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Critical Fujita exponents to a class of non-Newtonian filtration equations with fast diffusion

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Abstract

We consider the Cauchy problem to a class of fast-diffusion non-Newtonian filtration equations. Besides the usual degeneracy in the fast-diffusion non-Newtonian filtration, the equation is degenerate or singular at infinity, depending on the sign of the parameter related to the coefficient of diffusion. Fujita type theorems are established and the critical Fujita exponent is determined. Specially, we also prove that the nontrivial solution blows up in a finite time on the critical situation.

MSC: 35K55; 35B33

Keywords: critical Fujita exponent; degeneracy; singularity; non-Newtonian filtration equation; fast diffusion

1 Introduction

The purpose of this paper is to investigate the critical Fujita exponent for the following initial value problem:

$$\left(|x|+1\right)^{\mu_1}\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{q-1}\nabla u\right) + \left(|x|+1\right)^{\mu_2}u^p, \quad x\in\mathbb{R}^n, t>0, \tag{1}$$

$$u(x,0) = u_0(x), \quad x\in\mathbb{R}^n, \tag{2}$$

where $p>1$, $0<q<1$, $\max\{-n,(n-1)/q-(n+1)\}<\mu_1\le\mu_2<p\mu_1+(p-1)n$, and $0\le u_0\in C_0(\mathbb{R}^n)$.

The study of critical exponents began in 1966 by Fujita in [1], where it was proved for the initial value problem of

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x\in\mathbb{R}^n, t>0$$

that the problem admits no nontrivial nonnegative global solution if $1<p<p_c=1+2/n$, whereas if $p>p_c$, it admits both global (with small data) and non-global (with large initial data) solutions. Later, in 1981, Weissler [2] proved that the critical case $p=p_c$ is still a blow-up case.

In Fujita's work, the new phenomenon of nonlinear parabolic equations was discovered. From then on, there has been a lot of work on the critical Fujita exponents for various nonlinear evolution equations and systems (see, *e.g.*, the survey papers [3, 4] and the references therein, and also [5–15]). Among those, the Fujita type theorems for the slow-diffusion non-Newtonian filtration equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\big(|\nabla u|^{q-1}\nabla u\big) + u^p, \quad x \in \mathbb{R}^n, t > 0 \tag{3}$$

was investigated by Galaktionov in [16, 17], where $p, q > 1$. He proved that $p_c = q + (q + 1)/n$ by blow-up subsolutions and global supersolutions. Recently, the same problem for an interesting variant of (3) is studied by the authors [13]. The non-Newtonian filtration equations with fast diffusion were considered by Qi and Wang in [18], where the critical Fujita exponent was determined for the Cauchy problem of the equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\big(|\nabla u|^{q-1}\nabla u\big) + |x|^\sigma u^p, \quad x \in \mathbb{R}^n, t > 0 \tag{4}$$

with $p > 1$, $(n-1)/(n+1) < q < 1$, and $\sigma > n(1-q) - q - 1$. It is shown that $p_c = q + (q+1+\sigma)/n$ by energy functions. Obviously, they did not cover the portion $0 < q \le (n-1)/(n+1)$ of the fast-diffusion range.

In the present paper, we study the problem (1), (2) and formulate the critical Fujita exponent as

$$p_c = q + (q + 1 + \mu_2)/(n + \mu_1)$$

and the critical situation $p = p_c$ is still the blow-up case. The range of m considered in this paper is $0 < m < 1$, the whole fast-diffusion range of (1). Like the non-Newtonian filtration equation with fast diffusion, (1) is singular at points where $|\nabla u| = 0$. In addition, (1) is degenerate at $|x| = +\infty$ for $\mu_1 > 0$ and singular for $\mu_1 < 0$, different from both (3) and (4). Inspired by [11, 18, 19], to prove the solutions' blow-up, we analyze the interaction between the nonlinear source and nonlinear diffusion via precise estimates through constructing energy functions by use of the normalized principal eigenfunction of $-\Delta$ in the unit ball B_1 of $\mathbb{R}^n$ with homogeneous initial-boundary condition, rather than constructing subsolutions as the author did in [16, 17]. This method for equation (1) and its special case (4) basically depends upon the nonincreasing properties in the spatial variant of solutions, which is trivial with $\mu_1 = \mu_2$, while it may be invalid if $\mu_1 < \mu_2$. For all these reasons, we have to overcome some technical difficulties.

This paper is arranged as follows. Some preliminaries are introduced in Section 2, including the local existence theorem, the comparison principle, and a property of solutions from propagation of disturbances. The Fujita type theorems are established in Section 3. Finally in Section 4, the critical case will be concerned.

2 Preliminaries

Throughout this paper, we use B_r to indicate the ball in $\mathbb{R}^n$ with radius r and center at the origin. The solution considered here is taken in the following sense.

Definition 2.1 We call

$$0 \le u \in C\big([0,T); L^\infty(\mathbb{R}^n)\big) \cap L^{q+1}_{\mathrm{loc}}\big(0,T; W^{1,q+1}_{\mathrm{loc}}(\mathbb{R}^n)\big)$$

a solution to the problem (1), (2) in $(0,T)$ with $0 < T \leq +\infty$ if

$$\int_0^T \int_{\mathbb{R}^n} \left(\left(|x|+1\right)^{\mu_1} u \frac{\partial \phi}{\partial t} - |\nabla u|^{q-1} \nabla u \cdot \nabla \phi + \left(|x|+1\right)^{\mu_2} u^p \phi \right) dx\,dt = 0$$

holds for any $\phi \in C_0^\infty(\mathbb{R}^n \times (0,T))$ and

$$\lim_{t\to 0^+} \int_{\mathbb{R}^n} u(x,t)\zeta(x)\,dx = \int_{\mathbb{R}^n} u_0(x)\zeta(x)\,dx$$

for any $\zeta \in C_0^\infty(\mathbb{R}^n)$.

Like the non-Newtonian filtration equation, it is not hard to prove the well-posedness to the problem (1), (2), one can see, *e.g.*, [20].

