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A fourth-order elliptic Riemann type problem in $\mathbb{R}^3$

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Abstract

This article is concerned with a fourth-order elliptic equation *i.e.*, $(\Delta^2 - \kappa^2\Delta)[u] = 0$ ($\kappa > 0$) coupled by Riemann boundary value conditions in Clifford analysis. In the framework of a Clifford algebra $\mathrm{Cl}(V_{3,3})$, we obtain factorizations of the fourth-order elliptic equation and construct the explicit expressions of higher-order kernel functions. Some integral representation formulas and properties of the null solution of the fourth-order elliptic equations in Clifford analysis are presented. Based on these integral representation formulas, the boundary behavior of some singular integral operators, and the Clifford analytic approach, we prove that the fourth-order elliptic Riemann type problem in $\mathbb{R}^3$ is solvable. The explicit representation formula of the solution is also established.

MSC: 30G35; 45J05

Keywords: Clifford analysis; Riemann type problem; fourth-order elliptic equations; micro-electro-mechanical systems

1 Introduction

Fourth-order elliptic equations have become a very important and useful area of mathematics over the last few decades, which is caused both by the intensive development of the theory of partial differential equations and their applications in various fields of physics and engineering such as theory of elasticity, micro-electro-mechanical systems, bi-harmonic systems, and so on. We refer to [1–10]. Recently the fourth-order elliptic equations

$$B\Delta^2[u] - T\Delta[u] = 0 \quad (B > 0, T > 0),$$

arise in the modeling of micro-electro-mechanical systems; see [5, 6]. The equations combining bi-harmonic equations with harmonic equations can be rewritten as

$$\left(\Delta^2 - \kappa^2\Delta\right)[u] = 0, \tag{1.1}$$

with $\kappa = \sqrt{\frac{T}{B}}$.

The Riemann type problem is one of the famous problems in complex analysis and Clifford analysis; see [2–4, 11–23]. It is natural and important to study fourth-order elliptic equations coupled by the Riemann boundary conditions in $\mathbb{R}^n$ ($n \geq 3$). In general, two methods are used to deal with higher-order boundary value problems. One approach is to transform the boundary value problems for k-regular functions and poly-harmonic functions into equivalent boundary value problems for regular functions in Clifford analysis by the Almansi type decomposition theorem [15]. The other is to make use of higher-order integral representation formulas and a Clifford algebra approach [3, 4, 10]. Obviously, the first method fails to solve a system of the fourth-order elliptic equation *i.e.*, $(\Delta^2 - \kappa^2\Delta)u = 0$, coupled by the Riemann boundary conditions. Using the second method, we need to investigate factorizations of the fourth-order elliptic operator in the framework of a Clifford algebra. Furthermore, we will construct higher-order kernels. The key idea is to choose an appropriate framework of the Clifford algebra. A lot of boundary value problems for some functions with the Clifford algebra $\mathrm{Cl}(V_{n,0})$ $(n \geq 3)$ have been studied; for example, see [2, 3, 11–13, 15, 16, 24]. However, we fail to obtain factorizations of the fourth-order elliptic equation $(\Delta^2 - \kappa^2\Delta)u = 0$ $(\kappa > 0)$ using the Clifford algebra $\mathrm{Cl}(V_{n,0})$. In this article, using a Clifford algebra $\mathrm{Cl}(V_{3,3})$, we get the decomposition of the fourth-order elliptic operator *i.e.*, $(\Delta^2 - \kappa^2\Delta)$, and, moreover, construct higher-order kernel functions, which is different from [17, 25] due to choosing different Clifford algebra.

The article is organized as follows. In Section 2, we recall some basic facts about the Clifford analysis needed in the sequel. In Section 3, in the framework of the Clifford algebra $\mathrm{Cl}(V_{3,3})$, we construct the explicit expressions of the kernel functions and obtain some integral representation formulas, we study some properties of null solutions for the fourth-order elliptic equations $(\Delta^2 - \kappa^2\Delta)u = 0$, for instance, the mean value formula, the Painlevé principle, and so on. Section 4, on the basis of the above results, considers a Riemann boundary value problem for the fourth-order elliptic equation.

2 Preliminaries and notations

Let $V_{3,3}$ be an 3-dimensional real linear space with basis $\{e_1, e_2, e_3\}$, $\mathrm{Cl}(V_{3,3})$ be the Clifford algebra over $V_{3,3}$ and the 8-dimensional real linear space with basis

$$\{e_A, A = \{l_1, \ldots, l_r\} \in \mathcal{P}N, 1 \leq l_1 < \cdots < l_r \leq 3\},$$

where N stands for the set $\{1, 2, 3\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N in the above way. We denote $e_\emptyset$ by e_0 and e_A by $e_{l_1 \cdots l_r}$ for $A = \{l_1, \ldots, l_r\} \in \mathcal{P}N$. The product on $\mathrm{Cl}(V_{3,3})$ is defined by

$$\begin{cases} e_A e_B = (-1)^{n((A\cap B)\setminus N)}(-1)^{P(A,B)} e_{A\Delta B}, & \text{if } A, B \in \mathcal{P}N, \\ \lambda\mu = \sum_{A,B\in\mathcal{P}N} \lambda_A \mu_B e_A e_B, & \text{if } \lambda = \sum_{A\in\mathcal{P}N} \lambda_A e_A, \mu = \sum_{B\in\mathcal{P}N} \mu_B e_B, \end{cases}$$

where $n(A)$ is the cardinal number of the set A, the number $P(A,B) = \sum_{j\in B} P(A,j)$, $P(A,j) = n\{i, i \in A, i > j\}$, the symmetric difference set $A\Delta B$ is order-preserving in the above way, and $\lambda_A \in \mathbb{R}$ is the coefficient of the e_A-component of the Clifford number λ. It follows from the multiplication rule above that e_0 is the identity element written now as 1 and, in particular, $e_i e_j + e_j e_i = 2\delta_{ij}$, $i, j = 1, 2, 3$. Thus $\mathrm{Cl}(V_{3,3})$ is a real linear, associative, but non-commutative algebra. An involution is defined by

$$\begin{cases} \overline{e}_A = (-1)^{\frac{n(A)(n(A)+3)}{2}} e_A, & \text{if } A \in \mathcal{P}N, \\ \overline{\lambda} = \sum_{A\in\mathcal{P}N} \lambda_A \overline{e}_A, & \text{if } \lambda = \sum_{A\in\mathcal{P}N} \lambda_A e_A. \end{cases} \tag{2.1}$$

The norm of λ is defined by $\|\lambda\| = (\sum_{A\in\mathcal{P}N}|\lambda_A|^2)^{\frac{1}{2}}$. Throughout this article, suppose that Ω is an open bounded non-empty subset of $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$, we denote $\Omega^+ = \Omega$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$. We now introduce the Dirac operator $D = \sum_{i=1}^3 e_i \frac{\partial}{\partial x_i}$. In particular, we have $DD = \Delta$ where Δ is the Laplacian over $\mathbb{R}^3$. A function $u : \Omega \mapsto \mathrm{Cl}(V_{3,3})$ is said to be left monogenic if it satisfies the equation $D[u](\mathbf{x}) = 0$ for each $\mathbf{x} \in \Omega$. A similar definition can be given for right monogenic functions.

Denote

$$z_j = x_j e_0 - x_1 e_1 e_j, \quad j = 2, 3,$$

and

$$V_{l_1,\ldots,l_p}(\mathbf{x}) = \frac{1}{p!}\sum_{\pi(l_r,\ldots,l_p)} z_{l_1}\cdots z_{l_p},$$

where $(l_1,\ldots,l_p) \in \{2,3\}^p$, the sum is taken over all permutations with repetition of the sequence $(l_1,\ldots,l_p)$. In particular we define $V_{l_1,\ldots,l_p}(\mathbf{x}) = 1$ for $p = 0$ and $V_{l_1,\ldots,l_p}(\mathbf{x}) = 0$ for $p < 0$.

Lemma 2.1 [26] *Let $C_{j,p}$ and $V_{l_1,\ldots,l_p}(\mathbf{x})$ be as above, then for $j \in \mathbf{N}^*$,*

$$D\bigl[C_{j,p}\mathbf{x}^j V_{l_1,\ldots,l_p}(\mathbf{x})\bigr] = C_{j-1,p}\mathbf{x}^{j-1}V_{l_1,\ldots,l_p}(\mathbf{x}), \tag{2.2}$$

where $\mathbf{x} = \sum_{i=1}^3 x_i e_i$,

$$C_{j,p} = \begin{cases} 1, & j = 0, \\ \frac{1}{2^{[\frac{j}{2}]}([\frac{j}{2}])!\prod_{\mu=0}^{[\frac{j-1}{2}]}(3+2p+2\mu)}, & j \in \mathbf{N}^* = \mathbf{N}\setminus\{0\}. \end{cases}$$

In the following, we define

$$\Lambda(r,u) \triangleq \max_{\|\mathbf{x}\|\le r}\bigl\{\bigl\|u(\mathbf{x})\bigr\|\bigr\}, \qquad M(r,u) \triangleq \max_{\|\mathbf{x}\|=r}\bigl\{\bigl\|u(\mathbf{x})\bigr\|\bigr\}.$$

Lemma 2.2 [4, 27] *Let $D[u] = 0$ in $\mathbb{R}^3$ and $\liminf_{r\to\infty}\frac{M(r,u)}{r^m} = L < \infty$, $m \in \mathbf{N}^*$. Then*

$$u(\mathbf{x}) = \sum_{p=0}^{m}\sum_{(l_1,\ldots,l_p)} V_{l_1,\ldots,l_p}(\mathbf{x})C_{l_1,\ldots,l_p}. \tag{2.3}$$

For more information as regards the properties of Dirac operators and left monogenic functions can be found in [2, 3, 27–29].

