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Existence,uniqueness, and nonexistence of solutions to nonlinear diffusion equations with $p(x,t)$-Laplacian operator

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Abstract

The aim of this paper is to deal with the existence and nonexistence of weak solutions to the initial and boundary value problem for $u_t = \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u + b(x,t)\nabla u) + f(u)$. By constructing suitable function spaces and applying the method of Galerkin's approximation as well as weak convergence techniques, the authors prove the existence of local solutions. Furthermore, we choose a suitable test-function, make integral estimates, and apply Gronwall's inequality to prove the uniqueness of weak solutions. At the end of this paper, the authors construct a suitable energy functional, obtain a new energy inequality, and apply a convex method to prove the nonexistence of solutions. Especially, it is worth pointing out that the results are obtained with the assumption that $p_t(x,t)$ is only negative and integrable, which is weaker than most of the other papers required.

Keywords: nonstandard growth condition; nonexistence of solutions; Galerkin's approximation

1 Introduction

Consider the following initial and boundary value problem:

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u + b(x,t)\nabla u) + f(u), & (x,t) \in \Omega \times (0,T) := Q_T, \\ u(x,t) = 0, & (x,t) = \partial\Omega \times (0,T) := \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain, $\partial\Omega$ is Lipschitz continuous, and f is a continuous function satisfying

$$|f(s)| \leq a_0|s|^{\alpha-1}, \quad 0 < a_0 = \text{constant}, 1 < \alpha = \text{constant}. \tag{1.2}$$

It will be assumed throughout the paper that the exponent $p(x,t)$ is continuous in $Q = \overline{Q_T}$ with logarithmic module of continuity:

$$1 < p^- = \inf_{(x,t)\in Q} p(x,t) \leq p(x,t) \leq p^+ = \sup_{(x,t)\in Q} p(x,t) < \infty, \tag{1.3}$$

$$\forall z = (x,t), \xi = (y,s) \in Q_T, |z-\xi| < 1, \quad \bigl|p(z) - p(\xi)\bigr| \le \omega\bigl(|z-\xi|\bigr), \tag{1.4}$$

where

$$\limsup_{\tau \to 0^+} \omega(\tau) \ln \frac{1}{\tau} = C < +\infty,$$

and the coefficient $b(x,t)$ is a Carathéodory function.

Model (1.1) proposed by Růžička may describe some properties of electro-rheological fluids which change their mechanical properties dramatically when an external electric field is applied [1, 2]. The variable exponent p in Model (1.1) is a function of the external electric field $|E|^2$ which is subject to the quasi-static Maxwell equations

$$\operatorname{div}(\varepsilon_0 \vec{E} + \vec{P}) = 0, \qquad \operatorname{Curl}(\vec{E}) = 0,$$

where ε_0 is the dielectric constant in vacuum and the electric polarization $\vec{P}$ is linear in $\vec{E}$, *i.e.* $\vec{P} = \lambda \vec{E}$. Another important application is the image processing where the anisotropy and nonlinearity of the diffusion operator are used to underline the borders of the distorted image and to eliminate the noise [3, 4]. For more physical background, the interested reader may refer to [4–17].

In the case when $p(x,t)$ is a fixed constant, there have been many results about the existence, uniqueness, nonexistence, extinction of the solutions [10, 18, 19]. For nonconstant case, the authors of [7–9] and the authors of [17] studied the existence and uniqueness of weak solutions of the initial and Dirichlet boundary value problem with variable exponent of nonlinearity. Besides, the authors of [20] applied the differential and variational techniques to prove the existence of solutions when the exponent p only depends on the spatial variable. Motivated by the work above, we consider the existence and uniqueness of solutions to Problem (1.1). However, since the coefficient $b(x,t)$ is degenerate or singular, it is natural to ask: which kind of conditions on the coefficient $b(x,t)$ guarantees that Problem (1.1) admits a local solution? In our paper, we construct suitable function spaces and apply Galerkin's method to prove the existence of weak solutions to Problem (1.1) with necessary uniform estimates and compactness argument. In addition, there exist some difficulties such as the failure of the monotonicity of the energy functional, the anisotropy of the diffusive operator and the gap between the norm and the modular, which make the methods used in [19] fail. In order to overcome such difficulties, we have to search a new technique or method. In this paper, by constructing a revised energy functional and combining a new energy estimate with convex method, we obtain the nonexistence of weak solutions when the exponents p is a function with respect to time and spatial variables. Especially, it is worth pointing out that the results are obtained with the assumption that $p_t(x,t)$ is only negative and integrable which is weaker than those the most of the other papers required.

