *and* $\tau\le\tau_1(t,\hat{D})$, *where* $r_2(t)$ *is a positive constant which is independent of the initial data*.

Proof Like the proof of Proposition 5 in [16], we can obtain

$$\begin{aligned}\frac{\mu}{2}\bigl|Au(t)\bigr|_2 &\le \bigl|u_t(t)\bigr|_2 + C\bigl\|u(t)\bigr\|^3 + C\bigl|u(t)\bigr|_{\beta+1}^{\frac{\beta^2+4\beta+3}{10-2\beta}} + \bigl|f(t)\bigr|_2\\ &\le \bigl(r_1(t)\bigr)^{1/2} + \left(Ce^{-\sigma t}\int_{-\infty}^t e^{\sigma\xi}\bigl|f(\xi)\bigr|_2^2\,\mathrm{d}\xi\right)^{3/2}\end{aligned}$$

$$+\left(Ce^{-\sigma t}\int_{-\infty}^{t}e^{\sigma\xi}\left|f(\xi)\right|_2^2\,\mathrm{d}\xi\right)^{\frac{\beta+3}{10-2\beta}}+\left|f(t)\right|_2$$
$$\leq r_1(t)^{1/2}+C\left(r_0(t)\right)^{3/2}+C\left(r_0(t)\right)^{\frac{\beta+3}{10-2\beta}}+\left|f(t)\right|_2\equiv r_2(t). \qquad \square$$

Lemma 3.7 *Under the assumptions* (4)-(5) *and* $f\in L^2_{\mathrm{loc}}(\mathbb{R};H)$, $\frac{\mathrm{d}f}{\mathrm{d}t}\in L^2_b(\mathbb{R};H)$. *Let* $3<\beta\leq 5$, $\tau\in\mathbb{R}$, *and* $u(t)$ *be the solution of problem* (1). *Then for any* $\hat{D}\in\mathcal{D}$ *and* $t\in\mathbb{R}$, *there exists a family of positive constants* $\{r_3(t):t\in\mathbb{R}\}$ *such that*

$$\left\|u_t(t+1)\right\|^2\leq r_3(t),$$

for any $u_\tau\in D(\tau)$ *and* $\tau\leq\tau_1(t,\hat{D})$, *where* $r_3(t)$ *is a positive constant which is independent of the initial data.*

Proof From inequality (28) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_t|_2^2+\mu\|u_t\|^2\leq C\left(1+\|u\|^4\right)|u_t|_2^2+|f_t|_2^2. \tag{31}$$

Integrating the above inequality from t to $t+1$, then we have

$$\begin{aligned}\mu\int_t^{t+1}\left\|u_t(s)\right\|^2\mathrm{d}s&\leq\left|u_t(t)\right|_2^2+C\int_t^{t+1}\left(1+\left\|u(s)\right\|^4\right)\left|u_t(s)\right|_2^2\mathrm{d}s+\int_t^{t+1}\left|f_t(s)\right|_2^2\mathrm{d}s\\&\leq r_1(t)+C\int_t^{t+1}\left(1+\left(r_0(s)\right)^2\right)r_1(s)\,\mathrm{d}s+\int_t^{t+1}\left|f_t(s)\right|_2^2\mathrm{d}s\\&\equiv\rho_1(t).\end{aligned} \tag{32}$$

By Lemma 3.6, we know that

$$\left\|u(t)\right\|_{D(A)}\leq r_2(t),$$

so using the Agmon inequality we obtain

$$\left|u(t)\right|_\infty\leq C\left\|u(t)\right\|^{1/2}\left\|u(t)\right\|_{D(A)}^{1/2}\leq C\left(r_0(t)\right)^{1/4}\left(r_2(t)\right)^{1/2}\equiv\rho_2(t).$$

We now differentiate $(3)_1$ with respect to t, then taking the inner product with Au_t in H to obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^2+\mu|Au_t|_2^2&\leq\left|b(u_t,u,Au_t)\right|+\left|b(u,u_t,Au_t)\right|\\&\quad+\int_\Omega\left(F'(u)u_t\right)\cdot Au_t\,\mathrm{d}x+(f_t,Au_t).\end{aligned} \tag{33}$$

According to Lemma 2.4 in [16], for any $u,v,w\in\mathbb{R}^3$, $|(F'(u)v)\cdot w|\leq C|u|^{\beta-1}|v||w|$, so

$$\begin{aligned}\int_\Omega\left(F'(u)u_t\right)\cdot Au_t\,\mathrm{d}x&\leq C\int_\Omega|u|^{\beta-1}|u_t||Au_t|\,\mathrm{d}x\leq C|u|_\infty^{\beta-1}|u_t|_2|Au_t|_2\\&\leq C\left(\rho_2(t)\right)^{\beta-1}\|u_t\||Au_t|_2\leq\frac{\mu}{8}|Au_t|_2^2+C\left(\rho_2(t)\right)^{2(\beta-1)}\|u_t\|^2.\end{aligned} \tag{34}$$