3 Some integral representation formulas in Clifford analysis

The fourth-order elliptic partial differential operator $\Delta^2 - \kappa^2\Delta$, for $\kappa > 0$, corresponds to the fourth-order elliptic equation:

$$\bigl(\Delta^2 - \kappa^2\Delta\bigr)[u] = 0. \tag{3.1}$$

Using the multiplication rule on the Clifford algebra $\mathrm{Cl}(V_{3,3})$, equation (3.1) may also be written as

$$DDL_\kappa L_{-\kappa}[u] = DDL_{-\kappa}L_\kappa[u] = L_{-\kappa}L_\kappa DD[u] = L_\kappa L_{-\kappa}DD[u] = 0,$$

where $L_\kappa = D + \kappa$ and $L_{-\kappa} = D - \kappa$.

Lemma 3.1 [7, 8] *Let*

$$E(\kappa,\mathbf{x}) = \frac{1}{4\pi\kappa^2}\frac{e^{-\kappa\|\mathbf{x}\|}-1}{\|\mathbf{x}\|}. \tag{3.2}$$

Then the kernel function $E(\kappa,\mathbf{x})$ *is the fundamental solution to* (3.1) *in* $\mathbb{R}^3$.

Let

$$\begin{cases} H_1(\kappa,\mathbf{x}) = \frac{1}{4\pi}\frac{\mathbf{x}}{\|\mathbf{x}\|^3}, \\ H_2(\kappa,\mathbf{x}) = -\frac{1}{4\pi}\frac{1}{\|\mathbf{x}\|}, \\ H_3(\kappa,\mathbf{x}) = \frac{1}{4\pi\kappa^2}[(\frac{\mathbf{x}}{\|\mathbf{x}\|^3}+\kappa\frac{1}{\|\mathbf{x}\|})(e^{-\kappa\|\mathbf{x}\|}-1)+\kappa\frac{\mathbf{x}}{\|\mathbf{x}\|^2}e^{-\kappa\|\mathbf{x}\|}], \\ H_4(\kappa,\mathbf{x}) = -E(\kappa,\mathbf{x}) = -\frac{1}{4\pi\kappa^2}\frac{e^{-\kappa\|\mathbf{x}\|}-1}{\|\mathbf{x}\|}, \end{cases} \tag{3.3}$$

and

$$\begin{cases} E_1(\kappa,\mathbf{x}) = \frac{1}{4\pi}[\frac{\mathbf{x}}{\|\mathbf{x}\|^3}+\kappa\frac{\mathbf{x}}{\|\mathbf{x}\|^2}+\kappa\frac{1}{\|\mathbf{x}\|}]e^{-\kappa\|\mathbf{x}\|}, \\ E_2(\kappa,\mathbf{x}) = -\frac{1}{4\pi}\frac{e^{-\kappa\|\mathbf{x}\|}}{\|\mathbf{x}\|}, \\ E_3(\kappa,\mathbf{x}) = \frac{1}{4\pi\kappa^2}[\frac{\mathbf{x}}{\|\mathbf{x}\|^3}(e^{-\kappa\|\mathbf{x}\|}-1)+\kappa\frac{\mathbf{x}}{\|\mathbf{x}\|^2}e^{-\kappa\|\mathbf{x}\|}], \\ E_4(\kappa,\mathbf{x}) = -E(\kappa,\mathbf{x}) = -\frac{1}{4\pi\kappa^2}\frac{e^{-\kappa\|\mathbf{x}\|}-1}{\|\mathbf{x}\|}, \end{cases} \tag{3.4}$$

where $\kappa > 0$, $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$.

Lemma 3.2 *Let* $H_i(\kappa,\mathbf{x})$ *and* $E_i(\kappa,\mathbf{x})$ *be as in* (3.3) *and* (3.4), $i = 1,2,3,4$. *Then*

$$\begin{cases} L_{-\kappa}[H_4(\kappa,\mathbf{x})] = [H_4(\kappa,\mathbf{x})]L_{-\kappa} = H_3(\kappa,\mathbf{x}), \\ L_\kappa[H_3(\kappa,\mathbf{x})] = [H_3(\kappa,\mathbf{x})]L_\kappa = H_2(\kappa,\mathbf{x}), \\ L_{-\kappa}[E_2(\kappa,\mathbf{x})] = [E_2(\kappa,\mathbf{x})]L_{-\kappa} = E_1(\kappa,\mathbf{x}), \\ L_\kappa[E_1(\kappa,\mathbf{x})] = [E_1(\kappa,\mathbf{x})]L_\kappa = 0, \end{cases}$$

and

$$\begin{cases} D[E_4(\kappa,\mathbf{x})] = [E_4(\kappa,\mathbf{x})]D = E_3(\kappa,\mathbf{x}), \\ D[E_3(\kappa,\mathbf{x})] = [E_3(\kappa,\mathbf{x})]D = E_2(\kappa,\mathbf{x}), \\ D[H_2(\kappa,\mathbf{x})] = [H_2(\kappa,\mathbf{x})]D = H_1(\kappa,\mathbf{x}), \\ D[H_1(\kappa,\mathbf{x})] = [H_1(\kappa,\mathbf{x})]D = 0, \end{cases}$$

here $\kappa > 0$, $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$.

Remark 3.3 Let

$$H_3^*(\kappa,\mathbf{x}) = \frac{1}{4\pi\kappa^2}\left[\left(\frac{\mathbf{x}}{\|\mathbf{x}\|^3}-\kappa\frac{1}{\|\mathbf{x}\|}\right)\left(e^{-\kappa\|\mathbf{x}\|}-1\right)+\kappa\frac{\mathbf{x}}{\|\mathbf{x}\|^2}e^{-\kappa\|\mathbf{x}\|}\right] \tag{3.5}$$

and

$$E_1^*(\kappa,\mathbf{x}) = \frac{1}{4\pi}\left[\frac{\mathbf{x}}{\|\mathbf{x}\|^3} + \kappa\frac{\mathbf{x}}{\|\mathbf{x}\|^2} - \kappa\frac{1}{\|\mathbf{x}\|}\right]e^{-\kappa\|\mathbf{x}\|}. \tag{3.6}$$

When $H_3^*(\kappa,\mathbf{x})$, $E_1^*(\kappa,\mathbf{x})$ replace $H_3(\kappa,\mathbf{x})$, $E_1(\kappa,\mathbf{x})$ in (3.3), (3.4), respectively, we have the following results:

$$\begin{cases} L_\kappa[H_4(\kappa,\mathbf{x})] = [H_4(\kappa,\mathbf{x})]L_\kappa = H_3^*(\kappa,\mathbf{x}), \\ L_{-\kappa}[H_3^*(\kappa,\mathbf{x})] = [H_3^*(\kappa,\mathbf{x})]L_{-\kappa} = H_2(\kappa,\mathbf{x}), \\ L_\kappa[E_2(\kappa,\mathbf{x})] = [E_2(\kappa,\mathbf{x})]L_\kappa = E_1^*(\kappa,\mathbf{x}), \\ L_{-\kappa}[E_1^*(\kappa,\mathbf{x})] = [E_1^*(\kappa,\mathbf{x})]L_{-\kappa} = 0. \end{cases} \tag{3.7}$$

Let Ω be an open non-empty subset of $\mathbb{R}^3$ with a Lyapunov boundary, $u(\mathbf{x}) = \sum_A e_A u_A(\mathbf{x})$, where $u_A(\mathbf{x})$ are real functions. $u(\mathbf{x})$ is called a Hölder continuous function on $\overline{\Omega}$ if the following condition is satisfied:

$$\|u(\mathbf{x}_1) - u(\mathbf{x}_2)\| = \left[\sum_A \|u_A(\mathbf{x}_1) - u_A(\mathbf{x}_2)\|\right]^{\frac{1}{2}} \leq C\|\mathbf{x}_1 - \mathbf{x}_2\|^\alpha,$$

where, for any $\mathbf{x}_1, \mathbf{x}_2 \in \overline{\Omega}$, $\mathbf{x}_1 \neq \mathbf{x}_2$, $0 < \alpha \leq 1$, C is a positive constant independent of $\mathbf{x}_1$, $\mathbf{x}_2$.