The outline of this paper is the following: In Section 2, we shall introduce the function spaces of Orlicz-Sobolev type, give the definition of the weak solution to the problem, and prove the existence of weak solutions with Galerkin's method and the uniqueness of the solution. In Section 3, we establish sufficient condition of nonexistence of weak solutions of Problem (1.1) with the assumption that the exponent $p(x,t)$ depends on the time and spatial variables.

2 Existence of local solutions

In this section, the existence of weak solutions will be studied. First of all, we introduce some Banach spaces:

$$L^{p(x)}(\Omega) = \left\{u(x)\middle| u \text{ is measurable in } \Omega, A_{p(\cdot)}(u) = \int_\Omega |u|^{p(\cdot)}\,dx < \infty\right\},$$

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0, A_{p(\cdot)}(u/\lambda) \le 1\};$$

$$W^{1,p(x)}(\Omega) := \{u : u \in L^{p(\cdot)}(\Omega), |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot),\Omega} + \|\nabla u\|_{p(\cdot),\Omega};$$

$$V_t(\Omega) = \{u | u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |u|^{p(x,t)} \in L^1(\Omega), |\nabla u|^{p(x,t)} \in L^1(\Omega)\},$$

$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p(\cdot),\Omega};$$

$$H(Q_T) = \{u : [0,T] \mapsto V_t(\Omega) | u \in L^2(Q_T), |\nabla u| \in L^{p(x,t)}(Q_T), u = 0 \text{ on } \Gamma_T\},$$

$$\|u\|_{H(Q_T)} = \|u\|_{2,Q_T} + \|\nabla u\|_{p(\cdot),Q_T},$$

and denote by $H'(Q_T)$ the dual of $H(Q_T)$ with respect to the inner product in $L^2(Q_T)$. From [11], we know that condition (1.4) implies that $M \triangleq \{u : u \in W^{1,p(x)}(\Omega), u = 0 \text{ on } \partial\Omega\}$ is equivalent to $W_0^{1,p(x)}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$).

Lemma 2.1 [11, 12] *For any $u \in L^{p(x)}(\Omega)$,*

(1) $\|u\|_{p(\cdot)} < 1\ (= 1; > 1) \quad \Leftrightarrow \quad A_{p(\cdot)}(u) < 1\ (= 1; > 1)$;

(2) $\|u\|_{p(\cdot)} < 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^+} \le A_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-}$;

$\|u\|_{p(\cdot)} \ge 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^-} \le A_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^+}$;

(3) $\|u\|_{p(\cdot)} \to 0 \quad \Leftrightarrow \quad A_{p(\cdot)}(u) \to 0$;

$\|u\|_{p(\cdot)} \to \infty \quad \Leftrightarrow \quad A_{p(\cdot)}(u) \to \infty$.

Lemma 2.2 [11, 12] (Hölder's inequality) *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ with $q(x) = \frac{p(x)}{p(x)-1}$,*

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|u\|_{p(\cdot)}\|v\|_{q(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{q(\cdot)}.$$

Because of the degeneracy, Problem (1.1) does not admit classical solutions in general. So we introduce weak solutions in the following sense.

Definition 2.1 A function $u(x,t) \in H(Q_T) \cap L^\infty(0,T;L^2(\Omega))$, $b(x,t)|\nabla u| \in L^2(0,T;L^2(\Omega))$ is called a weak solution of Problem (1.1) if for every test-function

$$\xi \in \mathcal{Z} \triangleq \{\eta(z) : \eta \in H(Q_T) \cap L^2(0,T;H_0^1(\Omega)), \eta_t \in H'(Q_T)\},$$

and every $t_1, t_2 \in [0, T]$ the following identity holds:

$$\int_{t_1}^{t_2} \int_\Omega \big[u\xi_t - \big(b(x,t)\nabla u + |\nabla u|^{p(x,t)-2}\nabla u\big)\nabla\xi + f(u)\xi\big]\,dx\,dt = \int_\Omega u\xi\,dx\big|_{t_1}^{t_2}. \tag{2.1}$$