Next, we will prove the following proposition on a property of solutions from propagation of disturbances.

Proposition 2.1 *Assume that u is a solution to the problem* (1), (2) *with $0 \leq u_0 \in C_0(\mathbb{R}^n)$ nontrivial, then $u(0,t_0) > 0$ for some $t_0 > 0$.*

Proof That u_0 is nontrivial shows that there exists $0 \neq x_0 \in \mathbb{R}^n$ and $\kappa, \rho > 0$ such that

$$u_0(x) \geq \kappa \left(1 - \left(\frac{|x-x_0|^{q+1}}{\rho^{q+1}}\right)^{1/q}\right)_+^2, \quad x \in \mathbb{R}^n,$$

where $s_+ = \max\{s, 0\}$. Let

$$\Phi(x,t) = \frac{\kappa \rho^{(q+1)\xi}}{R^\xi(t)} \left(1 - \left(\frac{|x-x_0|^{q+1}}{R(t)}\right)^{1/q}\right)_+^2, \quad x \in \mathbb{R}^n, t > 0,$$

$$D = \left\{(x,t) \in \mathbb{R}^{n+1} \times \mathbb{R}_+ : |x-x_0| < 2|x_0|, |x-x_0|^{q+1} < R(t), 0 < t < \frac{\kappa^{1-q}\rho^{q+1}}{\xi}\right\},$$

with $R(t) = \kappa^{q-1}t + \rho^{q+1}$, and $\xi > 1$ independent of κ and ρ to be chosen later.

Denote

$$\|z\| = \frac{|x-x_0|^{q+1}}{R(t)}, \qquad H = 1 - \|z\|^{1/q}.$$

A direct calculation within D shows

$$\frac{\partial \Phi}{\partial t} = -\frac{\xi \kappa^q \rho^{(q+1)\xi}}{R^{\xi+1}(t)} H^2 + \frac{2}{q} \frac{\kappa^q \rho^{(q+1)\xi}}{R^{\xi+1}(t)} H \|z\|^{1/q},$$

$$\operatorname{div}\left(|\nabla \Phi|^{q-1} \nabla \Phi\right) = -\left(\frac{2(q+1)}{q}\right)^q \frac{1}{R(t)} \left(\frac{\kappa \rho^{(q+1)\xi}}{R^\xi(t)} H\right)^q \left(n - (q+1)\frac{\|z\|^{1/q}}{H}\right).$$

Setting

$$\mathscr{L}[\Phi] = \frac{R^{\xi+1}(t)}{\kappa^q \rho^{(q+1)\xi} H} \left(\left(|x|+1\right)^{\mu_1} \frac{\partial \Phi}{\partial t} - \operatorname{div}\left(|\nabla \Phi|^{q-1} \nabla \Phi\right) \right), \quad (x,t) \in D,$$

then

$$\mathscr{L}[\Phi] = -\big(|x|+1\big)^{\mu_1}\xi H + \frac{2}{q}\big(|x|+1\big)^{\mu_1}\|z\|^{1/q} + \left(\frac{2(q+1)}{q}\right)^q \left(\frac{\rho^{(q+1)\xi}}{R^\xi(t)}H\right)^{q-1}\left(n-(q+1)\frac{\|z\|^{1/q}}{H}\right).$$

Divide D into two sets

$$D^{(1)} = \big\{(x,t)\in D : H<\delta\big\} \quad \text{and} \quad D^{(2)} = \big\{(x,t)\in D : H\geq\delta\big\}$$

with $\delta>0$ satisfying

$$\delta^{q-1}\left((q+1)\left(\frac{1}{\delta}-1\right)-n\right) \geq \frac{2}{q}\Lambda_1, \tag{5}$$

where

$$\Lambda_1 = \begin{cases} (3|x_0|+1)^{\mu_1}, & \mu_1\geq 0, \\ 1, & \mu_1<0. \end{cases}$$

Then in $D^{(1)}$,

$$\begin{aligned} \mathscr{L}[\Phi] &\leq \frac{2}{q}\Lambda_1 + \left(\frac{2(q+1)}{q}\right)^q \delta^{q-1}\left(n-(q+1)\left(\frac{1}{\delta}-1\right)\right) \\ &\leq \frac{2}{q}\Lambda_1 + \delta^{q-1}\left(n+q+1-\frac{q+1}{\delta}\right) \\ &\leq 0. \end{aligned}$$

For the chosen $\delta>0$, we have in $D^{(2)}$

$$\mathscr{L}[\Phi] \leq -\Lambda_2\xi\delta + \frac{2}{q}\Lambda_1 + n\left(\frac{2(q+1)}{q}\right)^q\left(\frac{\rho^{(q+1)\xi}}{R^\xi(t)}\delta\right)^{q-1},$$

where

$$\Lambda_2 = \begin{cases} 1, & \mu_1\geq 0, \\ (3|x_0|+1)^{\mu_1}, & \mu_1<0. \end{cases}$$

Due to

$$\left(\frac{\rho^{(q+1)\xi}}{R^\xi(t)}\delta\right)^{q-1} \leq \left(1+\frac{1}{\xi}\right)^{\xi(1-q)}\delta^{q-1} \leq \left(\frac{\mathrm{e}}{\delta}\right)^{1-q}, \quad (x,t)\in D,$$

we know

$$\mathscr{L}[\Phi] \leq -\Lambda_2\xi\delta + \frac{2}{q}\Lambda_1 + n\left(\frac{2(q+1)}{q}\right)^m\left(\frac{\mathrm{e}}{\delta}\right)^{1-q}, \quad (x,t)\in D.$$

So for fixed $\delta > 0$ satisfying (5) and $\xi > 1$ satisfying

$$\Lambda_2 \xi \delta \geq \frac{2}{q}\Lambda_1 + n\left(\frac{2(q+1)}{q}\right)^q \left(\frac{\mathrm{e}}{\delta}\right)^{1-q},$$

we obtain

$$\left(|x|+1\right)^{\mu_1}\frac{\partial \Phi}{\partial t} - \operatorname{div}\left(|\nabla \Phi|^{q-1}\nabla \Phi\right) \leq 0, \quad x \in \mathbb{R}^n, 0 < t < \frac{\kappa^{1-q}\rho^{q+1}}{\xi}.$$