Because

$$\begin{aligned}
\big|b(u_t,u,Au_t)\big| &\le C\|u_t\|\|u\|^{1/2}|Au|_2^{1/2}|Au_t|_2 \le \frac{\mu}{8}|Au_t|_2^2 + C\|u\||Au|_2\|u_t\|^2 \\
&\le \frac{\mu}{8}|Au_t|_2^2 + C\big(r_0(t)\big)^{1/2}r_2(t)\|u_t\|^2, \qquad (35)\\
\big|b(u,u_t,Au_t)\big| &\le C\|u\|\|u_t\|^{1/2}|Au_t|_2^{1/2}|Au_t|_2 \le \frac{\mu}{8}|Au_t|_2^2 + C\|u\|^4\|u_t\|^2 \\
&\le \frac{\mu}{8}|Au_t|_2^2 + C\big(r_0(t)\big)^2\|u_t\|^2, \qquad (36)\\
(f_t,Au_t) &\le \frac{\mu}{8}|Au_t|_2^2 + \frac{2}{\mu}|f_t|_2^2, \qquad (37)
\end{aligned}$$

combining (34)-(37) with (33), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|^2 \le C\Big[\big(\rho_2(t)\big)^{2(\beta-1)} + \big(r_0(t)\big)^{1/2}r_2(t) + \big(r_0(t)\big)^2\Big]\|u_t\|^2 + \frac{4}{\mu}|f_t|_2^2. \qquad (38)$$

Thanks to (32), applying the uniform Gronwall lemma to (38) on interval $[t,t+1]$, we get

$$\begin{aligned}
\big\|u_t(t+1)\big\|^2 &\le \left(\frac{\rho_1(t)}{\mu} + \frac{4}{\mu}\int_t^{t+1}\big|f_t(s)\big|_2^2\,\mathrm{d}s\right) \\
&\quad\cdot \exp\left\{C\int_t^{t+1}\Big[\big(\rho_2(s)\big)^{2(\beta-1)} + \big(r_0(s)\big)^{1/2}r_2(s) + \big(r_0(s)\big)^2\Big]\,\mathrm{d}s\right\} \\
&= r_3(t). \qquad (39)
\end{aligned}$$

□

4 Existence of pullback attractors

In Section 2, we have known that the process $\{U(t,\tau)\}_{t\ge\tau}$ associated with (1) is norm-to-weak continuous in V. In this section, we prove the existence of pullback attractors in V and $\mathbf{H}^2(\Omega)$ for the non-autonomous Navier-Stokes equation with nonlinear damping.

Theorem 4.1 *Under the assumptions* (4)-(5) *and* $f \in L^2_{\mathrm{loc}}(\mathbb{R};H)$, $\frac{\mathrm{d}f}{\mathrm{d}t} \in L^2_b(\mathbb{R};H)$. *Let* $3 < \beta \le 5$ *and* $\tau \in \mathbb{R}$, *then the process* $\{U(t,\tau)\}_{t\ge\tau}$ *associated with* (1) *has a pullback* $\mathcal{D}$*-attractor* $\mathcal{A}_1$ *in* V.

Proof Let $B_0 = \{B(t) : t \in \mathbb{R}\}$ and $C_0 = \{C(t) : t \in \mathbb{R}\}$ be pullback $\mathcal{D}$-absorbing sets in V and in $D(A)$ obtained by Lemma 3.3 and Lemma 3.6, respectively. Since $D(A) \hookrightarrow V$ is compact, we have $\{U(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}$-asymptotically compact in V. Then by Lemma 2.1, $\{U(t,\tau)\}_{t\ge\tau}$ has a minimal pullback $\mathcal{D}$-attractor $\mathcal{A}_1$ in V. □

Lemma 4.1 *The process* $\{U(t,\tau)\}_{t\ge\tau}$ *is norm-to-weak continuous in* $\mathbf{H}^2(\Omega)$.

Proof We know $i : D(A) \hookrightarrow V$, $i^* : V^* \hookrightarrow (D(A))^*$ and i, i^* are dense. From Section 2, we know that $\{U(t,\tau)\}_{t\ge\tau} : V \to V$ is norm-to-weak continuous. From Lemma 3.6, we find that $\{U(t,\tau)\}_{t\ge\tau}$ has a pullback $\mathcal{D}$-absorbing set in $D(A)$. That is to say, $\{U(t,\tau)\}_{t\ge\tau}$ maps a bounded set in V into a bounded set in $D(A)$, therefore $\{U(t,\tau)\}_{t\ge\tau}$ maps a compact set in $D(A)$ into a bounded set in $D(A)$. By Theorem 2.1, the proof is completed. □

Theorem 4.2 *Under the assumptions* (4)-(5) *and* $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, $\frac{\mathrm{d}f}{\mathrm{d}t} \in L^2_b(\mathbb{R}; H)$. *Let* $3 < \beta \leq 5$ *and* $\tau \in \mathbb{R}$, *then the process* $\{U(t,\tau)\}_{t\geq\tau}$ *associated with* (1) *has a pullback* $\mathcal{D}$*-attractor* $\mathcal{A}_2$ *in* $\mathbf{H}^2(\Omega)$.