Remark 2.1 On one hand, $u \in H(Q_T)$, $\xi \in \mathcal{Z}$, implies that

$$\int_{t_1}^{t_2}\int_\Omega u\xi_t\,dx\,dt < \infty,$$
$$\int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p(x,t)-2}\nabla u\nabla\xi\,dx\,dt < \infty,$$
$$\int_\Omega u\xi\,dx\big|_{t_1}^{t_2} < \infty.$$

On the other hand,

$$u \in L^\alpha(Q_T), \quad \big|f(u)\big| \le a_0|u|^{\alpha-1} \quad \text{and} \quad \xi \in L^\alpha(Q_T) \quad \Longrightarrow \quad \int_{t_1}^{t_2}\int_\Omega f(u)\xi\,dx\,dt < \infty;$$
$$b(x,t)|\nabla u| \in L^2\big(0,T;L^2(\Omega)\big), \qquad \xi \in H(Q_T) \quad \Longrightarrow \quad \int_{t_1}^{t_2}\int_\Omega b(x,t)\nabla u\nabla\xi\,dx\,dt < \infty,$$

which imply that the definition of weak solutions is well defined.

The main theorem in this section is the following.

Theorem 2.1 *Suppose that the continuous function* $f(s)$ *and the exponents* $p(x,t)$, α *satisfy conditions* (1.2)-(1.4). *If the following conditions hold*:

$$(\mathrm{H}_1) \quad 2 < p^- < p^+ < \max\left\{N, \frac{Np^-}{N-p^-}\right\}, \qquad 2 < \alpha < p^-;$$
$$(\mathrm{H}_2) \quad u_0 \in L^2(\Omega), \qquad b(x,t) \in L^{\frac{p^-}{p^--2}}\big(0,\infty;L^{\frac{p^-}{p^--2}}(\Omega)\big),$$

then Problem (1.1) *has at least one weak solution* $u(x,t) \in H(Q_T) \cap L^\infty(0,T;L^2(\Omega))$, $b(x,t)|\nabla u| \in L^2(0,T;L^2(\Omega))$.

Due to $p^- > 2$ and $b(x,t) \in L^{\frac{p^-}{p^--2}}(0,\infty;L^{\frac{p^-}{p^--2}}(\Omega))$, the term $b(x,t)|\nabla u|^2$ can be controlled by the nonlinear diffusion term $|\nabla u|^{p(x,t)}$, and then one may follow the lines of the proof of Theorem 3.1(a) in [6] or Theorem 4.1(a) of Chapter 4 in [7] to complete the rest of the proof.

Corollary 2.1 *Let the conditions of Theorem* 2.1 *be fulfilled, then the solution* $u \in H(Q_T)$ *to Problem* (1.1) *satisfies the identity*

$$\iint_{Q_T} u_t\xi\,dx + \iint_{Q_T}\big[|\nabla u|^{p(x,t)-2}\nabla u\nabla\xi + b(x,t)\nabla u\nabla\xi - f(u)\xi\big]\,dx\,dt$$
$$= 0, \quad \forall\xi \in \mathcal{Z}. \tag{2.2}$$

When $b(x,t) \geq 0$, we follow the line of the proof of Theorem 5.1 in [6] to obtain the following theorem.

Theorem 2.2 *Suppose that the conditions in Theorem* 2.1 *are fulfilled and* $b(x,t) \geq 0$, *then the bounded solution of Problem* (1.1) *is unique provided that the following condition holds*

(H_3) *the function* $f(s)$ *is decreasing in* $s \in \mathbb{R}$.

Furthermore, we have the following comparison theorem.

Corollary 2.2 (Comparison principle) *Let* $u, v \in H(Q_T) \cap L^2(0,T;H_0^1(\Omega))$ *be two bounded weak solutions of Problem* (1.1) *such that* $u(x,0) \leq v(x,0)$ *a.e. in* Ω. *If the nonlinearity exponents and the function* $f(s)$ *satisfy the conditions of Theorem* 3.1, *then* $u(x,t) \leq v(x,t)$ *a.e. in* Q_T.

3 Nonexistence of global weak solutions

In this section, we concentrate on the study of nonexistence of weak solutions to Problem (1.1). For convenience, we first state that the function $f(s)$ and the coefficient $b(x,t)$ satisfy the following conditions:

$$b(x,t) \geq 0, \qquad b_t(x,t) \leq 0, \quad \forall (x,t) \in Q_T; \tag{3.1}$$

$$f(u) \in C(R), \qquad f(u)u - p^+ G(u) \geq 0, \quad \forall u \in \mathbb{R}, \tag{3.2}$$

with $G(u) = \int_0^u f(s)\,ds$. Before stating the main results, we give the definition of global solutions.