Clearly, the constant $\xi > 1$ is independent of κ and ρ. The comparison principle implies

$$u(x,t) \geq \Phi(x,t), \quad (x,t) \in D.$$

In particular,

$$u(x,t_1) > 0, \quad x \in \Gamma_1$$

with $t_1 = \frac{\kappa^{1-q}\rho^{q+1}}{\xi}$ and

$$\Gamma_1 = \left\{ x \in \mathbb{R}^n : |x - x_0| < 2|x_0|, |x - x_0|^{q+1} < \frac{\xi + 1}{\xi}\rho^{q+1} \right\}.$$

If $0 \in \Gamma_1$, the proof is complete. Otherwise,

$$u(x,t_1) \geq \Phi(x,t_1) = \kappa_1 \left(1 - \left(\frac{|x-x_0|^{q+1}}{\rho_1^{q+1}}\right)^{1/q}\right)_+^2, \quad x \in \mathbb{R}^n,$$

where

$$\kappa_1 = \kappa\left(\frac{\xi}{\xi+1}\right)^{\xi}, \qquad \rho_1 = \rho\left(\frac{\xi+1}{\xi}\right)^{1/(q+1)}.$$

From the above argument, we have

$$u(x,t) \geq \Phi_1(x,t), \quad (x,t) \in D_1,$$

where

$$\Phi_1(x,t) = \frac{\kappa_1 \rho_1^{(q+1)\xi}}{R_1^{\xi}(t)}\left(1 - \left(\frac{|x-x_0|^{q+1}}{R_1(t)}\right)^{1/q}\right)_+^2, \quad x \in \mathbb{R}^n, t > t_1,$$

$$D_1 = \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ : |x-x_0| < 2|x_0|, |x-x_0|^{q+1} < R_1(t), t_1 < t < t_1 + \frac{\kappa_1^{1-q}\rho_1^{q+1}}{\xi} \right\},$$

with $R_1(t) = \kappa_1^{q-1}(t - t_1) + \rho_1^{q+1}$. In particular,

$$u(x,t_2) > 0, \quad x \in \Gamma_2,$$

with $t_2 = t_1 + \frac{\kappa_1^{1-q}\rho_1^{q+1}}{\xi}$ and

$$\Gamma_2 = \left\{x \in \mathbb{R}^n : |x - x_0| < 2|x_0|, |x - x_0|^{q+1} < \frac{\xi + 1}{\xi}\rho_1^{q+1}\right\}.$$

If $0 \in \Gamma_2$, the proof is complete. Otherwise, repeat the above procedure. We get the conclusion in finite steps. □

3 Fujita type theorems

Let us establish the Fujita type theorems.

Definition 3.1 We call u the blow-up solution to equation (1) if there exists some $0 < T_* < +\infty$ such that

$$\lim_{t\to T_*^-} \left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} = \lim_{t\to T_*^-} \sup_{x\in\mathbb{R}^n} u(x,t) = +\infty.$$

Theorem 3.1 *Assume that* $1 < p < p_c = q + (q + 1 + \mu_2)/(n + \mu_1)$ *and* $0 \le u_0 \in C_0(\mathbb{R}^n)$ *is nontrivial. Then the problem* (1), (2) *admits a blow-up solution.*

Proof Due to Proposition 2.1, we may assume $u_0(0) > 0$. By the comparison principle, we only need to prove the conclusion for radial and nonincreasing $u_0(x)$, *i.e.*,

$$u_0(x) = h_0\bigl(|x|\bigr), \quad x \in \mathbb{R}^n,$$

where $h_0 \in C_0^1([0,+\infty))$ satisfies $h_0'(0) = 0$ and $h_0'(r) \le 0$ for $r > 0$. With such initial data, the solution u is also radial, namely

$$u(x,t) = h\bigl(|x|,t\bigr), \quad x \in \mathbb{R}^n, t \ge 0. \tag{6}$$

If $\mu_1 = \mu_2$, it is easy to know that u is also nonincreasing by a standard regularization argument and the maximum principle. However, this method is invalid if $\mu_1 < \mu_2$. In the following discussion, we will first of all consider a nonincreasing u, namely $h(r,t)$ is nonincreasing with respect to $r \in [0,+\infty)$ for any $t \ge 0$, and we treat the general case finally.

Let

$$\psi(x) = \begin{cases} 1, & 0 \le |x| \le 1, \\ f(|x| - 1), & 1 < |x| < 2, \\ 0, & |x| \ge 2, \end{cases}$$

where f is the principal eigenfunction of $-\Delta$ in the unit ball B_1 of $\mathbb{R}^n$ with homogeneous initial-boundary condition, normalized by $\|f\|_{L^\infty(B_1)} = 1$. For $l > 1$, define

$$\psi_l(x) = \psi(x/l), \quad x \in \mathbb{R}^n.$$

Then

$$|\nabla\psi_l| \le \frac{M_0}{l}, \qquad |\Delta\psi_l| \le \frac{M_0}{l^2}, \qquad \frac{|\Delta\psi_l|}{\psi_l} \le \frac{M_0}{l^2}, \quad x \in B_{2l} \setminus B_l,$$

with $M_0 > 0$ independent of l. Set

$$\eta_l(t) = \int_{\mathbb{R}^n} \big(|x| + 1\big)^{\mu_1} u\psi_l \, dx, \quad t \geq 0. \tag{7}$$

Definition 2.1 gives

$$\frac{d\eta_l}{dt} = -\int_{B_{2l}} |\nabla u|^{q-1} \nabla u \cdot \nabla \psi_l \, dx + \int_{\mathbb{R}^n} \big(|x| + 1\big)^{\mu_2} u^p \psi_l \, dx.$$