Proof Let $C = \{C(t) : t \in \mathbb{R}\}$ be a pullback $\mathcal{D}$-absorbing set in $D(A)$ obtained in Lemma 3.6. Then we need only to show that for any $t \in \mathbb{R}$, any $\tau_n \to -\infty$, and $u_{0n} \in C(\tau_n)$, $\{u_n(\tau_n)\}_{n=0}^{\infty}$ is precompact in $\mathbf{H}^2(\Omega)$, where $u_n(\tau_n) = u(t; \tau_n, u_{0n}) = U(t, \tau_n)u_{0n}$.

Because $V \hookrightarrow H$ is compact, from Lemma 3.7 we know that

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}u_n(\tau_n)\right\}_{n=0}^{\infty} \text{ is precompact in } L^2(\Omega). \tag{40}$$

In the following, we prove that $\{u_n(\tau_n)\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbf{H}^2(\Omega)$. From (3) we have

$$\begin{aligned}\mu\big(Au_{nk}(\tau_{nk}) - Au_{nj}(\tau_{nj})\big) = {}& -\frac{\mathrm{d}}{\mathrm{d}t}u_{nk}(\tau_{nk}) + \frac{\mathrm{d}}{\mathrm{d}t}u_{nj}(\tau_{nj}) - B\big(u_{nk}(\tau_{nk})\big) + B\big(u_{nj}(\tau_{nj})\big)\\ & - G\big(u_{nk}(\tau_{nk})\big) + G\big(u_{nj}(\tau_{nj})\big).\end{aligned} \tag{41}$$

Taking the inner product of (41) with $Au_{nk}(\tau_{nk}) - Au_{nj}(\tau_{nj})$ we can obtain

$$\begin{aligned}&\mu\big|Au_{nk}(\tau_{nk}) - Au_{nj}(\tau_{nj})\big|_2^2\\ &\quad\leq \left|\frac{\mathrm{d}}{\mathrm{d}t}u_{nk}(\tau_{nk}) - \frac{\mathrm{d}}{\mathrm{d}t}u_{nj}(\tau_{nj})\right|_2 \cdot \big|Au_{nk}(\tau_{nk}) - Au_{nj}(\tau_{nj})\big|_2\\ &\qquad + \big|B\big(u_{nk}(\tau_{nk})\big) - B\big(u_{nj}(\tau_{nj})\big)\big|_2 \cdot \big|Au_{nk}(\tau_{nk}) - Au_{nj}(\tau_{nj})\big|_2\\ &\qquad + \big|G\big(u_{nk}(\tau_{nk})\big) - G\big(u_{nj}(\tau_{nj})\big)\big|_2 \cdot \big|Au_{nk}(\tau_{nk}) - Au_{nj}(\tau_{nj})\big|_2.\end{aligned} \tag{42}$$

Like the proof of Lemma 4.2 in [16], we can also prove

$$\begin{aligned}&\big|G\big(u_{nk}(\tau_{nk})\big) - G\big(u_{nj}(\tau_{nj})\big)\big|_2 \to 0,\\ &\big|B\big(u_{nk}(\tau_{nk})\big) - B\big(u_{nj}(\tau_{nj})\big)\big|_2 \to 0, \quad \text{as } k, j \to +\infty.\end{aligned} \tag{43}$$

Taking into account (40), (42), and (43), we have

$$\big|Au_{nk}(\tau_{nk}) - Au_{nj}(\tau_{nj})\big|_2 \to 0. \tag{44}$$

Now, because

$$\|w\|_{H^2(\Omega)} \leq C(\Omega)|Aw|_2, \quad \forall w \in D(A),$$

we have

$$\big\|u_{nk}(\tau_{nk}) - u_{nj}(\tau_{nj})\big\|_{H^2(\Omega)} \to 0, \quad \text{as } k, j \to +\infty. \tag{45}$$

Equation (45) implies that the process $\{U(t,\tau)\}_{t\geq\tau}$ is pullback $\mathcal{D}$-asymptotically compact in $\mathbf{H}^2(\Omega)$. Combining Lemma 3.6, Lemma 4.1, and Theorem 2.1, yields Theorem 4.2 immediately. □

5 Conclusions

In this paper, we consider the 3D Navier-Stokes equations with nonlinear damping $\alpha|u|^{\beta-1}u$ with initial and Dirichlet boundary conditions which arises in the fluid dynamics. Under suitable assumptions on the external force function f, we obtain the existence of pullback $\mathcal{D}$-attractors of solutions in V and $\mathbf{H}^2(\Omega)$, respectively. In [16] and [17], we have discussed the existence of global attractors and uniform attractors of such 3D NSEs in V and $\mathbf{H}^2(\Omega)$ with $\frac{7}{2} \leq \beta \leq 5$. From this paper, we find that the pullback $\mathcal{D}$-attractors can exist in V and $\mathbf{H}^2(\Omega)$ with $\beta \in (3, 5]$. Obviously, this improves the results in [16] and [17].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of the three authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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