Lemma 3.4 *Let* $f, g \in C^1(\Omega, \mathrm{Cl}(V_{3,3})) \cap C(\overline{\Omega}, \mathrm{Cl}(V_{3,3}))$. *Then*

$$\begin{aligned} \int_{\partial\Omega} f\, d\sigma_{\mathbf{y}} g &= \int_\Omega [f]L_\kappa g\, dV + \int_\Omega f L_{-\kappa}[g]\, dV \\ &= \int_\Omega [f]L_{-\kappa} g\, dV + \int_\Omega f L_\kappa[g]\, dV, \end{aligned}$$

where dV *denotes Lebesgue volume measure,* $d\sigma$ *stands for* $\mathrm{Cl}(V_{3,3})$*-valued* 2*-differential form.*

Proof From Stokes' theorem in Clifford analysis in [29], the results can be directly proved. □

Theorem 3.5 *Suppose that* Ω *is an open bounded non-empty subset of* $\mathbb{R}^3$ *with a Lyapunov boundary* $\partial\Omega$, $u \in C^4(\Omega, \mathrm{Cl}(V_{3,3})) \cap C^3(\overline{\Omega}, \mathrm{Cl}(V_{3,3}))$. *Then*

$$\begin{aligned} &\sum_{i=1}^3 (-1)^{i-1} \int_{\partial\Omega} H_i(\kappa, \mathbf{y}-\mathbf{x})\, d\sigma_{\mathbf{y}} D^{i-1}[u](\mathbf{y}) \\ &\quad - \int_{\partial\Omega} H_4(\kappa, \mathbf{y}-\mathbf{x})\, d\sigma_{\mathbf{y}} L_{-\kappa}\Delta[u](\mathbf{y}) \\ &\quad + \int_\Omega H_4(\kappa, \mathbf{y}-\mathbf{x})\big(\Delta^2 - \kappa^2\Delta\big)[u](\mathbf{y})\, dV \\ &= \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}, \end{cases} \end{aligned} \tag{3.8}$$

where $H_i(\kappa, \mathbf{y}-\mathbf{x})$ $(i = 1, 2, 3, 4)$ *are as in* (3.3).

Proof Let $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$. Using Lemma 3.4 and Lemma 3.2, we get

$$\begin{aligned}
&\int_{\Omega} H_4(\kappa, \mathbf{y}-\mathbf{x})\big(\Delta^2 - \kappa^2\Delta\big)[u](\mathbf{y})\,dV \\
&\quad= \int_{\Omega} H_4(\kappa, \mathbf{y}-\mathbf{x}) L_\kappa [L_{-\kappa}\Delta u](\mathbf{y})\,dV \\
&\quad= \int_{\partial\Omega} H_4(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}} L_{-\kappa}\Delta[u](\mathbf{y}) \\
&\qquad- \int_{\Omega} \big[H_4(\kappa, \mathbf{y}-\mathbf{x})\big] L_{-\kappa} \cdot L_{-\kappa}\Delta[u](\mathbf{y})\,dV \\
&\quad= \int_{\partial\Omega} H_4(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}} L_{-\kappa}\Delta[u](\mathbf{y}) \\
&\qquad- \int_{\Omega} H_3(\kappa, \mathbf{y}-\mathbf{x}) L_{-\kappa}\Delta[u](\mathbf{y})\,dV. \qquad (3.9)
\end{aligned}$$

Assume

$$I_1 = \int_{\Omega} H_3(\kappa, \mathbf{y}-\mathbf{x}) L_{-\kappa}\Delta[u](\mathbf{y})\,dV.$$

Applying Lemma 3.4 and Lemma 3.2 once again, we continue to calculate the integral I_1 and get

$$\begin{aligned}
I_1 &= \int_{\partial\Omega} H_3(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y}) - \int_{\Omega}\big[H_3(\kappa, \mathbf{y}-\mathbf{x})\big] L_\kappa \cdot \Delta[u](\mathbf{y})\,dV \\
&= \int_{\partial\Omega} H_3(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y}) - \int_{\Omega} H_2(\kappa, \mathbf{y}-\mathbf{x})\Delta[u](\mathbf{y})\,dV \\
&= \int_{\partial\Omega} H_3(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y}) - \int_{\partial\Omega} H_2(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}} D[u](\mathbf{y}) \\
&\quad+ \int_{\Omega}\big[H_2(\kappa, \mathbf{y}-\mathbf{x})\big] D \cdot D[u](\mathbf{y})\,dV \\
&= \int_{\partial\Omega} H_3(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y}) - \int_{\partial\Omega} H_2(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}} D[u](\mathbf{y}) \\
&\quad+ \int_{\partial\Omega} H_1(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}} u(\mathbf{y}), \qquad (3.10)
\end{aligned}$$

From (3.9) and (3.10), in this case, the result follows.

Now, let $\mathbf{x} \in \Omega$ and take $r > 0$ such that $B(\mathbf{x}, r) \subset \Omega$. Invoking the previous case, we may write

$$\begin{aligned}
&\sum_{i=1}^{3} (-1)^{i-1} \int_{\partial(\Omega\setminus B(\mathbf{x},r))} H_i(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}} D^{i-1}[u](\mathbf{y}) \\
&\quad- \int_{\partial(\Omega\setminus B(\mathbf{x},r))} H_4(\kappa, \mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}} L_{-\kappa}\Delta[u](\mathbf{y}) \\
&\quad+ \int_{\Omega\setminus B(\mathbf{x},r)} H_4(\kappa, \mathbf{y}-\mathbf{x})\big(\Delta^2 - \kappa^2\Delta\big)[u](\mathbf{y})\,dV = 0. \qquad (3.11)
\end{aligned}$$

Here we take the limits for $r \to 0$. As regards the weak singularity of $H_4(\kappa, \mathbf{y}-\mathbf{x})$, it follows that

$$\begin{aligned}&\lim_{r\to 0}\int_{\Omega\setminus B(\mathbf{x},r)} H_4(\kappa,\mathbf{y}-\mathbf{x})\left(\Delta^2-\kappa^2\Delta\right)[u](\mathbf{y})\,dV\\&\quad=\int_{\Omega} H_4(\kappa,\mathbf{y}-\mathbf{x})\left(\Delta^2-\kappa^2\Delta\right)[u](\mathbf{y})\,dV.\end{aligned}\tag{3.12}$$

Furthermore we write

$$\begin{aligned}&\sum_{i=1}^{3}(-1)^{i-1}\int_{\partial(\Omega\setminus B(\mathbf{x},r))} H_i(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y})\\&\qquad-\int_{\partial(\Omega\setminus B(\mathbf{x},r))} H_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y})\\&\quad=\sum_{i=1}^{3}(-1)^{i-1}\int_{\partial\Omega} H_i(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y})\\&\qquad-\int_{\partial\Omega} H_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y})\\&\qquad-\sum_{i=1}^{3}(-1)^{i-1}\int_{\partial B(\mathbf{x},r)} H_i(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y})\\&\qquad+\int_{\partial B(\mathbf{x},r)} H_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y}).\end{aligned}\tag{3.13}$$

We denote

$$\begin{aligned}I_2 &\triangleq -\sum_{i=1}^{3}(-1)^{i-1}\int_{\partial B(\mathbf{x},r)} H_i(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y})\\&\qquad+\int_{\partial B(\mathbf{x},r)} H_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{y}L_{-\kappa}\Delta[u](\mathbf{y}).\end{aligned}$$

Applying the Stokes formula and the Lebesgue differentiation theorem, we have

$$\lim_{r\to 0} I_2 = -u(\mathbf{x}).\tag{3.14}$$

Combining (3.11) with (3.12)-(3.14), we get the desired result. □

Theorem 3.6 *Suppose that Ω is an open bounded non-empty subset of $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$, $u \in C^4(\Omega, \mathrm{Cl}(V_{3,3})) \cap C^3(\overline{\Omega}, \mathrm{Cl}(V_{3,3}))$. Then*

$$\begin{aligned}&\int_{\partial\Omega} E_1(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}u(\mathbf{y})-\int_{\partial\Omega} E_2(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{-\kappa}[u](\mathbf{y})\\&\qquad+\int_{\partial\Omega} E_3(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{\kappa}L_{-\kappa}[u](\mathbf{y})\\&\qquad-\int_{\partial\Omega} E_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}DL_{\kappa}L_{-\kappa}[u](\mathbf{y})\end{aligned}$$

$$+\int_{\Omega} E_4(\kappa,\mathbf{y}-\mathbf{x})\Delta\big(\Delta-\kappa^2\big)[u](\mathbf{y})\,dV$$
$$=\begin{cases} u(\mathbf{x}), & \mathbf{x}\in\Omega,\\ 0, & \mathbf{x}\in\mathbb{R}^3\setminus\overline{\Omega},\end{cases} \tag{3.15}$$

where $E_i(\kappa,\mathbf{y}-\mathbf{x})$ $(i=1,2,3,4)$ *are as in* (3.4).

Proof The result can be similarly proved to Theorem 3.5. □

Applying Theorems 3.5 and 3.6, we directly have the following results.

Theorem 3.7 *Suppose that* Ω *is an open bounded non-empty subset of* $\mathbb{R}^3$ *with a Lyapunov boundary* $\partial\Omega$, $u\in C^4(\Omega,\mathrm{Cl}(V_{3,3}))\cap C^3(\overline{\Omega},\mathrm{Cl}(V_{3,3}))$, *and* $(\Delta^2-\kappa^2\Delta)[u]=0$ *in* Ω. *Then*

$$\sum_{i=1}^{3}(-1)^{i-1}\int_{\partial\Omega} H_i(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y})$$
$$-\int_{\partial\Omega} H_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y})$$
$$=\begin{cases} u(\mathbf{x}), & \mathbf{x}\in\Omega,\\ 0, & \mathbf{x}\in\mathbb{R}^3\setminus\overline{\Omega},\end{cases} \tag{3.16}$$

where $H_i(\kappa,\mathbf{y}-\mathbf{x})$ $(i=1,2,3,4)$ *are as in* (3.3).