For radial and nonincreasing $u(x,t)$, one has

$$\begin{aligned}\int_{B_{2l}} |\nabla u|^{q-1} \nabla u \cdot \nabla \psi_l \, dx &= \int_{B_{2l}} |\nabla u|^q |\nabla \psi_l| \, dx \\ &\leq \left(\int_{B_{2l}} |\nabla u| \cdot |\nabla \psi_l| \, dx\right)^q \left(\int_{B_{2l}} |\nabla \psi_l| \, dx\right)^{1-q} \\ &= \left(\int_{B_{2l}} \nabla u \cdot \nabla \psi_l \, dx\right)^q \left(\int_{B_{2l}} |\nabla \psi_l| \, dx\right)^{1-q} \\ &\leq M_0 l^{(n-1)(1-q)} \left(\int_{B_{2l}} \nabla u \cdot \nabla \psi_l \, dx\right)^q\end{aligned}$$

and

$$0 \leq \int_{B_{2l}} \nabla u \cdot \nabla \psi_l \, dx = \int_{\partial B_{2l}} u \nabla \psi_l \cdot \boldsymbol{\nu} \, d\sigma - \int_{B_{2l}} u \Delta \psi_l \, dx \leq -\int_{B_{2l}} u \Delta \psi_l \, dx,$$

where $\boldsymbol{\nu}$ is the unit outer normal to ∂B_{2l}. Hence

$$\frac{d\eta_l}{dt} \geq -M_0 l^{(1-q)(n-1)} \left|\int_{B_{2l}} u \Delta \psi_l \, dx\right|^q + \int_{\mathbb{R}^n} \big(|x| + 1\big)^{\mu_2} u^p \psi_l \, dx. \tag{8}$$

The Hölder inequality yields

$$\begin{aligned}\left|\int_{B_{2l}} u \Delta \psi_l \, dx\right|^q &\leq \left(\int_{B_{2l} \setminus B_l} u |\Delta \psi_l| \, dx\right)^q \\ &\leq \left(\int_{B_{2l} \setminus B_l} \big(|x| + 1\big)^{-\mu_2/(p-1)} |\Delta \psi_l|^{p/(p-1)} \psi_l^{-1/(p-1)} \, dx\right)^{q(p-1)/p} \\ &\quad \cdot \left(\int_{B_{2l} \setminus B_l} \big(|x| + 1\big)^{\mu_2} u^p \psi_l \, dx\right)^{q/p} \\ &\leq M_1 l^{q(n-2)-q(n+\mu_2)/p} \left(\int_{\mathbb{R}^n} \big(|x| + 1\big)^{\mu_2} u^p \psi_l \, dx\right)^{q/p}\end{aligned}$$

with $M_1 > 0$ independent of l, which, together with (8), implies

$$\begin{aligned}\frac{d\eta_l}{dt} \geq{}& \left(\int_{\mathbb{R}^n} \big(|x| + 1\big)^{\mu_2} u^p \psi_l \, dx\right)^{q/p} \\ &\cdot \left\{-M_0 M_1 l^{n-q-1-q(n+\mu_2)/p} + \left(\int_{\mathbb{R}^n} \big(|x| + 1\big)^{\mu_2} u^p \psi_l \, dx\right)^{(p-q)/p}\right\}.\end{aligned} \tag{9}$$

By the Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}^n}\big(|x|+1\big)^{\mu_1}u\psi_l\,dx &\le \left(\int_{B_{2l}}\big(|x|+1\big)^{(p\mu_1-\mu_2)/(p-1)}\psi_l\,dx\right)^{(p-1)/p}\\ &\quad\cdot\left(\int_{\mathbb{R}^n}\big(|x|+1\big)^{\mu_2}u^p\psi_l\,dx\right)^{1/p}\\ &\le M_2 l^{(n+\mu_1)-(n+\mu_2)/p}\left(\int_{\mathbb{R}^n}\big(|x|+1\big)^{\mu_2}u^p\psi_l\,dx\right)^{1/p},\end{aligned}$$

and hence

$$\int_{\mathbb{R}^n}\big(|x|+1\big)^{\mu_2}u^p\psi_l\,dx \ge M_2^{-p}l^{-p(n+\mu_1)+(n+\mu_2)}\eta_l^p \tag{10}$$

with $M_2 > 0$ independent of l. Equations (9) and (10) show that

$$\begin{aligned}\frac{d\eta_l}{dt} &\ge \big(M_2^{-p}l^{-p(n+\mu_1)+(n+\mu_2)}\big)^{q/p}\eta_l^q\\ &\quad\cdot\big\{-M_0M_1l^{n-q-1-q(n+\mu_2)/p}+M_2^{q-p}l^{[-p(n+\mu_1)+(n+\mu_2)](p-q)/p}\eta_l^{p-q}\big\}.\end{aligned} \tag{11}$$

We mention that the above discussion holds provided that $p > 1$.

If $p < p_c = q + (q + 1 + \mu_2)/(n + \mu_1)$, then

$$n-q-1-(n+\mu_2)q/p < \big[-p(n+\mu_1)+(n+\mu_2)\big](p-q)/p.$$

Notice that η_l is nondecreasing with respect to $l \in (1, +\infty)$ and $\sup\{\eta_l(0) : l \in (1, +\infty)\} > 0$, and from (11) one shows that, for $l > 1$ large enough, there exists a constant $\delta > 0$ such that

$$\begin{aligned}\frac{d\eta_l}{dt} &\ge \big(M_2^{-p}l^{-p(n+\mu_1)+(n+\mu_2)}\big)^{q/p}\eta_l^q\\ &\quad\cdot\left(\frac{1}{2}M_2^{q-p}l^{[-p(n+\mu_1)+(n+\mu_2)](p-q)/p}\eta_l^{p-q}\right)\\ &\ge \delta\eta_l^p.\end{aligned}$$

So there exists some $0 < T_* < +\infty$ such that

$$\lim_{t\to T_*^-}\eta_l(t) = +\infty.$$

Due to $\operatorname{supp}\psi_l = B_{2l}$, we obtain

$$\lim_{t\to T_*^-}\big\|u(\cdot,t)\big\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