Definition 3.1 A function $u(x,t)$ is called a global solution to Problem (1.1) if $\forall T > 0$ the following property holds:

$$\sup_{t \in (0,T)} \left\| u(x,t) \right\|_{L^\infty(\Omega)} < +\infty.$$

Otherwise, we say that Problem (1.1) does not admit global weak solutions.

First, we consider the case $p(x,t) \equiv p(x)$. Our main result is as follows.

Theorem 3.1 *Assume that* $u(x,t) \in H(Q_T) \cap L^\infty(0,T;L^2(\Omega))$, $b(x,t)|\nabla u| \in L^2(0,T;L^2(\Omega))$ *is the local solution to Problem* (1.1). *If* (3.1) *and* (3.2) *are fulfilled and* $u_0 \in W_0^{1,p(x)}(\Omega)$, $p^+ > 2$, *such that*

$$\begin{aligned} \int_\Omega G(u_0)\,dx > \int_\Omega \frac{b(x,0)}{2} |\nabla u_0|^2\,dx + \int_\Omega \frac{1}{p(x)} |\nabla u_0|^{p(x)}\,dx \\ + \frac{4(p^+ - 1)}{Tp^+(p^+ - 2)^2} \int_\Omega |u_0|^2\,dx. \end{aligned} \tag{3.3}$$

Then there exists a $T^* \in (0,T]$ *such that*

$$\lim_{t \to T^{*-}} \left\| u(\cdot,t) \right\|_{\infty,\Omega} = +\infty.$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.1 *Assume that $u \in H(Q_T)$ is the solution to Problem* (1.1), *then $u(x,t)$ satisfies the following relation*:

$$\begin{aligned}&\int_\Omega \frac{1}{p(x)}\left|\nabla u(x,t)\right|^{p(x)}dx + \int_\Omega \frac{b(x,t)}{2}\left|\nabla u(x,t)\right|^2 dx - \int_\Omega G\big(u(x,t)\big)\,dx \\ &\qquad + \int_0^t\int_\Omega (u_\tau)^2\,dx\,d\tau - \int_0^t\int_\Omega \frac{b_\tau|\nabla u|^2}{2}\,dx\,d\tau \\ &\quad = \int_\Omega \frac{1}{p(x)}\left|\nabla u_0(x)\right|^{p(x)}dx + \int_\Omega \frac{b(x,0)}{2}\left|\nabla u_0(x)\right|^2 dx - \int_\Omega G\big(u_0(x)\big)\,dx. \end{aligned}\tag{3.4}$$

Proof Following the lines of the proof of Lemma 3.1 and Theorem 6.1 in [7], we know that $u_t \in L^2(Q_T)$. Noting that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{|\nabla u|^{p(x)}}{p(x)}\right) = |\nabla u|^{p(x)-2}\nabla u\nabla u_t, \qquad \frac{\partial}{\partial t}\big(G\big(u(x,t)\big)\big) = f\big(u(x,t)\big)u_t, \\ &\frac{\partial}{\partial t}\left(b(x,t)\frac{|\nabla u|^2}{2}\right) = b(x,t)\nabla u\nabla u_t + b_t\frac{|\nabla u|^2}{2}\end{aligned}$$

and using the idea of the proof of Lemma 1 in [8], we arrive at the relation

$$\frac{d}{dt}\int_\Omega\left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{b(x,t)}{2}\left|\nabla u(x)\right|^2 - G\big(u(x,t)\big)\right)dx - \int_\Omega \frac{b_t|\nabla u|^2}{2}\,dx = -\int_\Omega u_t^2\,dx.$$

After integrating over $(0,t)$, it is obvious that Lemma 3.1 holds. □

Proof of Theorem 3.1 Let

$$\begin{aligned}&\beta = \frac{2}{T(p^+-2)^2}\int_\Omega u_0^2\,dx, \qquad t_0 = \frac{T(p^+-2)}{2}, \\ &K(t) = \frac{1}{2}\int_0^t\int_\Omega u^2\,dx\,d\tau + (T-t)\int_\Omega \frac{1}{2}u_0^2\,dx + \beta(t+t_0)^2.\end{aligned}$$