Theorem 3.8 *Suppose that* Ω *is an open bounded non-empty subset of* $\mathbb{R}^3$ *with a Lyapunov boundary* $\partial\Omega$, $u\in C^4(\Omega,\mathrm{Cl}(V_{3,3}))\cap C^3(\overline{\Omega},\mathrm{Cl}(V_{3,3}))$, *and* $(\Delta^2-\kappa^2\Delta)[u]=0$ *in* Ω. *Then*

$$\int_{\partial\Omega} E_1(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}u(\mathbf{y})-\int_{\partial\Omega} E_2(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{-\kappa}[u](\mathbf{y})$$
$$+\int_{\partial\Omega} E_3(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{\kappa}L_{-\kappa}[u](\mathbf{y})$$
$$-\int_{\partial\Omega} E_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}DL_{\kappa}L_{-\kappa}[u](\mathbf{y})$$
$$=\begin{cases} u(\mathbf{x}), & \mathbf{x}\in\Omega,\\ 0, & \mathbf{x}\in\mathbb{R}^3\setminus\overline{\Omega},\end{cases} \tag{3.17}$$

where $E_i(\kappa,\mathbf{y}-\mathbf{x})$ $(i=1,2,3,4)$ *are as in* (3.4).

In this article, as usual dS denotes the Lebesgue surface measure. Using Theorem 3.7 or 3.8, we have the following result.

Corollary 3.9 *Suppose that* $(\Delta^2-\kappa^2\Delta)[u](\mathbf{x})=0$ *in* $\mathbb{R}^3$. *Then*

$$u(\mathbf{x})=\frac{1+\kappa R}{4\pi R^2(1+\kappa R+\frac{\kappa^2R^2}{3})}\int_{\partial B(\mathbf{x},R)}u(\mathbf{y})\,dS$$
$$+\frac{\kappa^2}{4\pi R(1+\kappa R+\frac{\kappa^2R^2}{3})}\int_{B(\mathbf{x},R)}u(\mathbf{y})\,dV+\frac{\frac{1}{\kappa^2}+\frac{R}{\kappa}+\frac{R^2}{3}-\frac{e^{\kappa R}}{\kappa^2}}{1+\kappa R+\frac{\kappa^2R^2}{3}}\Delta[u](\mathbf{x}).$$

Proof For arbitrary $\mathbf{x} \in \mathbb{R}^3$, Theorem 3.7 and Stokes' formula imply

$$\begin{aligned} u(\mathbf{x}) &= \frac{1}{4\pi R^2} \int_{\partial B(\mathbf{x},R)} u(\mathbf{y})\,dS + \frac{1}{4\pi R} \int_{B(\mathbf{x},R)} \Delta[u](\mathbf{y})\,dV \\ &\quad + \left(\frac{e^{-\kappa R} - 1}{4\pi\kappa^2 R^2} + \frac{e^{-\kappa R}}{4\pi\kappa R} \right) \int_{\partial B(\mathbf{x},R)} \Delta[u](\mathbf{y})\,dS \\ &\quad + \frac{e^{-\kappa R} - 1}{4\pi\kappa^2 R} \int_{B(\mathbf{x},R)} \Delta^2[u](\mathbf{y})\,dV. \end{aligned} \tag{3.18}$$

Using the mean value formula for harmonic functions and the condition $(\Delta^2 - \kappa^2\Delta)[u] = 0$, from (3.18), we have

$$\begin{aligned} u(\mathbf{x}) &= e^{-\kappa R} \frac{1 + \kappa R}{4\pi R^2} \int_{\partial B(\mathbf{x},R)} u(\mathbf{y})\,dS + e^{-\kappa R} \frac{\kappa^2}{4\pi R} \int_{B(\mathbf{x},R)} u(\mathbf{y})\,dV \\ &\quad + \left(\frac{1 - e^{\kappa R}}{\kappa^2} + \frac{R}{\kappa} + \frac{R^2}{3} \right) e^{-\kappa R} \Delta[u](\mathbf{x}) \\ &\quad - \left(1 - e^{\kappa R} + \kappa R + \frac{R^2\kappa^2}{3} \right) e^{-\kappa R} u(\mathbf{x}). \end{aligned} \tag{3.19}$$

Thus the result follows. □

Theorem 3.10 *Suppose that* $(\Delta^2 - \kappa^2\Delta)[u] = 0$ *in* $\mathbb{R}^3$ *and* $\lim_{r\to\infty} \frac{\Lambda(r,u)}{r^m} = l < \infty$, $m \in \mathbf{N}^*$. *Then*

$$\begin{cases} \liminf_{r\to\infty} \frac{M(r,D[u])}{r^{m-1}} < \infty, \\ \Delta[u](\infty) = 0, \\ L_{\pm\kappa} \Delta[u](\infty) = 0. \end{cases} \tag{3.20}$$

Proof For arbitrary $\mathbf{x} \in \mathbb{R}^3$, by Corollary 3.9, we have

$$\begin{aligned} \Delta[u](\mathbf{x}) &= \frac{1 + \kappa R + \frac{\kappa^2 R^2}{3}}{\frac{1}{\kappa^2} + \frac{R}{\kappa} + \frac{R^2}{3} - \frac{e^{\kappa R}}{\kappa^2}} u(\mathbf{x}) \\ &\quad - \frac{(1 + \kappa R)}{4\pi R^2 (\frac{1}{\kappa^2} + \frac{R}{\kappa} + \frac{R^2}{3} - \frac{e^{\kappa R}}{\kappa^2})} \int_{\partial B(\mathbf{x},R)} u(\mathbf{y})\,dS \\ &\quad - \frac{\kappa^2}{4\pi R (\frac{1}{\kappa^2} + \frac{R}{\kappa} + \frac{R^2}{3} - \frac{e^{\kappa R}}{\kappa^2})} \int_{B(\mathbf{x},R)} u(\mathbf{y})\,dV. \end{aligned} \tag{3.21}$$

Taking $R = \|\mathbf{x}\|$ in (3.21), we obtain

$$\begin{aligned} \big\| \Delta[u](\mathbf{x}) \big\| &\le \left| \frac{1 + \kappa\|\mathbf{x}\| + \frac{\kappa^2\|\mathbf{x}\|^2}{3}}{\frac{1}{\kappa^2} + \frac{\|\mathbf{x}\|}{\kappa} + \frac{\|\mathbf{x}\|^2}{3} - \frac{e^{\kappa\|\mathbf{x}\|}}{\kappa^2}} \right| \big\| u(\mathbf{x}) \big\| \\ &\quad + \left| \frac{1 + \kappa\|\mathbf{x}\|}{\frac{1}{\kappa^2} + \frac{\|\mathbf{x}\|}{\kappa} + \frac{\|\mathbf{x}\|^2}{3} - \frac{e^{\kappa\|\mathbf{x}\|}}{\kappa^2}} \right| \max_{\|\mathbf{y}\|\le 2\|\mathbf{x}\|} \big\{ \big\| u(\mathbf{y}) \big\| \big\} \\ &\quad + \left| \frac{\kappa^2\|\mathbf{x}\|^2}{\frac{3}{\kappa^2} + \frac{3\|\mathbf{x}\|}{\kappa} + \|\mathbf{x}\|^2 - 3\frac{e^{\kappa\|\mathbf{x}\|}}{\kappa^2}} \right| \max_{\|\mathbf{y}\|\le 2\|\mathbf{x}\|} \big\{ \big\| u(\mathbf{y}) \big\| \big\}. \end{aligned} \tag{3.22}$$

Denoting $\|\mathbf{x}\| = r$, we get from (3.22)

$$\begin{aligned}\max_{\|\mathbf{x}\|=r}\{\|\Delta[u](\mathbf{x})\|\} \leq{}& \left|\frac{1+\kappa r+\frac{\kappa^2r^2}{3}}{\frac{1}{\kappa^2}+\frac{r}{\kappa}+\frac{r^2}{3}-\frac{e^{r\kappa}}{\kappa^2}}\right| \max_{\|\mathbf{x}\|=r}\{\|u(\mathbf{x})\|\} \\ &+ \left|\frac{1+r\kappa}{\frac{1}{\kappa^2}+\frac{r}{\kappa}+\frac{r^2}{3}-\frac{e^{r\kappa}}{\kappa^2}}\right| \max_{\|\mathbf{y}\|\leq 2r}\{\|u(\mathbf{y})\|\} \\ &+ \left|\frac{\kappa^2r^2}{\frac{3}{\kappa^2}+\frac{3r}{\kappa}+r^2-3\frac{e^{r\kappa}}{\kappa^2}}\right| \max_{\|\mathbf{y}\|\leq 2r}\{\|u(\mathbf{y})\|\}.\end{aligned} \tag{3.23}$$

The inequality (3.23) can be rewritten as

$$\begin{aligned}M\big(r,\Delta[u]\big) \leq{}& \left|\frac{r^m+\kappa r^{m+1}+\frac{\kappa^2r^{m+2}}{3}}{\frac{1}{\kappa^2}+\frac{r}{\kappa}+\frac{r^2}{3}-\frac{e^{r\kappa}}{\kappa^2}}\right| \frac{M(r,u)}{r^m} \\ &+ \left|\frac{r^m+r^{m+1}\kappa}{\frac{1}{\kappa^2}+\frac{r}{\kappa}+\frac{r^2}{3}-\frac{e^{r\kappa}}{\kappa^2}}\right| \frac{\Lambda(2r,u)}{r^m} \\ &+ \left|\frac{\kappa^2r^{m+2}}{\frac{3}{\kappa^2}+\frac{3r}{\kappa}+r^2-3\frac{e^{r\kappa}}{\kappa^2}}\right| \frac{\Lambda(2r,u)}{r^m}.\end{aligned} \tag{3.24}$$

In view of (3.23) and $\lim_{r\to\infty}\frac{\Lambda(r,u)}{r^m} = l < \infty$, when $r \to \infty$, we get $\Delta[u](\infty) = 0$.