Next, for the general case without the assumption that $u(x,t)$ is nonincreasing, define

$$\underline{u}(x,t) = \min_{0\le r\le|x|}h(r,t), \quad x\in\mathbb{R}^n, t\ge 0. \tag{12}$$

Then $\underline{u}(x,t)$ is nonincreasing,

$$0 \le \underline{u}(x,t) \le u(x,t), \quad x \in \mathbb{R}^n, t \ge 0, \tag{13}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^n}\left(|x|+1\right)^{\mu_1}\underline{u}\psi_l\,dx \ge -\int_{B_{2l}}|\nabla\underline{u}|^{q-1}\nabla\underline{u}\cdot\nabla\psi_l\,dx + \int_{\mathbb{R}^n}\left(|x|+1\right)^{\mu_2}\underline{u}^p\psi_l\,dx. \tag{14}$$

From the above argument, we get

$$\lim_{t\to\tilde{T}_*^-}\left\|\underline{u}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} = +\infty$$

for some $0 < \tilde{T}_* < +\infty$, and (13) ensures that u is a blow-up solution. □

Let us turn to the case $p > p_c$. Suppose that

$$U(x,t) = (t+1)^{-\alpha}V\left((t+1)^{-\beta}\left(|x|+1\right)\right), \tag{15}$$

where

$$\alpha = \frac{q+1+\mu_2}{\mu_1(p-q)+\mu_2(q-1)+(q+1)(p-1)}, \qquad \beta = \frac{p-q}{q+1+\mu_2}\alpha, \tag{16}$$

is a self-similar solution to (1). It is easy to show that $V(r)$ solves

$$\left(\left|V'\right|^{q-1}V'\right)' + \frac{n-1}{r}\left|V'\right|^{q-1}V' + \beta r^{\mu_1+1}V' + \alpha r^{\mu_1}V + r^{\mu_2}V^p = 0, \quad r > 0. \tag{17}$$

Lemma 3.1 *Assume that $p > p_c = q + (q+1+\mu_2)/(n+\mu_1)$. Then for $\varepsilon > 0$ small enough, the function*

$$V(r) = \varepsilon\left(1+\rho(\varepsilon)r^\lambda\right)^{-\gamma}, \quad r > 0,$$

where

$$\lambda = 1 + \frac{\mu_1+1}{q}, \qquad \gamma = \frac{q}{1-q}, \qquad \rho(\varepsilon) = \frac{1}{\gamma\lambda}\beta^{1/q}\varepsilon^{(1-q)/q},$$

is a supersolution to equation (17), *i.e.*

$$\left(\left|V'\right|^{q-1}V'\right)' + \frac{n-1}{r}\left|V'\right|^{q-1}V' + \beta r^{\mu_1+1}V' + \alpha r^{\mu_1}V + r^{\mu_2}V^p \le 0, \quad r > 0.$$

Proof It is not hard to show that it suffices to verify

$$\begin{aligned}
&q(\gamma+1)\rho(\varepsilon)\lambda\left[\varepsilon\gamma\lambda\rho(\varepsilon)\right]^q\left[1+\rho(\varepsilon)r^\lambda\right]^{-q(\gamma+1)-1}r^{(q+1)(\lambda-1)} \\
&\quad - q(\lambda-1)\left[\varepsilon\gamma\lambda\rho(\varepsilon)\right]^q\left[1+\rho(\varepsilon)r^\lambda\right]^{-q(\gamma+1)}r^{q(\lambda-1)-1} \\
&\quad - (n-1)\left[\varepsilon\gamma\lambda\rho(\varepsilon)\right]^q\left[1+\rho(\varepsilon)r^\lambda\right]^{-q(\gamma+1)}r^{q(\lambda-1)-1}
\end{aligned}$$

$$- \beta\varepsilon\gamma\lambda\rho(\varepsilon)\big[1+\rho(\varepsilon)r^{\lambda}\big]^{-(\gamma+1)}r^{\mu_1+\lambda} + \alpha\varepsilon\big[1+\rho(\varepsilon)r^{\lambda}\big]^{-\gamma}r^{\mu_1}$$
$$+ \varepsilon^{p}\big[1+\rho(\varepsilon)r^{\lambda}\big]^{-p\gamma}r^{\mu_2} \le 0, \quad r > 0,$$

namely

$$\begin{aligned} &\big\{\big[\varepsilon\gamma\lambda\rho(\varepsilon)\big]^{q} - \beta\varepsilon\big\}\gamma\lambda\rho(\varepsilon)\big[1+\rho(\varepsilon)r^{\lambda}\big]^{-(\gamma+1)}r^{\mu_1+\lambda} \\ &\quad - \big\{\big[q(\lambda-1)+n-1\big]\varepsilon^{-1}\big[\varepsilon\gamma\lambda\rho(\varepsilon)\big]^{q} - \alpha - \varepsilon^{p-1}\big[1+\rho(\varepsilon)r^{\lambda}\big]^{-(p-1)\gamma}r^{\mu_2-\mu_1}\big\} \\ &\quad \cdot \varepsilon\big[1+\rho(\varepsilon)r^{\lambda}\big]^{-\gamma}r^{\mu_1} \le 0, \quad r > 0. \end{aligned} \tag{18}$$

From the definition of $[\varepsilon\gamma\lambda\rho(\varepsilon)]^{q} = \beta\varepsilon$, we have

$$\begin{aligned} &\big[q(\lambda-1)+n-1\big]\varepsilon^{-1}\big[\varepsilon\gamma\lambda\rho(\varepsilon)\big]^{q} - \alpha - \varepsilon^{p-1}\big[1+\rho(\varepsilon)r^{\lambda}\big]^{-(p-1)\gamma}r^{\mu_2-\mu_1} \\ &\quad = (n+\mu_1)\beta - \alpha - \varepsilon^{p-1}\big[1+\rho(\varepsilon)r^{\lambda}\big]^{-(p-1)\gamma}r^{\mu_2-\mu_1} \\ &\quad \ge (n+\mu_1)\beta - \alpha - \varepsilon^{p-1}\big[\rho(\varepsilon)\big]^{-(p-1)\gamma}r^{\mu_2-\mu_1-\lambda(p-1)\gamma}. \end{aligned}$$