Clearly

$$\begin{aligned}K'(t) &= \frac{1}{2}\int_\Omega u^2\,dx\,dt - \frac{1}{2}\int_\Omega u_0^2\,dx + 2\beta(t+t_0) \\ &= \iint_{Q_t}\big(-b(x,\tau)|\nabla u|^2 - |\nabla u|^{p(x)} + f(u)u\big)\,dx\,dt + 2\beta(t+t_0); \\ K''(t) &= \int_\Omega\big(-b(x,\tau)|\nabla u|^2 - |\nabla u|^{p(x)} + f(u)u\big)\,dx + 2\beta.\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega u^2\,dx\,dt - \frac{1}{2}\int_\Omega u_0^2\,dx &= \frac{1}{2}\left|\int_0^t\int_\Omega \big(u^2\big)_\tau\,dx\,d\tau\right| \\ &\le \left(\int_0^t\int_\Omega u^2\,dx\,d\tau\right)^{1/2}\left(\int_0^t\int_\Omega |u_\tau|^2\,dx\,d\tau\right)^{1/2}.\end{aligned}$$

Thus by Schwarz's inequality and the definition of $K(t)$, we have

$$\begin{aligned}
\big(K'(t)\big)^2 &\leq \left(\int_0^t\int_\Omega u^2\,dx\,d\tau\right)\left(\int_0^t\int_\Omega u_\tau^2\,dx\,d\tau\right) + 4\beta^2(t+t_0)^2 \\
&\quad + 4\beta(t+t_0)\left(\int_0^t\int_\Omega u^2\,dx\,d\tau\right)^{1/2}\left(\int_0^t\int_\Omega u_\tau^2\,dx\,d\tau\right)^{1/2} \\
&\leq \left(\int_0^t\int_\Omega u^2\,dx\,d\tau\right)\left(\int_0^t\int_\Omega u_\tau^2\,dx\,d\tau\right) + 4\beta^2(t+t_0)^2 \\
&\quad + 2\int_0^t\int_\Omega u_\tau^2\,dx\,d\tau\left[(T-t)\int_\Omega \frac{1}{2}u_0^2\,dx + \beta(t+t_0)^2\right] \\
&\quad + 4\beta^2(t+t_0)^2\left(\int_0^t\int_\Omega \frac{1}{2}u^2\,dx\,d\tau\right)\left((T-t)\int_\Omega \frac{1}{2}u_0^2\,dx + \beta(t+t_0)^2\right)^{-1} \\
&\leq K(t)\left(2\int_0^t\int_\Omega u_\tau^2\,dx\,d\tau + 4\beta\right).
\end{aligned}$$

Therefore by Lemma 3.1, we obtain the following inequality:

$$\begin{aligned}
&K(t)K''(t) - \frac{p^+}{2}\big(K'(t)\big)^2 \\
&\quad\geq K(t)\left(\int_\Omega b(x,t)\left(\frac{p^+}{2}-1\right)|\nabla u|^2\,dx + \int_\Omega\left(\frac{p^+}{p(x)}-1\right)|\nabla u|^{p(x)}\,dx\right. \\
&\qquad \left. + \int_\Omega \big(uf(u) - p^+G(u)\big)\,dx\right) - 2\beta\big(p^+-1\big) \\
&\qquad + \frac{p^+}{2}\int_\Omega\left[2G(u_0) - b(x,0)|\nabla u_0|^2 - \frac{2}{p(x)}|\nabla u_0|^{p(x)}\right]dx \\
&\qquad - \frac{p^+}{2}\int_0^t\int_\Omega \frac{b_t|\nabla u|^2}{2}\,dx\,dt.
\end{aligned} \tag{3.5}$$

Noticing $p^+ > 2$, $K(t) > 0$, we conclude from (3.1), (3.2), (3.3) that

$$K(t)K''(t) - \frac{p^+}{2}\big(K'(t)\big)^2 \geq 0, \quad \text{for } t \in (0,T),$$

which implies

$$\big(K^{1-\frac{p^+}{2}}(t)\big)'' \leq 0, \quad \text{for } t \in (0,T).$$

Noting that $K^{1-\frac{p^+}{2}}(0) > 0$, $(K^{1-\frac{p^+}{2}})'(0) \leq 0$, then

$$K^{1-\frac{p^+}{2}}\big(T^*\big) = 0, \quad \text{for some } T^* \in \left(0, \frac{-K^{1-\frac{p^+}{2}}(0)}{(K^{1-\frac{p^+}{2}})'(0)}\right).$$