Using Lemma 3.13 in [19] and $(\Delta - \kappa^2)\Delta[u](\mathbf{x}) = 0$, we obtain

$$L_{-\kappa}\Delta[u](\infty) = 0 \tag{3.25}$$

and

$$L_{\kappa}\Delta[u](\infty) = 0. \tag{3.26}$$

From (3.25) and (3.26), we further obtain $D^3[u](\infty) = 0$.

We finally verify the remaining of the result of the theorem. For all $\mathbf{x} \in \mathbb{R}^3$, from Theorem 3.7, Lemma 3.2, Remark 3.3, and Stokes' formula it follows that

$$\begin{aligned}D[u](\mathbf{x}) ={}& \frac{1}{4\pi}\int_{\partial B(\mathbf{x},R)} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}\,d\sigma_{\mathbf{y}}D[u](\mathbf{y}) \\ &+ \frac{1}{4\pi}\int_{\partial B(\mathbf{x},R)} \frac{1}{\|\mathbf{y}-\mathbf{x}\|}e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\,d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y}) \\ &+ \frac{1}{4\pi\kappa^2}\int_{\partial B(\mathbf{x},R)} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}\big(e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}-1\big)\,d\sigma_{\mathbf{y}}D^3[u](\mathbf{y}) \\ &+ \frac{1}{4\pi\kappa}\int_{\partial B(\mathbf{x},R)} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2}e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\,d\sigma_{\mathbf{y}}D^3[u](\mathbf{y}) \\ ={}& \frac{3}{4\pi R^3}\int_{\partial B(\mathbf{x},R)} d\sigma_{\mathbf{y}}u(\mathbf{y}) + \frac{1}{4\pi R^3}\int_{B(\mathbf{x},R)} (\mathbf{y}-\mathbf{x})\Delta[u](\mathbf{y})\,dV \\ &+ \frac{e^{-\kappa R}}{4\pi R}\int_{\partial B(\mathbf{x},R)} d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y}) \\ &+ \frac{3(e^{-\kappa R}-1)}{4\pi\kappa^2R^3}\int_{\partial B(\mathbf{x},R)} d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y})\end{aligned}$$

$$
\begin{aligned}
&+ \frac{e^{-\kappa R}-1}{4\pi R^3}\int_{B(\mathbf{x},R)} (\mathbf{y}-\mathbf{x})\Delta[u](\mathbf{y})\,dV \\
&+ \frac{3e^{-\kappa R}}{4\pi\kappa R^2}\int_{\partial B(\mathbf{x},R)} d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y}) \\
&+ \frac{\kappa e^{-\kappa R}}{4\pi R^2}\int_{B(\mathbf{x},R)} (\mathbf{y}-\mathbf{x})\Delta[u](\mathbf{y})\,dV \\
={}& \frac{3}{4\pi R^3}\int_{\partial B(\mathbf{x},R)} d\sigma_{\mathbf{y}}u(\mathbf{y}) + \left(\frac{1+R\kappa}{4\pi R^3}\right)e^{-\kappa R}\int_{B(\mathbf{x},R)} (\mathbf{y}-\mathbf{x})\Delta[u](\mathbf{y})\,dV \\
&+ \left[\frac{e^{-\kappa R}}{4\pi R} + \frac{3(e^{-\kappa R}-1)}{4\pi\kappa^2 R^3} + \frac{3e^{-\kappa R}}{4\pi\kappa R^2}\right]\int_{\partial B(\mathbf{x},R)} d\sigma_{\mathbf{y}}\Delta[u](\mathbf{y}).
\end{aligned}
\tag{3.27}
$$

Taking $R = \|\mathbf{x}\|$ in (3.27), in view of the maximum principle of the modified Helmholtz equation in [7, 18], it immediately follows that

$$
\begin{aligned}
\big\|D[u](\mathbf{x})\big\| \le{}& 24\frac{\Lambda(2\|\mathbf{x}\|,u)}{\|\mathbf{x}\|} + \frac{e^{-\kappa\|\mathbf{x}\|}}{3}\|\mathbf{x}\|M\big(2\|\mathbf{x}\|,\Delta[u]\big) \\
&+ \kappa\frac{e^{-\kappa\|\mathbf{x}\|}}{3}\|\mathbf{x}\|^2M\big(2\|\mathbf{x}\|,\Delta[u]\big) + e^{-\kappa\|\mathbf{x}\|}\|\mathbf{x}\|M\big(2\|\mathbf{x}\|,\Delta[u]\big) \\
&+ \frac{3(1-e^{-\kappa\|\mathbf{x}\|})}{\kappa^2\|\mathbf{x}\|}M\big(2\|\mathbf{x}\|,\Delta[u]\big) + \frac{3e^{-\kappa\|\mathbf{x}\|}}{\kappa}M\big(2\|\mathbf{x}\|,\Delta[u]\big).
\end{aligned}
\tag{3.28}
$$

Denoting $\|\mathbf{x}\| = r$, we conclude from (3.28)

$$
\begin{aligned}
M\big(r,D[u]\big) \le{}& 24\frac{\Lambda(2r,u)}{r} + e^{-\kappa r}\left(\frac{4r}{3} + \frac{\kappa r^2}{3} + \frac{3}{\kappa}\right)M\big(2r,\Delta[u]\big) \\
&+ \frac{3(1-e^{-\kappa r})}{\kappa^2 r}M\big(2r,\Delta[u]\big).
\end{aligned}
\tag{3.29}
$$

Then by (3.29), we have

$$
\begin{aligned}
\frac{M(r,D[u])}{r^{m-1}} \le{}& 24\cdot\frac{\Lambda(2r,u)}{r^m} + \frac{e^{-\kappa r}}{r^{m-1}}\left(\frac{4r}{3} + \frac{\kappa r^2}{3} + \frac{3}{\kappa}\right)M\big(2r,\Delta[u]\big) \\
&+ \frac{3(1-e^{-\kappa r})}{\kappa^2 r^m}M\big(2r,\Delta[u]\big).
\end{aligned}
\tag{3.30}
$$

Applying the maximum principle of the modified Helmholtz equation and the condition $\lim_{r\to\infty}\frac{\Lambda(r,u)}{r^m} = l < \infty$, we have $\liminf_{r\to\infty}\frac{M(r,D[u])}{r^{m-1}} < \infty$. The proof is completed. □

Next, denote some integral operators as follows:

$$
\begin{aligned}
(\mathcal{F}u)(\mathbf{x}) \triangleq{}& \sum_{i=1}^{3}(-1)^{i-1}\int_{\partial\Omega} H_i(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y}) \\
&- \int_{\partial\Omega} H_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y}), \quad \mathbf{x}\in\mathbb{R}^3\setminus\partial\Omega,
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
(\mathcal{S}u)(\mathbf{x}) \triangleq{}& \sum_{i=1}^{3}(-1)^{i-1}\int_{\partial\Omega} H_i(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y}) \\
&- \int_{\partial\Omega} H_4(\kappa,\mathbf{y}-\mathbf{x})\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y}), \quad \mathbf{x}\in\partial\Omega,
\end{aligned}
\tag{3.32}
$$

where $H_i(\kappa, \mathbf{y} - \mathbf{x})$ ($i = 1, 2, 3, 4$) are as in (3.3) and the above singular integral is taken in the principal sense.

Lemma 3.11 *Let Ω be an open, bounded non-empty subset of $\mathbb{R}^3$ with Lyapunov boundary $\partial\Omega$, $u(\mathbf{x}) \in H^\alpha(\partial\Omega, \mathrm{Cl}(V_{3,3}))$, $0 < \alpha \leq 1$. Then, for $\mathbf{x} \in \partial\Omega$,*

$$\begin{cases} \lim_{\substack{\mathbf{x}\to\mathbf{x}_0\in\partial\Omega \\ \mathbf{x}\in\Omega}} (\mathcal{F}u)(\mathbf{x}) = \frac{u(\mathbf{x}_0)}{2} + (\mathcal{S}u)(\mathbf{x}_0), \\ \lim_{\substack{\mathbf{x}\to\mathbf{x}_0\in\partial\Omega \\ \mathbf{x}\in\mathbb{R}^3\setminus\overline{\Omega}}} (\mathcal{F}u)(\mathbf{x}) = -\frac{u(\mathbf{x}_0)}{2} + (\mathcal{S}u)(\mathbf{x}_0). \end{cases} \tag{3.33}$$

Proof Applying the Plemelj formulas with parameter (Theorem 2.3 in [10]), the result follows. □

In the following, we denote

$$u^{\pm}(\mathbf{x}) = \lim_{\substack{\mathbf{y}\to\mathbf{x}\in\partial\Omega \\ \mathbf{x}\in\Omega^{\pm}}} u(\mathbf{y}), \tag{3.34}$$

where $\Omega = \Omega^+$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$.