Due to $p > p_c = q + (q+1+\mu_2)/(n+\mu_1)$ and $(n-1)/q - (n+1) < \mu_1 \le \mu_2$,

$$(n+\mu_1)\beta > \alpha, \quad \lambda(p-1)\gamma \ge \mu_2 - \mu_1 \ge 0.$$

Hence,

$$(n+\mu_1)\beta - \alpha - \varepsilon^{p-1}\big[\rho(\varepsilon)\big]^{-(p-1)\gamma}r^{\mu_2-\mu_1-\lambda(p-1)\gamma} > 0, \quad r > 0$$

holds for sufficiently small $\varepsilon > 0$, and (18) is obtained. □

Theorem 3.2 *Assume that* $p > p_c = q + (q+1+\mu_2)/(n+\mu_1)$. *Then the solution to the problem* (1), (2) *exists globally with small* u_0, *or blows up with large* u_0.

Proof Let $V(r)$ be a supersolution to equation (17) in Lemma 3.1. From

$$V'(r) < 0, \quad r > 0,$$

one can show that $U(x,t)$ given in (15) is a supersolution to equation (1) with α and β given in (16). Therefore, the comparison principle implies the problem (1), (2) has a nontrivial global solution with small u_0.

Let us turn to the case of large u_0. Denote by the radial u a solution to the problem (1), (2). Temporarily suppose u is nonincreasing. Then (11) holds with η_l defined by (7). If u_0 is so large that

$$M_0M_1l_0^{n-q-1-(n+\mu_2)q/p} < \frac{1}{2}M_2^{q-p}l_0^{[-p(n+\mu_1)+(n+\mu_2)](p-q)/p}\eta_{l_0}^{p-q}(0)$$

holds for some $l_0 > 1$, then from (11), we get

$$\eta_{l_0}(t) \ge \eta_{l_0}(0), \quad t > 0,$$

and

$$\begin{aligned}\frac{d\eta_{l_0}}{dt} &\geq \left(M_2^{-p} l_0^{-p(n+\mu_1)+(n+\mu_2)}\right)^{q/p} \eta_{l_0}^q \\ &\quad \cdot \left(\frac{1}{2} M_2^{q-p} l_0^{[-p(n+\mu_1)+(n+\mu_2)](p-q)/p} \eta_{l_0}^{p-q}\right) \\ &= \delta_0 \eta_{l_0}^p\end{aligned}$$

for some $\delta_0 > 0$. Therefore, u is a blow-up solution.

For the general case without the assumption that $u(x,t)$ is nonincreasing, considering a new function just as in the proof of Theorem 3.1, one can also see that u is a blow-up solution. □

4 Critical case

Now, let us deal with the critical case $p = p_c = q + (q+1+\mu_2)/(n+\mu_1)$. Let ψ_l, η_l, M_1, M_2 be defined as in the previous section.

Lemma 4.1 *Assume that* $p = p_c = q + (q+1+\mu_2)/(n+\mu_1)$ *and* u *is a nontrivial, global, radial, and nonincreasing solution to the problem* (1), (2). *Then*

$$\int_{\mathbb{R}^n} \left(|x|+1\right)^{\mu_1} u(x,t)\,dx \leq M, \quad t > 0, \tag{19}$$

holds for some constant $M > 0$ *independent of* t.

Proof $p = p_c$ yields

$$n - q - 1 - (n+\mu_2)q/p_c = \left[-p_c(n+\mu_1) + (n+\mu_2)\right](p_c - q)/p_c.$$

Then, for the global, radial and nonincreasing solution u, from (11), we have

$$M_2^{q-p_c} \eta_l^{p_c-q}(t) \leq 2M_0 M_1, \quad l > 1, t > 0.$$

Otherwise, u blows up in a finite time. Therefore (19) holds for some constant $M > 0$ owing to

$$\lim_{l\to+\infty} \eta_l(t) = \int_{\mathbb{R}^n} \left(|x|+1\right)^{\mu_1} u(x,t)\,dx.$$ □

Lemma 4.2 *Assume that* $p = p_c = q + (q+1+\mu_2)/(n+\mu_1)$, u *be a nontrivial, radial, and nonincreasing solution to the problem* (1), (2) *and* $0 < \theta < 1$. *Then*

$$\begin{aligned}\frac{d\eta_l}{dt} &\geq M_2^{-q(1-\theta)} l^{(n+\mu_2)-p_c(n+\mu_1)} \eta_l^{q(1-\theta)} \\ &\quad \cdot \left\{-M_3 \left(\int_{B_{2l}\setminus B_l} \left(|x|+1\right)^{\mu_1} u\psi_l\,dx\right)^{q\theta} + M_2^{q(1-\theta)-p_c} \eta_l^{p_c-q(1-\theta)}\right\}\end{aligned} \tag{20}$$

holds for any $l > 1$ *with a constant* $M_3 > 0$ *independent of* l.

Proof For any $l > 1$, the Hölder inequality yields

$$\begin{aligned}
&\left|\int_{B_{2l}} u\Delta\psi_l\,dx\right|^q \\
&\quad\leq \left(\int_{B_{2l}\setminus B_l} u|\Delta\psi_l|\,dx\right)^q \\
&\quad\leq \left(\int_{B_{2l}\setminus B_l} \bigl(|x|+1\bigr)^{-(\theta p_c\mu_1+(1-\theta)\mu_2)/[(p_c-1)(1-\theta)]}\right. \\
&\qquad\left.\cdot |\Delta\psi_l|^{p_c/[(p_c-1)(1-\theta)]}\psi_l^{-(1-\theta+p_c\theta)/[(p_c-1)(1-\theta)]}\,dx\right)^{q(p_c-1)(1-\theta)/p_c} \\
&\qquad\cdot\left(\int_{B_{2l}\setminus B_l}\bigl(|x|+1\bigr)^{\mu_2}u^{p_c}\psi_l\,dx\right)^{q(1-\theta)/p_c}\left(\int_{B_{2l}\setminus B_l}\bigl(|x|+1\bigr)^{\mu_1}u\psi_l\,dx\right)^{q\theta} \\
&\quad\leq Ml^{q(n-2)-q(n+\mu_2)/p_c+q\theta(\mu_2-p_c\mu_1-(p_c-1)n)/p_c} \\
&\qquad\cdot\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\mu_2}u^{p_c}\psi_l\,dx\right)^{q(1-\theta)/p_c}\left(\int_{B_{2l}\setminus B_l}\bigl(|x|+1\bigr)^{\mu_1}u\psi_l\,dx\right)^{q\theta},
\end{aligned}$$