Here

$$\frac{-K^{1-\frac{p^+}{2}}(0)}{(K^{1-\frac{p^+}{2}})'(0)} = \frac{T\int_\Omega u_0^2\,dx + 2\beta t_0^2}{2(p^+-2)\beta t_0^2} \leq T.$$

Thus, $0 < T^* \leq T$ and

$$\lim_{t\to T^{*-}} \big|u(\cdot,t)\big|_{\infty,\Omega} = +\infty.$$

This completes the proof of Theorem 3.1. □

Next, we consider the case when p is dependent of t. Before stating the conclusion, we first give a useful lemma.

Lemma 3.2 *Assume $u_0 \in W_0^{1,p(x,0)}(\Omega)$, $p^+ > 2$, $p_t \leq 0$. Then the solution of Problem* (1.1) *satisfies*

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x,t)}\big|\nabla u(x,t)\big|^{p(x,t)}\,dx + \int_\Omega \frac{b(x,t)}{2}\big|\nabla u(x,t)\big|^2\,dx - \int_\Omega G\big(u(x,t)\big)\,dx \\
&\quad + \int_0^t\int_\Omega |u_\tau|^2\,dx\,d\tau - \int_0^t\int_\Omega \frac{b_\tau|\nabla u|^2}{2}\,dx\,d\tau \\
&\leq \int_\Omega \frac{1}{p(x,0)}\big|\nabla u_0(x)\big|^{p(x,0)}\,dx + \int_\Omega \frac{b(x,0)}{2}\big|\nabla u_0(x)\big|^2\,dx - \int_\Omega G\big(u_0(x)\big)\,dx \\
&\quad + \int_\Omega\left(\frac{1}{p(x,t)} - \frac{1}{p(x,0)}\right)dx.
\end{aligned} \tag{3.6}$$

Proof Following the lines of the proof of Lemmas 3.1 and Lemma 6.1 of [7], we know $u_t \in L^2(Q_T)$ and

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{|\nabla u|^{p(x,t)}}{p(x,t)}\right) = |\nabla u|^{p(x,t)-2}\nabla u\nabla u_t + \frac{p_t}{p^2}|\nabla u|^{p(x,t)}\big(\ln|\nabla u|^{p(x,t)} - 1\big),\\
&\frac{\partial}{\partial t}\big(G\big(u(x,t)\big)\big) = f\big(u(x,t)\big)u_t, \qquad \frac{\partial}{\partial t}\left(b(x,t)\frac{|\nabla u|^2}{2}\right) = b(x,t)\nabla u\nabla u_t + b_t\frac{|\nabla u|^2}{2}.
\end{aligned}$$

On one hand, a simple analysis shows that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(\frac{|\nabla u|^{p(x,t)}}{p(x,t)} + \frac{b(x,t)}{2}\big|\nabla u(x)\big|^2 - G\big(u(x,t)\big)\right)dx - \int_\Omega \frac{b_t|\nabla u|^2}{2}\,dx \\
&\quad = \int_\Omega\left[-u_t^2 + \frac{p_t}{p^2}|\nabla u|^{p(x,t)}\big(\ln|\nabla u|^{p(x,t)} - 1\big)\right]dx.
\end{aligned} \tag{3.7}$$

On the other hand, we apply the condition $p_t \leq 0$ to obtain

$$\begin{aligned}
&\int_{\{|\nabla u|^p \leq e\}} \frac{|\nabla u|^{p(x,t)}}{p^2(x,t)}\big(\ln|\nabla u|^{p(x,t)} - 1\big)p_t(x,t)\,dx \\
&\quad \leq \int_{\{|\nabla u|^p \leq e\}} \frac{-p_t(x,t)}{p^2(x,t)}\,dx \leq \int_\Omega \frac{-p_t(x,t)}{p^2(x,t)}\,dx.
\end{aligned} \tag{3.8}$$

The second inequality above follows from

$$-\frac{1}{e} \leq s\ln s \leq 0, \qquad 0 \leq s \leq 1.$$

Lemma 3.2 follows from (3.7) and (3.8). □

Our main result is as follows.