Theorem 3.12 *Assume that Ω is an open, bounded non-empty subset of $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$, $u \in C^4(\Omega, \mathrm{Cl}(V_{3,3})) \cap C^3(\overline{\Omega}, \mathrm{Cl}(V_{3,3}))$, $u \in C^4(\Omega^-, \mathrm{Cl}(V_{3,3})) \cap C^3(\overline{\Omega}^-, \mathrm{Cl}(V_{3,3}))$ and $u(\mathbf{x})$ satisfies the following conditions*:

$$\begin{cases} (\Delta^2 - \kappa^2\Delta)[u](\mathbf{x}) = 0, \quad \textit{in } \mathbb{R}^3 \setminus \partial\Omega, \\ u^+(\mathbf{x}) = u^-(\mathbf{x}) \in H^{\alpha_1}(\partial\Omega, \mathrm{Cl}(V_{3,3})), \quad \forall \mathbf{x} \in \partial\Omega, \\ D[u]^+(\mathbf{x}) = D[u]^-(\mathbf{x}) \in H^{\alpha_2}(\partial\Omega, \mathrm{Cl}(V_{3,3})), \quad \forall \mathbf{x} \in \partial\Omega, \\ \Delta[u]^+(\mathbf{x}) = \Delta[u]^-(\mathbf{x}) \in H^{\alpha_3}(\partial\Omega, \mathrm{Cl}(V_{3,3})), \quad \forall \mathbf{x} \in \partial\Omega, \\ L_{-\kappa}\Delta[u]^+(\mathbf{x}) = L_{-\kappa}\Delta[u]^-(\mathbf{x}) \in H^{\alpha_4}(\partial\Omega, \mathrm{Cl}(V_{3,3})), \quad \forall \mathbf{x} \in \partial\Omega, \end{cases} \tag{3.35}$$

where $0 < \alpha_i \leq 1$, $i = 1, 2, 3, 4$, then $(\Delta^2 - \kappa^2\Delta)[u] = 0$ in $\mathbb{R}^3$.

Proof We only need to prove that for $\forall \mathbf{x}_0 \in \partial\Omega$, $(\Delta^2 - \kappa^2\Delta)[u] = 0$. Taking a constant $r > 0$, $B(\mathbf{x}_0, r)$ is an open ball with the center at $\mathbf{x}_0$ and radius r such that $\Omega \subset B(\mathbf{x}_0, r)$. It is clear that $\partial\Omega \cup \partial B(\mathbf{x}_0, r)$ is a Lyapunov boundary.

Let

$$\begin{cases} u(\mathbf{x}) = u^+(\mathbf{x}) = u^-(\mathbf{x}), \\ D[u](\mathbf{x}) = D[u]^+(\mathbf{x}) = D[u]^-(\mathbf{x}), \\ \Delta[u](\mathbf{x}) = \Delta[u]^+(\mathbf{x}) = \Delta[u]^-(\mathbf{x}), \\ L_{-\kappa}\Delta[u](\mathbf{x}) = L_{-\kappa}\Delta[u]^+(\mathbf{x}) = L_{-\kappa}\Delta[u]^-(\mathbf{x}), \end{cases}$$

here $\mathbf{x} \in \partial\Omega$. In view of Theorem 3.7, it follows that

$$\begin{aligned} u(\mathbf{x}_1) = {} & \sum_{i=1}^{3} (-1)^{i-1} \int_{\partial\Omega} H_i(\kappa, \mathbf{y} - \mathbf{x}_1)\, d\sigma_{\mathbf{y}} D^{i-1}[u](\mathbf{y}) \\ & - \int_{\partial\Omega} H_4(\kappa, \mathbf{y} - \mathbf{x}_1)\, d\sigma_{\mathbf{y}} L_{-\kappa}\Delta[u](\mathbf{y}), \quad \mathbf{x}_1 \in \Omega, \end{aligned} \tag{3.36}$$

$$\begin{aligned} u(\mathbf{x}_2) &= \sum_{i=1}^{3}(-1)^{i-1}\int_{\partial\Omega\cup\partial B(\mathbf{x}_0,r)} H_i(\kappa,\mathbf{y}-\mathbf{x}_2)\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y}) \\ &\quad - \int_{\partial\Omega\cup\partial B(\mathbf{x}_0,r)} H_4(\kappa,\mathbf{y}-\mathbf{x}_2)\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y}), \\ &\quad \mathbf{x}_2 \in B(\mathbf{x}_0,r)\setminus\overline{\Omega}. \end{aligned} \tag{3.37}$$

Combining (3.36) with (3.37) and using Lemma 3.11, we obtain

$$\begin{aligned} u^+(\mathbf{x}_0) &= \lim_{\mathbf{x}_1\to\mathbf{x}_0} u(\mathbf{x}_1) \\ &= \frac{u(\mathbf{x}_0)}{2} + \sum_{i=1}^{3}(-1)^{i-1}\int_{\partial\Omega} H_i(\kappa,\mathbf{y}-\mathbf{x}_0)\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y}) \\ &\quad - \int_{\partial\Omega} H_4(\kappa,\mathbf{y}-\mathbf{x}_0)\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y}), \end{aligned} \tag{3.38}$$

$$\begin{aligned} u^-(\mathbf{x}_0) &= \lim_{\mathbf{x}_2\to\mathbf{x}_0} u(\mathbf{x}_0) \\ &= \frac{u(\mathbf{x}_0)}{2} + \sum_{i=1}^{3}(-1)^{i-1}\int_{\partial\Omega\cup\partial B(\mathbf{x}_0,r)} H_i(\kappa,\mathbf{y}-\mathbf{x}_0)\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y}) \\ &\quad - \int_{\partial\Omega\cup\partial B(\mathbf{x}_0,r)} H_4(\kappa,\mathbf{y}-\mathbf{x}_0)\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y}). \end{aligned} \tag{3.39}$$

From (3.38) and (3.39), we derive

$$\begin{aligned} u(\mathbf{x}_0) &= \sum_{i=1}^{3}(-1)^{i-1}\int_{\partial B(\mathbf{x}_0,r)} H_i(\kappa,\mathbf{y}-\mathbf{x}_0)\,d\sigma_{\mathbf{y}}D^{i-1}[u](\mathbf{y}) \\ &\quad - \int_{\partial B(\mathbf{x}_0,r)} H_4(\kappa,\mathbf{y}-\mathbf{x}_0)\,d\sigma_{\mathbf{y}}L_{-\kappa}\Delta[u](\mathbf{y}). \end{aligned} \tag{3.40}$$

Therefore $\Delta(\Delta-\kappa^2)[u](\mathbf{x}_0)=0$, the result follows. □

Theorem 3.13 *Let Ω be an open bounded non-empty subset of $\mathbb{R}^3$ with Lyapunov boundary $\partial\Omega$, $u\in C^4(\Omega^-,\mathrm{Cl}(V_{3,3}))\cap C^3(\overline{\Omega}^-,\mathrm{Cl}(V_{3,3}))$, $(\Delta^2-\kappa^2\Delta)[u]=0$ in Ω^-, and*

$$\begin{cases} u(\mathbf{x})\in H^{\alpha_1}(\partial\Omega,\mathrm{Cl}(V_{3,3})), \\ D[u](\mathbf{x})\in H^{\alpha_2}(\partial\Omega,\mathrm{Cl}(V_{3,3})), \\ \Delta[u](\mathbf{x})\in H^{\alpha_3}(\partial\Omega,\mathrm{Cl}(V_{3,3})), \\ L_{-\kappa}\Delta[u](\mathbf{x})\in H^{\alpha_4}(\partial\Omega,\mathrm{Cl}(V_{3,3})), \\ \lim_{r\to\infty}\frac{\Delta(r,u)}{r^m}=l<\infty, \quad m\in\mathbf{N}^*, \end{cases}$$

where $0<\alpha_i\leq 1$, $i=1,2,3,4$. Then

$$\begin{cases} \liminf_{r\to\infty}\frac{M(r,D[u])}{r^{m-1}}<\infty, \\ \Delta[u](\infty)=0, \\ L_{-\kappa}[u]\Delta(\infty)=0, \qquad L_{\kappa}\Delta[u](\infty)=0. \end{cases}$$

Proof For $\mathbf{y} \in \partial\Omega$, let

$$\begin{cases} u(\mathbf{y}) = -f_1(\mathbf{y}), \\ D[u](\mathbf{y}) = -f_2(\mathbf{y}), \\ \Delta[u](\mathbf{y}) = -f_3(\mathbf{y}), \\ L_{-\kappa}\Delta[u](\mathbf{y}) = -f_4(\mathbf{y}). \end{cases}$$