with $M > 0$ a constant independent of l, which, together with (10) and (8), implies

$$\begin{aligned}
\frac{d\eta_l}{dt} &\geq \left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\mu_2}u^{p_c}\psi_l\,dx\right)^{q(1-\theta)/p_c} \\
&\quad\cdot\left\{-M_0Ml^{(n-1)(1-q)+q(n-2)-q(n+\mu_2)/p_c+q\theta[\mu_2-p_c\mu_1-(p_c-1)n]/p_c}\right. \\
&\quad\left.\cdot\left(\int_{B_{2l}\setminus B_l}\bigl(|x|+1\bigr)^{\mu_1}u\psi_l\,dx\right)^{q\theta}+\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\mu_2}u^{p_c}\psi_l\,dx\right)^{[p_c-q(1-\theta)]/p_c}\right\} \\
&\geq \bigl(M_2^{-p_c}l^{-p_c(n+\mu_1)+(n+\mu_2)}\bigr)^{q(1-\theta)/p_c}\eta_l^{q(1-\theta)} \\
&\quad\cdot\left\{-M_0Ml^{(n-1)(1-q)+q(n-2)-q(n+\mu_2)/p_c+q\theta[\mu_2-p_c\mu_1-(p_c-1)n]/p_c}\right. \\
&\quad\cdot\left(\int_{B_{2l}\setminus B_l}\bigl(|x|+1\bigr)^{\mu_1}u\psi_l\,dx\right)^{q\theta} \\
&\quad\left.+M_2^{q(1-\theta)-p_c}\eta_l^{p_c-q(1-\theta)}l^{[-p_c(n+\mu_1)+(n+\mu_2)][p_c-q(1-\theta)]/p_c}\right\}.
\end{aligned}$$

Then (20) holds due to

$$\begin{aligned}
&(n-1)(1-q)+q(n-2)-q(n+\mu_2)/p_c+q\theta\bigl[\mu_2-p_c\mu_1-(p_c-1)n\bigr]/p_c \\
&\quad=\bigl[-p_c(n+\mu_1)+(n+\mu_2)\bigr]\bigl[p_c-q(1-\theta)\bigr]/p_c.
\end{aligned}$$

□

Lemma 4.3 *Assume that $p = p_c = q + (q + 1 + \mu_2)/(n + \mu_1)$, u be a nontrivial, radial and nonincreasing solution to the problem* (1), (2). *Then*

$$\frac{d\eta_l}{dt} \geq -M_4 l^{[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)}, \quad l > 1 \tag{21}$$

holds for some constant $M_4 > 0$ independent of l.

Proof From (9), we have

$$\frac{d\eta_l}{dt} \geq \int_{\mathbb{R}^n} \big(|x|+1\big)^{\mu_2} u^{p_c} \psi_l \, dx - M l^{(n-1)(1-q)+q[n-2-(n+\mu_2)/p_c]} \left(\int_{\mathbb{R}^n} \big(|x|+1\big)^{\mu_2} u^{p_c} \psi_l \, dx \right)^{q/p_c}.$$

Then the Young inequality gives

$$\begin{aligned} \frac{d\eta_l}{dt} &\geq \int_{\mathbb{R}^n} \big(|x|+1\big)^{\mu_2} u^{p_c} \psi_l \, dx - \frac{q}{p_c} \int_{\mathbb{R}^n} \big(|x|+1\big)^{\mu_2} u^{p_c} \psi_l \, dx \\ &\quad - \frac{p_c - q}{p_c} M^{p_c/(p_c-q)} l^{[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)} \\ &\geq -M_4 l^{[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)}. \end{aligned}$$

□

We are ready to prove the blow-up theorem of Fujita type for the critical case $p = p_c = q + (q+1+\mu_2)/(n+\mu_1)$.

Theorem 4.1 *Assume that $p = p_c = q + (q+1+\mu_2)/(n+\mu_1)$ and u is a solution to the problem* (1), (2) *with $0 \leq u_0 \in C_0(\mathbb{R}^n)$ nontrivial. Then the problem* (1), (2) *admits a blow-up solution.*

Proof Similarly to the proof of Theorem 3.1, at first assume u_0 is radial and nonincreasing. Then u is radial, given by (6). Denote

$$\Lambda = \sup_{l>1, t>0} \eta_l(t) = \sup_{t>0} \int_{\mathbb{R}^n} \big(|x|+1\big)^{\mu_1} u(x,t) \, dx. \tag{22}$$

From (19) and the nontriviality of u, $0 < \Lambda < +\infty$. For any $0 < \sigma < \Lambda$, due to (22) and η_l being nondecreasing with respect to $l \in (1,+\infty)$, there exist $\omega_0 \geq 0$ and $l_0 > 2$ such that

$$\eta_{l_0/2}(\omega_0) \geq \Lambda - \sigma.$$

Then it follows from (21) that

$$\begin{aligned} &\int_{\mathbb{R}^n} \big(|x|+1\big)^{\mu_1} u(x,s) \psi_{l_0/2}(x) \, dx \\ &\quad \geq \int_{\mathbb{R}^n} |x|^{\mu_1} u(x,\omega_0) \psi_{l_0/2}(x) \, dx - M_4 (l_0/2)^{[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)} (s-\omega_0) \\ &\quad \geq \Lambda - \sigma - M_4 (l_0/2)^{[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)} (s-\omega_0), \quad s \geq \omega_0. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{B_{2l_0} \setminus B_{l_0}} \big(|x|+1\big)^{\mu_1} u(x,s) \psi_{l_0}(x) \, dx \\ &\quad \leq \int_{\mathbb{R}^n} \big(|x|+1\big)^{\mu_1} u(x,s) \, dx - \int_{\mathbb{R}^n} \big(|x|+1\big)^{\mu_1} u(x,s) \psi_{l_0/2}(x) \, dx \\ &\quad \leq \sigma + M_4 (l_0/2)^{[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)} (s-\omega_0), \quad s \geq \omega_0. \end{aligned}$$