Theorem 3.2 *Suppose that* (3.1) *and* (3.2) *hold and* $p^+ > 2$, $p_t \le 0$. *If* $u_0 \in W_0^{1,p(x,0)}(\Omega)$ *satisfies*

$$\int_\Omega G(u_0)\,dx > \int_\Omega \frac{b(x,0)}{2}|\nabla u_0|^2\,dx + \int_\Omega \frac{1}{p(x,0)}|\nabla u_0|^{p(x,0)}\,dx + \frac{4(p^+-1)}{Tp^+(p^+-2)^2}\int_\Omega |u_0|^2\,dx + \int_\Omega\left(\frac{2}{p^-} - \frac{1}{p(x,0)}\right)dx, \tag{3.9}$$

then there exists $T^* \in (0,T]$ *such that*

$$\lim_{t\to T^{*-}} \left|u(\cdot,t)\right|_{\infty,\Omega} = +\infty.$$

Proof We argue by contradiction. Define

$$\beta = \frac{2}{T(p^+-2)^2}\int_\Omega u_0^2\,dx, \qquad t_0 = \frac{T(p^+-2)}{2},$$
$$K(t) = \frac{1}{2}\int_0^t\int_\Omega u^2\,dx\,d\tau + (T-t)\int_\Omega \frac{1}{2}u_0^2\,dx + \beta(t+t_0)^2.$$

It is easy to verify that

$$\begin{aligned} K'(t) &= \frac{1}{2}\int_\Omega u^2\,dx\,dt - \frac{1}{2}\int_\Omega u_0^2\,dx + 2\beta(t+t_0) \\ &= \iint_{Q_t}\left(-b(x,\tau)|\nabla u|^2 - |\nabla u|^{p(x,t)} + f(u)u\right)dx\,dt + 2\beta(t+t_0); \\ K''(t) &= \int_\Omega\left(-b(x,\tau)|\nabla u|^2 - |\nabla u|^{p(x,t)} + f(u)u\right)dx + 2\beta; \\ \left(K'(t)\right)^2 &\le K(t)\left(2\int_0^t\int_\Omega u_\tau^2\,dx\,d\tau + 4\beta\right). \end{aligned} \tag{3.10}$$

Therefore by Lemma 3.2 and (3.10), we obtain the following inequality:

$$\begin{aligned} &K(t)K''(t) - \frac{p^+}{2}\left(K'(t)\right)^2 \\ &\quad\ge K(t)\left[\int_\Omega b(x,t)\left(\frac{p^+}{2}-1\right)|\nabla u|^2\,dx + \int_\Omega\left(\frac{p^+}{p(x,t)}-1\right)|\nabla u|^{p(x,t)}\,dx\right. \\ &\qquad + \int_\Omega\left(uf(u) - p^+G(u)\right)dx - 2\beta\left(p^+-1\right) \\ &\qquad + \frac{p^+}{2}\int_\Omega\left[2G(u_0) - b(x,0)|\nabla u_0|^2 - \frac{2}{p(x,0)}|\nabla u_0|^{p(x,0)}\right]dx \\ &\qquad \left. - \frac{p^+}{2}\int_0^t\int_\Omega \frac{b_\tau|\nabla u|^2}{2}\,dx\,d\tau + \frac{p^+}{2}\int_\Omega\left(\frac{1}{p(x,t)} - \frac{1}{p(x,0)}\right)dx\right]. \end{aligned} \tag{3.11}$$

In the rest of the proof, we follow the lines of the proof of Theorem 3.1 to finish the proof of this theorem. □

At the end of this paper, we give an example to illustrate that the condition on $p_t(x,t)$ is weaker than that of [8, 9].

Example 3.1 Set $p(x,y,z,t) = \frac{\sqrt{t}\cos x}{100} + \frac{5}{2}$, $x \in (\frac{\pi}{2}, \pi)$, $y,z \in (0, \frac{\pi}{2})$, $0 < t < 1$. A simple computation shows that

$$p_t(x,y,z,t) = \frac{\cos x}{200\sqrt{t}} < 0,$$

$$\int_0^1 \int_{\frac{\pi}{2}}^{\pi} \iint_0^{\frac{\pi}{2}} \left|p_t(x,y,z,t)\right| \, dx\, dy\, dz\, dt = \frac{\pi^2}{400}, \quad p_t(x,y,z,t) \notin L^\infty.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this paper and they read and approved the final manuscript.

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