For $\mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega$, we get

$$F(\mathbf{x}) = \sum_{i=1}^{4}(-1)^{i-1}\int_{\partial\Omega} H_i(\kappa, \mathbf{y}-\mathbf{x})\, d\sigma_{\mathbf{y}} f_i(\mathbf{y})$$

and

$$\widetilde{F}(\mathbf{x}) = \begin{cases} -F(\mathbf{x}) & \mathbf{x} \in \Omega^+, \\ u(\mathbf{x}) - F(\mathbf{x}), & \mathbf{x} \in \Omega^-, \end{cases}$$

in view of Lemma 3.2, it is easy to check that $(\Delta^2 - \kappa^2\Delta)[\widetilde{F}] = 0$ in $\mathbb{R}^3 \setminus \partial\Omega$, combining Lemma 3.11, we get

$$\begin{cases} [\widetilde{F}]^+(\mathbf{x}) = [\widetilde{F}]^-(\mathbf{x}) \in H^{\tilde{\alpha}_1}(\partial\Omega, \mathrm{Cl}(V_{3,3})), \\ D[\widetilde{F}]^+(\mathbf{x}) = D[\widetilde{F}]^-(\mathbf{x}) \in H^{\tilde{\alpha}_2}(\partial\Omega, \mathrm{Cl}(V_{3,3})), \\ \Delta[\widetilde{F}]^+(\mathbf{x}) = \Delta[\widetilde{F}]^-(\mathbf{x}) \in H^{\tilde{\alpha}_3}(\partial\Omega, \mathrm{Cl}(V_{3,3})), \\ L_{-\kappa}\Delta[\widetilde{F}]^+(\mathbf{x}) = L_{-\kappa}\Delta[\widetilde{F}]^-(\mathbf{x}) \in H^{\tilde{\alpha}_4}(\partial\Omega, \mathrm{Cl}(V_{3,3})). \end{cases}$$

Thus $(\Delta^2 - \kappa^2\Delta)[\widetilde{F}] = 0$ in $\mathbb{R}^3$ where we use Theorem 3.12. Obviously, $\lim_{r\to\infty}\frac{\Delta(r,\widetilde{F}(\mathbf{x}))}{r^m} = l < \infty$, using Theorem 3.10, we arrive at

$$\begin{cases} \liminf_{r\to\infty}\frac{M(r,D[\widetilde{F}])}{r^{m-1}} < \infty, \\ \Delta[\widetilde{F}](\infty) = 0, \\ L_{-\kappa}\Delta[\widetilde{F}](\infty) = 0, \qquad L_{\kappa}\Delta[\widetilde{F}](\infty) = 0. \end{cases} \tag{3.41}$$

For $\mathbf{x} \in \Omega^-$,

$$\begin{cases} D[u](\mathbf{x}) = D[\widetilde{F}](\mathbf{x}) \\ \qquad + \frac{1}{4\pi}\int_{\partial\Omega}\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}\, d\sigma_{\mathbf{y}} f_2(\mathbf{y}) \\ \qquad + \frac{1}{4\pi\kappa}\int_{\partial\Omega}[(\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \kappa\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} + \kappa\frac{1}{\|\mathbf{y}-\mathbf{x}\|})e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}]\, d\sigma_{\mathbf{y}} f_3(\mathbf{y}) \\ \qquad + \frac{1}{4\pi\kappa^2}\int_{\partial\Omega}[\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}(e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - 1) + \kappa\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2}e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}]\, d\sigma_{\mathbf{y}} f_4(\mathbf{y}), \\ \Delta[u](\mathbf{x}) = \Delta[\widetilde{F}](\mathbf{x}) \\ \qquad + \frac{1}{4\pi}\int_{\partial\Omega}[\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \kappa\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} + \kappa\frac{1}{\|\mathbf{y}-\mathbf{x}\|}]e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} f_3(\mathbf{y}) \\ \qquad + \frac{1}{4\pi}\int_{\partial\Omega}\frac{e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}}{\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} f_4(\mathbf{y}), \\ L_{-\kappa}\Delta[u](\mathbf{x}) = L_{-\kappa}\Delta[\widetilde{F}](\mathbf{x}) \\ \qquad + \frac{1}{4\pi}\int_{\partial\Omega}[\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \kappa\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} + \kappa\frac{1}{\|\mathbf{y}-\mathbf{x}\|}]e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} f_4(\mathbf{y}). \end{cases} \tag{3.42}$$

Equations (3.41) and (3.42) imply that the result holds. □

4 Riemann type problem for the fourth-order elliptic equation

In this section we will find solutions to

$$\begin{cases} (\Delta^2 - \kappa^2 \Delta)[u] = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ u^+(\mathbf{x}) = u^-(\mathbf{x})A + g_1(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \\ D[u]^+(\mathbf{x}) = D[u]^-(\mathbf{x})B + g_2(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \\ \Delta[u]^+(\mathbf{x}) = \Delta[u]^-(\mathbf{x})C + g_3(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \\ L_\kappa \Delta[u]^+(\mathbf{x}) = L_\kappa \Delta[u]^-(\mathbf{x})D + g_4(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \\ \lim_{r\to\infty} \frac{\Delta(r,u)}{r^m} = l < \infty, & m \in \mathbf{N}^*, \end{cases} \tag{4.1}$$

where A, B, C, D are invertible Clifford constants and $g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x}), g_4(\mathbf{x}) \in H^\alpha(\partial\Omega, \mathrm{Cl}(V_{3,3}))$, $0 < \alpha \leq 1$, $\kappa > 0$.

Theorem 4.1 *The Riemann type problem* (4.1) *is solvable and the solution can be written as*

$$u(\mathbf{x}) = \sum_{i=1}^{4} u_i(\mathbf{x}), \tag{4.2}$$

where

$$u_1(\mathbf{x}) = \begin{cases} \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} \frac{e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}-1}{\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} g_4(\mathbf{y}), & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} \frac{e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}-1}{\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} g_4(\mathbf{y}) D^{-1}, & \mathbf{x} \in \Omega^-, \end{cases} \tag{4.3}$$

$$u_2(\mathbf{x}) = \begin{cases} \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} (\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} - \kappa \frac{1}{\|\mathbf{y}-\mathbf{x}\|})(e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - 1)\, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) & \\ \quad + \frac{1}{4\pi\kappa} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}), & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} (\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} - \kappa \frac{1}{\|\mathbf{y}-\mathbf{x}\|})(e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - 1)\, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) C^{-1} & \\ \quad + \frac{1}{4\pi\kappa} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) C^{-1}, & \mathbf{x} \in \Omega^-, \end{cases} \tag{4.4}$$

$$u_3(\mathbf{x}) = \begin{cases} \frac{1}{4\pi} \int_{\partial\Omega} \frac{1}{\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} \tilde{g}_2(\mathbf{y}) & \\ \quad + \sum_{p=1}^{m} \sum_{(l_1,\ldots,l_{p-1})} C_{1,p-1} \mathbf{x} V_{l_1,\ldots,l_{p-1}}(\mathbf{x}) C^1_{l_1,\ldots,l_{p-1}}, & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi} \int_{\partial\Omega} \frac{1}{\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} \tilde{g}_2(\mathbf{y}) B^{-1} & \\ \quad + \sum_{p=1}^{m} \sum_{(l_1,\ldots,l_{p-1})} C_{1,p-1} \mathbf{x} V_{l_1,\ldots,l_{p-1}}(\mathbf{x}) C^1_{l_1,\ldots,l_{p-1}} B^{-1}, & \mathbf{x} \in \Omega^-, \end{cases} \tag{4.5}$$

$$u_4(\mathbf{x}) = \begin{cases} \frac{1}{4\pi} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}\, d\sigma_{\mathbf{y}} \tilde{g}_1(\mathbf{y}) & \\ \quad + \sum_{p=0}^{m} \sum_{(l_1,\ldots,l_p)} V_{l_1,\ldots,l_p}(\mathbf{x}) C_{l_1,\ldots,l_p}, & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}\, d\sigma_{\mathbf{y}} \tilde{g}_1(\mathbf{y}) A^{-1} & \\ \quad + \sum_{p=0}^{m} \sum_{(l_1,\ldots,l_p)} V_{l_1,\ldots,l_p}(\mathbf{x}) C_{l_1,\ldots,l_p} A^{-1}, & \mathbf{x} \in \Omega^-, \end{cases} \tag{4.6}$$

and

$$\tilde{g}_3(\mathbf{x}) = g_3(\mathbf{x}) + \frac{1}{4\pi} \int_{\partial\Omega} \frac{e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}}{\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} g_4(\mathbf{y}) \bigl(-1 + D^{-1}C\bigr), \quad \mathbf{x} \in \partial\Omega, \tag{4.7}$$

$$\begin{aligned} \tilde{g}_2(\mathbf{x}) = {}& \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} \bigl(e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - 1\bigr)\, d\sigma_{\mathbf{y}} g_4(\mathbf{y}) \bigl(-1 + D^{-1}B\bigr) \\ & + \frac{1}{4\pi\kappa} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} g_4(\mathbf{y}) \bigl(-1 + D^{-1}B\bigr) \\ & - \frac{1}{4\pi\kappa} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) \bigl(-1 + C^{-1}B\bigr) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4\pi} \int_{\partial\Omega} \left(\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} - \frac{1}{\|\mathbf{y}-\mathbf{x}\|} \right) e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) \left(-1 + C^{-1}B\right) \\
&+ \frac{1}{4\pi\kappa} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} \, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) \left(-1 + C^{-1}B\right) \\
&+ g_2(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}_1(\mathbf{x}) = {}& \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} \frac{e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - 1}{\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} g_4(\mathbf{y}) \left(-1 + D^{-1}A\right) \\
&+ \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} \left(\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} - \kappa \frac{1}{\|\mathbf{y}-\mathbf{x}\|} \right) e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) \left(-1 + C^{-1}A\right) \\
&- \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} \left(\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} - \kappa \frac{1}{\|\mathbf{y}-\mathbf{x}\|} \right) d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) \left(-1 + C^{-1}A\right) \\
&+ \frac{1}{4\pi\kappa} \int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^2} e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} \tilde{g}_3(\mathbf{y}) \left(-1 + C^{-1}A\right) \\
&+ \frac{1}{4\pi} \int_{\partial\Omega} \frac{1}{\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} \tilde{g}_2(\mathbf{y}) \left(-1 + B^{-1}A\right) \\
&+ \sum_{p=1}^{m} \sum_{(l_1,\ldots,l_{p-1})} C_{1,p-1} \mathbf{x} V_{l_1,\ldots,l_{p-1}}(\mathbf{x}) C^1_{l_1,\ldots,l_{p-1}} \left(-1 + B^{-1}A\right) \\
&+ g_1(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \qquad (4.9)
\end{aligned}
$$