Choosing $l = l_0$ in (20), we have

$$\begin{aligned}\frac{d\eta_{l_0}}{dt} &\geq M_2^{-q(1-\theta)} l_0^{(n+\mu_2)-p_c(n+\mu_1)} \eta_{l_0}^{q(1-\theta)} \\ &\quad \cdot \left\{-M_3\left(\int_{B_{2l_0}\setminus B_{l_0}} \left(|x|+1\right)^{\mu_1} u\psi_{l_0}\,dx\right)^{q\theta} + M_2^{q(1-\theta)-p_c}\eta_{l_0}^{p_c-q(1-\theta)}\right\} \\ &\geq M_2^{-q(1-\theta)} l_0^{(n+\mu_2)-p_c(n+\mu_1)} \eta_{l_0}^{q(1-\theta)} \\ &\quad \cdot \left\{-M_3\left(\sigma + M_4(l_0/2)^{[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)}(s-\omega_0)\right\}\right)^{q\theta} \\ &\quad + M_2^{q(1-\theta)-p_c}\eta_{l_0}^{p_c-q(1-\theta)}\}, \quad t > \omega_0.\end{aligned}$$

Fix $\sigma_0 \in (0, \Lambda)$ and $M_5 > 0$, independent of l_0, such that

$$M_3(\sigma_0 + M_5)^{q\theta} \leq \frac{1}{2}M_2^{q(1-\theta)-p_c}(\Lambda - \sigma_0)^{p_c-q(1-\theta)}.$$

Then

$$\frac{d\eta_{l_0}}{dt} \geq \frac{1}{2}M_2^{-p_c} l_0^{(n+\mu_2)-p_c(n+\mu_1)} \eta_{l_0}^{p_c}, \quad \omega_0 < t < \omega_1,$$

where

$$\omega_1 = \omega_0 + \frac{M_5}{M_4}(l_0/2)^{-[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)}.$$

Hence

$$\begin{aligned}\eta_{l_0}(\omega_1) &\geq \eta_{l_0}(\omega_0) + \frac{1}{2}M_2^{-p_c} l_0^{(n+\mu_2)-p_c(n+\mu_1)}(\Lambda - \sigma_0)^{p_c}(\omega_1 - \omega_0) \\ &\geq \eta_{l_0}(\omega_0) + \frac{1}{2}M_2^{-p_c} l_0^{(n+\mu_2)-p_c(n+\mu_1)}(\Lambda - \sigma_0)^{p_c} \\ &\quad \cdot \frac{M_5}{M_4}(l_0/2)^{-[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)}.\end{aligned}$$

Notice

$$(n+\mu_2) - p_c(n+\mu_1) - \left[p_c(n-q-1) - q(n+\mu_2)\right]/(p_c - q) = 0,$$

one gets

$$\int_{\mathbb{R}^n} \left(|x|+1\right)^{\mu_1} u(x,\omega_1)\,dx \geq \eta_{l_0}(\omega_1) \geq \eta_{l_0}(\omega_0) + \delta_0 \geq \Lambda - \sigma_0 + \delta_0,$$

with a positive constant

$$\delta_0 = \frac{M_2^{-p_c}M_5}{2M_4}(\Lambda - \sigma_0)^{p_c} 2^{[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)}$$

independent of l_0. Similarly, we reason

$$\eta_{(2l_0)/2}(\omega_1) = \eta_{l_0}(\omega_1) \geq \Lambda - \sigma_0 + \delta_0 \geq \Lambda - \sigma_0.$$

The same argument yields

$$\int_{\mathbb{R}^n} \bigl(|x|+1\bigr)^{\mu_1} u(x,t_2)\,dx \ge \eta_{2l_0}(\omega_2) \ge \eta_{2l_0}(\omega_1) + \delta_0 \ge \Lambda - \sigma_0 + 2\delta_0$$

with

$$\omega_2 = \omega_1 + \frac{M_5}{M_4} l_0^{-[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)}.$$

Repeating the procedure, one can show that

$$\int_{\mathbb{R}^n} \bigl(|x|+1\bigr)^{\mu_1} u(x,\omega_k)\,dx \ge \eta_{2^{k-1}l_0}(\omega_k) \ge \eta_{2^{k-1}l_0}(\omega_{k-1}) + \delta_0 \ge \Lambda - \sigma_0 + k\delta_0$$

with

$$\omega_k = \omega_{k-1} + \frac{M_5}{M_4} \bigl(2^{k-2} l_0\bigr)^{-[p_c(n-q-1)-q(n+\mu_2)]/(p_c-q)}, \quad k = 1, 2, \ldots.$$

Therefore

$$\sup_{t>0} \int_{\mathbb{R}^n} \bigl(|x|+1\bigr)^{\mu_1} u(x,t)\,dx = +\infty,$$

which contradicts (19).

Now, for the general case without the assumption that $u(x,t)$ is nonincreasing, consider $\underline{u}(x,t)$ defined by (12), which is nonincreasing and satisfies (13) and (14). Therefore, the conclusions of Lemmas 4.1-4.3 are all valid for $\underline{u}$. Similar to the above argument, one can show that $\underline{u}$ blows up in some $0 < T_* < +\infty$, and thus u is a blow-up solution. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed to each part of this study equally and approved the final version of the manuscript.

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