Proof Let $u(\mathbf{x})$ be the solution of (4.1) for $\mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega$, we denote $\omega(\mathbf{x}) = L_\kappa \Delta[u](\mathbf{x})$. Then

$$
\omega^+(\mathbf{x}) = \omega^-(\mathbf{x}) D + g_4(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \qquad (4.10)
$$

In view of Theorem 3.13, $(\Delta^2 - \kappa^2\Delta)[u](\mathbf{x}) = L_{-\kappa} L_\kappa \Delta[u](\mathbf{x}) = 0$, and $\lim_{r\to\infty} \frac{\Lambda(r,u)}{r^m} = l < \infty$, we have

$$
\omega(\mathbf{x}) = \begin{cases} \frac{1}{4\pi} \int_{\partial\Omega} \left[\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \frac{\kappa(\mathbf{y}-\mathbf{x})}{\|\mathbf{y}-\mathbf{x}\|^2} + \frac{\kappa}{\|\mathbf{y}-\mathbf{x}\|}\right] e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} g_4(\mathbf{y}), & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi} \int_{\partial\Omega} \left[\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \frac{\kappa(\mathbf{y}-\mathbf{x})}{\|\mathbf{y}-\mathbf{x}\|^2} + \frac{\kappa}{\|\mathbf{y}-\mathbf{x}\|}\right] e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} g_4(\mathbf{y}) D^{-1}, & \mathbf{x} \in \Omega^-. \end{cases}
$$

Let

$$
u_1(\mathbf{x}) = \begin{cases} \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} \frac{e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - 1}{\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} g_4(\mathbf{y}), & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi\kappa^2} \int_{\partial\Omega} \frac{e^{-\kappa\|\mathbf{y}-\mathbf{x}\|} - 1}{\|\mathbf{y}-\mathbf{x}\|} \, d\sigma_{\mathbf{y}} g_4(\mathbf{y}) D^{-1}, & \mathbf{x} \in \Omega^-. \end{cases} \qquad (4.11)
$$

It is easy to check that

$$
L_\kappa \Delta[u - u_1](\mathbf{x}) = 0, \quad \text{in } \mathbb{R}^3 \setminus \partial\Omega. \qquad (4.12)
$$

If we denote

$$
\Delta[u](\mathbf{x}) - \Delta[u_1](\mathbf{x}) \triangleq \Theta(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega, \qquad (4.13)
$$

and use boundary value condition

$$
\Delta[u]^+(\mathbf{x}) = \Delta[u]^-(\mathbf{x}) C + g_3(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega,
$$

we conclude

$$\Theta^+(\mathbf{x}) = \Theta^-(\mathbf{x})C + \tilde{g}_3(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{4.14}$$

here $\tilde{g}_3(\mathbf{x}) \in H^{\tilde{\alpha}}(\partial\Omega, \mathrm{Cl}(V_{3,3}))$, $0 < \tilde{\alpha} \leq 1$ is taken from (4.7). It follows that $\Theta(\infty) = 0$ from Theorem 3.13. Using (4.12), we get the representation formula

$$\Theta(\mathbf{x}) = \begin{cases} \frac{1}{4\pi}\int_{\partial\Omega}[\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \frac{\kappa(\mathbf{y}-\mathbf{x})}{\|\mathbf{y}-\mathbf{x}\|^2} - \frac{\kappa}{\|\mathbf{y}-\mathbf{x}\|}]e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}}\tilde{g}_3(\mathbf{y}), & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi}\int_{\partial\Omega}[\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3} + \frac{\kappa(\mathbf{y}-\mathbf{x})}{\|\mathbf{y}-\mathbf{x}\|^2} - \frac{\kappa}{\|\mathbf{y}-\mathbf{x}\|}]e^{-\kappa\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}}\tilde{g}_3(\mathbf{y})C^{-1}, & \mathbf{x} \in \Omega^-. \end{cases}$$

Analogously, we find with $u_2(\mathbf{x})$ from (4.4) that

$$\Delta[u - u_1 - u_2](\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega. \tag{4.15}$$

Denoting $D[u] - D[u_1] - D[u_2] \triangleq \Xi$, where $\mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega$ and using the boundary value condition

$$D[u]^+(\mathbf{x}) = D[u]^-(\mathbf{x})B + g_2(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{4.16}$$

it follows that

$$\Xi^+(\mathbf{x}) = \Xi^-(\mathbf{x})B + \tilde{g}_2(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{4.17}$$

where $\tilde{g}_2(\mathbf{x}) \in H^{\tilde{\alpha}}(\partial\Omega, \mathrm{Cl}(V_{3,3}))$, $0 < \tilde{\alpha} \leq 1$ is taken from (4.8). Again applying Theorem 3.13, we have $\liminf_{r\to\infty} \frac{M(r,\Xi)}{r^{m-1}} < \infty$. In view of (4.17) and Lemma 2.2, we obtain

$$\Xi(\mathbf{x}) = \begin{cases} \frac{1}{4\pi}\int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}\, d\sigma_{\mathbf{y}}\tilde{g}_2(\mathbf{y}) \\ \quad + \sum_{p=1}^{m}\sum_{(l_1,\ldots,l_{p-1})} V_{l_1,\ldots,l_{p-1}}(\mathbf{x})C^1_{l_1,\ldots,l_{p-1}}, & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi}\int_{\partial\Omega} \frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|^3}\, d\sigma_{\mathbf{y}}\tilde{g}_2(\mathbf{y})B^{-1} \\ \quad + \sum_{p=1}^{m}\sum_{(l_1,\ldots,l_{p-1})} V_{l_1,\ldots,l_{p-1}}(\mathbf{x})C^1_{l_1,\ldots,l_{p-1}}B^{-1}, & \mathbf{x} \in \Omega^-. \end{cases} \tag{4.18}$$

Finally, we use

$$u_3(\mathbf{x}) = \begin{cases} \frac{1}{4\pi}\int_{\partial\Omega} \frac{1}{\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}}\tilde{g}_2(\mathbf{y}) \\ \quad + \sum_{p=1}^{m}\sum_{(l_1,\ldots,l_{p-1})} C_{1,p-1}\mathbf{x}V_{l_1,\ldots,l_{p-1}}(\mathbf{x})C^1_{l_1,\ldots,l_{p-1}}, & \mathbf{x} \in \Omega^+, \\ \frac{1}{4\pi}\int_{\partial\Omega} \frac{1}{\|\mathbf{y}-\mathbf{x}\|}\, d\sigma_{\mathbf{y}}\tilde{g}_2(\mathbf{y})B^{-1} \\ \quad + \sum_{p=1}^{m}\sum_{(l_1,\ldots,l_{p-1})} C_{1,p-1}\mathbf{x}V_{l_1,\ldots,l_{p-1}}(\mathbf{x})C^1_{l_1,\ldots,l_{p-1}}B^{-1}, & \mathbf{x} \in \Omega^-. \end{cases} \tag{4.19}$$

We arrive at

$$D[u - u_1 - u_2 - u_3] = 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega.$$

Defining

$$u(\mathbf{x}) - u_1(\mathbf{x}) - u_2(\mathbf{x}) - u_3(\mathbf{x}) \triangleq \Upsilon(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega. \tag{4.20}$$

According to the boundary value condition

$$u^{+}(\mathbf{x}) = u^{-}(\mathbf{x})A + g_1(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega,$$

we get

$$\Upsilon^{+}(\mathbf{x}) = \Upsilon^{-}(\mathbf{x})A + \tilde{g}_1(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{4.21}$$

where $\tilde{g}_1(\mathbf{x}) \in H^{\tilde{\alpha}}(\partial\Omega, \mathrm{Cl}(V_{3,3}))$, $0 < \tilde{\alpha} \leq 1$, is as in (4.7). It is clear that $\liminf_{r\to\infty} \frac{M(r,\Upsilon)}{r^m} < \infty$. Using Lemma 2.2, we get

$$\Upsilon(\mathbf{x}) = u_4(\mathbf{x}).$$

On the other hand, we can directly prove that (4.2) is the solution of (4.1). The proof is completed. □

Competing interests
The author declares that they have no competing interests.

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